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RESEARCH ARTICLE

On the explicit solutions of fractional Bagley-Torvik equation arises in engineering

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ABSTRACT

In this work, Bagley-Torvik equation is considered with conformable derivatives. The analytical solutions will be obtained via Sine-Gordon expansion method and Bernouli equation method for the two cases of Bagley-Torvik equation. We will illustrate and discuss about the methodology and solutions therefore the proposed equation has meaning in different areas of science and engineering.

1. Introduction

For engineering and science, fractional calculus has become the important theory including both conservative and nonconservative phenomena [1] and to model realistic processes such as diffusion wave, electromagnetic waves, heat conduction, electro-electrolyte polarization [2, 3].

In this paper, the Bagley-Torvik equation, the specific type of fractional hyperbolic partial differential equation, is considered

$$u_t^{\,2} - u_{xx} + u_t^{\,\alpha} = f(x,t) \tag{1}$$

where $f(x,t)$ is continuous for $t>0, x<1$ and $m-1<\alpha<m$ is the order fractional derivative.

Generally, the 1/2-order derivative and 3/2-order derivative is common to determine the frequency-dependent damping materials [4,5]. Therefore, Eq. (1) with 1/2-order derivative or 3/2-order derivative is used to model the motion of real physical systems. The most known examples for each derivative are an immersed plate in a Newtonian fluid and a gas in a fluid, respectively [6, 7]. When α is between 0 and 2, it describes damping force. We will consider Eq. (1) for $\alpha = 1/2$.

The papers on the modelling physical phenomena via fractional Bagley-Torvik equation, the numeric [8-12] or analytical solutions [6, 13, 14] of Eq. (1) are seen in the literature commonly. A fractional mathematical model for a micro-electro-mechanical system (MEMS) device has been developed to measure the viscosity of fluids during oil well exploration by Fitt et al. [15]. There are many numerical methods based on Bernoulli polynomials [11], generalized form of the Bessel functions of the first kind [9], wavelet [10], the generalized Taylor series [8], spline methods [17, 18], finite difference scheme [12, 16] etc. to solve the fractional Bagley–Torvik equation. In addition, the approximations and analytical methods are proposed to solve the fractional Bagley-Torvik equation such as quadratic polynomial spline function [18], homogenous balanced principle [13], Adomian decomposition method [20, 21], first integral method [22], homotopy analysis method [19, 23], Lie group theory method [24, 25], invariant subspace method [14, 26, 27, 36], Fractional variational method [28- 30, 56], G'/G-expansion method [31], sub-equation method [32-35], transformed rational function method [37], multiple exp-function method [38].

Besides different approaches to solve fractional partial differential equation, the other most important tool is defination of the fractional derivative. So there are many approaches/definitions for fractional derivative such as Riemann–Liouville definition, Grünwald–Letnikov definition, Caputo definition, Riez–Feller definition, Miller-Ross sequential definition, Weyl definition, Jumarie's modified Riemann–Liouville definition [3, 39, 40]. Among them, the most knowns are Riemann–Liouville definition and Caputo

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definition whereas in the recent times, the most popular ones are conformable derivative [41, 42]. The most known definitions of fractional derivatives, Riemann–Liouville definition and Caputo definition, depend on Gamma function. Therefore, Gamma function can be defined as a definite integral and also it is seen as in the definitions whose behavior is asymptotic. Because of this reason, we consider the conformable derivative.

In the work, the fractional Bagley-Torvik equation with the one of the popular derivative definitions, conformable derivative is considered. The analytical solutions will be obtained via Sine-Gordon expansion method and Bernoulli approximation method. The obtained solutions will be compared with the exact solutions.

2. Definitions and methodologies

2.1. Basic definitions

As we mentioned in the introduction, various definitions for fractional derivative are seen and the most known and used ones are the Riemann–Liouville and the Caputo fractional derivative, there is a relation between the two. Generally, Caputo fractional derivative is preferred so it is not depended on initial conditions to give the physical meaning, but generally it can be said that it has advantages for fractional differential equations with initial conditions. These definitions are useful for modelling but they have lack of main properties for the computation as the product rule, quotient rule and the chain rules and etc. Because their definitions include the Gamma function which is a special function and has an asymptotic behavior. The transition between fractional derivative and Newton derivative is not exact. To overcome these problems, Abdeljawad [43] proposed the conformable derivative and its most properties correspond to classical derivative and with this definition the equations can be solved more easily.

Definition 1. $f:[0,\infty)\to R$ is a function, the conformable derivative of order α is given by

$$T_\alpha(f)(t)=\lim_{\varepsilon\to 0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon},\forall t>0,\alpha\in(0,1).$$

Therefore, if f is α- differentiable in some $(0,a), a>0$ and $\lim_{t\to 0^+} f^{\alpha}(t)$ exists, then define $f^{\alpha}(0)=\lim_{t\to 0^+} f^{\alpha}(t)$.

Properties. All properties of the classical derivatives are same as the conformable derivative such as linearity, sum, product, division, etc. In addition to these properties, assume that $\alpha\in(0,1)$ and f is differentiable $t>0$, the following property of the conformable derivative is given:

$T_\alpha\left(f\right)(t)=t^{1-\alpha}\frac{df}{dt}$, if f is differentiable.

2.2. Methodologies

There are various analytical methods to obtain the analytical/exact solutions of partial differential equations and also these methods can be applied to fractional partial differential equations with some modifications. The popular methods in the last decade are to obtain the exact solutions of NPDEs such as tanh-method [44, 45], G'/G -expansion method [46, 47], simplest equation method [48], auxiliary equation method [49, 50], sub-equation method [51], and so on. With the same view, the methods can also be applied to the fractional partial differential equations with the modification of the transformation [42, 52, 53, 57].

For the general case, the conformable fractional partial differential equation is considered

$$F\left(u,\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial u}{\partial x},\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}},\frac{\partial^2 u}{\partial x^2},...\right)=0. \tag{2}$$

To reduce Eq. (2) into nonlinear ODE, instead of $\zeta=\mu x+\beta t$ the classical wave transformation, the new transformation $\zeta=x+\beta\frac{t^\alpha}{\alpha}$ is used by many authors in the literature.

Proposition 1. Using the wave transformation $\zeta=\mu x+\beta t$ and properties of the conformable derivative especially $T_\alpha\left(u\right)(t)=t^{1-\alpha}\frac{du}{dt}$, if u is differentiable are used to reduce into nonlinear ODE respect to ζ

$$F(u,u',u'',u''',...)=0. \tag{3}$$

Remark 1. As a result, the obtained nonlinear ODE is generally variable coefficient nonlinear differential equation.

In the view of auxiliary equation method, the solution of Eq. (3) is considered as the finite sum of the solution of the proposed auxiliary equation

$$u(\zeta)=\sum_{i=1}^{N}a_i z^i(\zeta) \tag{4}$$

where $z(\zeta)$ is the solution of the proposed auxiliary equation, a_i are the parameters will be determined via obtained algebraic system, N is determined by the balancing principle [54]. The procedure is the same, substituting the proposed auxiliary equation and solution (Eq. 4) into the reduced equation (Eq. 3), then classify the obtained equation respect to the powers of $z(\zeta)$ Each coefficient of the power of $z(\zeta)$ is equal to zero, so the algebraic system is obtained and the solutions of system are parameters in Eq. (4). Substituting results and transformation in Eq. (4), the analytical solution of Eq. (2) is obtained.

In this work, two types of auxiliary equations which are different from the literature are considered. The first one is the case of the Sine-Gordon equation and the second one is the variable coefficient Bernoulli type differential equation [50, 55].

3. Results and discussion

The Bagley-Torvik equation with the conformable derivative of order $\alpha = 1/2$ is considered

$$u_t^{\,2} - u_{xx} + u_t^{1/2} = f(x,t) \qquad (5)$$

where $f(x,t) = \left(2 - \frac{8t^{3/2}}{3\sqrt{\pi}} + (\pi t)^2\right)\sin(\pi x)$. Its exact solution $u(x,t) = t^2 \sin(\pi x)$ is given via separation of variables by [12, 16].

Now we try to obtain analytical solutions by suggested methods with the proposed transformation $\zeta = \mu x + \beta t$ for $\alpha = 1/2$. With the proposed transformation, Eq. (5) is reduced into

$$\beta^2 (u')^2 - u'' + \beta t^{1-\alpha} u' = \left(2 - \frac{8t^{3/2}}{3\sqrt{\pi}} + (\pi t)^2\right)\sin(\pi x) \cdot$$

Case 1. The Sine-Gordon equation $u_{xx} - u_{tt} = m^2 \sin(u)$ is considered and its solution is obtained via the wave transformation as $\sin(w) = \sec h(\xi), \cos(w) = \tanh(\xi)$ .Therefore, the ansatz is

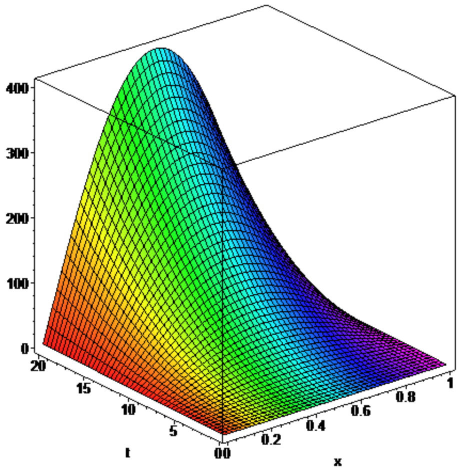
$$u(\xi) = a_0 + \sum_{i=1}^{N} \tanh^{i-1}(\xi)\left(b_i \sec h(\xi) + a_i \tanh(\xi)\right), \xi = \mu x + \beta t$$

When the given procedure is applied, as a solution of algebraic system, the parameters are obtained;

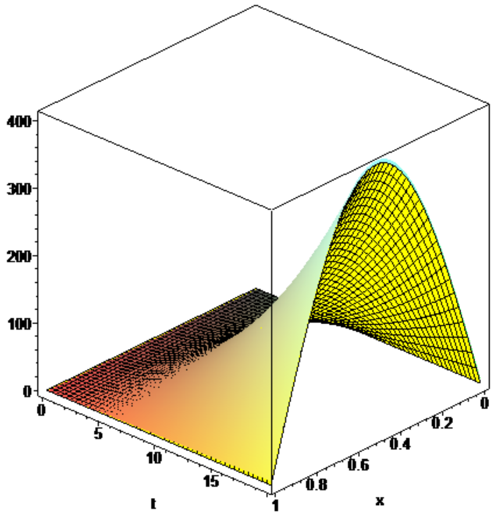
$$b_1 = \frac{I\mu^2}{2\beta^2}, a_2 = b_2 = -\frac{\mu^2}{2\beta^2}, a_1 = -b_1 - 2b_2 \tanh(\xi), a_0 = t^2 \sin(\pi x)$$

As a result, substituting the parameters and the solution of Sine-Gordon equation into Eq. (4), the analytical solution is given by Figure 1 for the special parameters

$$\beta = \frac{-144 + 42I}{\left((-5760 + 1680I)t\right)^{1/2}}, \mu = \left(-\frac{3}{10} + \frac{I}{10}\right)(-9 - 13I)^{1/2} \cdot$$



i) The solution of Eq. (5) via Sine-Gordon Expansion Method



ii) the comparison between analytical solution (surface) with the exact solution(surfacewireframe)

Figure 1. The solutions obtained via Sine-Gordon Expansion Method

In the following figures are obtained for $\alpha \in (0,1)$ and $x = 0.6$, the comparison of α values , comparison of approximate and exact solutions (see Figure 2).

Case 2. For the second we will consider the variable coefficient Bernoulli equation instead of the classical auxiliary equation

$$z'(\zeta) = P(\zeta)z(\zeta) + Q(\zeta)z^n(\zeta), n \neq 0,1 \qquad (6)$$

The solutions of Eq. (6) depends upon the coefficient functions $P(\zeta), Q(\zeta)$ and the degree of Eq. (6) n .

When the classical procedure is applied to Eq. (5), the coefficient functions and parameters are obtained

$$P(\zeta) = \frac{\beta}{1 + c_1\beta e^{-\beta\zeta}}, Q(\zeta) = \frac{8\beta^2 g_2}{3 + 3c_1\beta e^{-\beta\zeta}},$$

$$g_2 = \frac{5}{19}\beta^2 g_1^{\,2}, g_0 = \left(2 - \frac{8t^{3/2}}{3\sqrt{\pi}} + (\pi t)^2\right)\sin(\pi x)$$

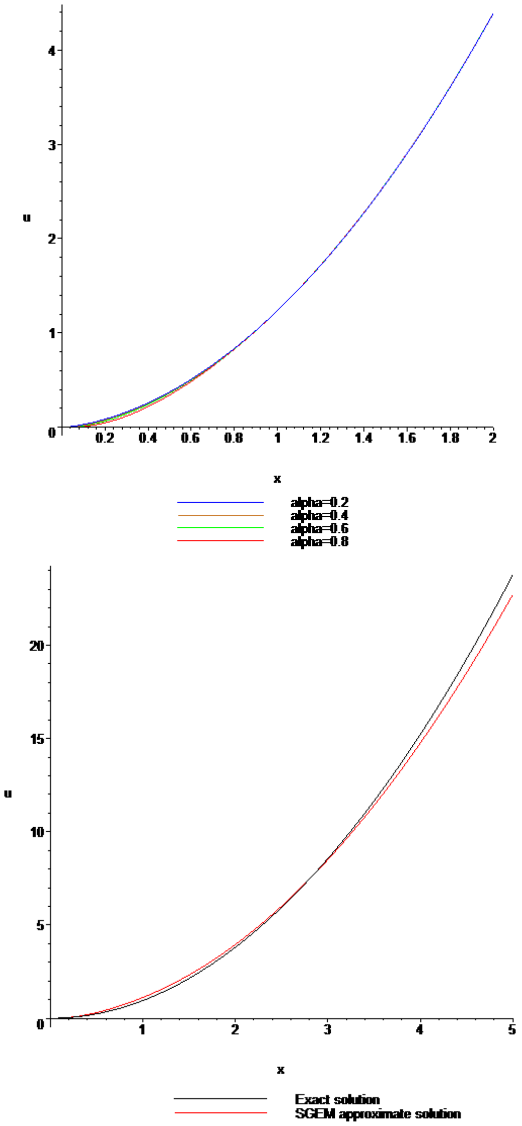
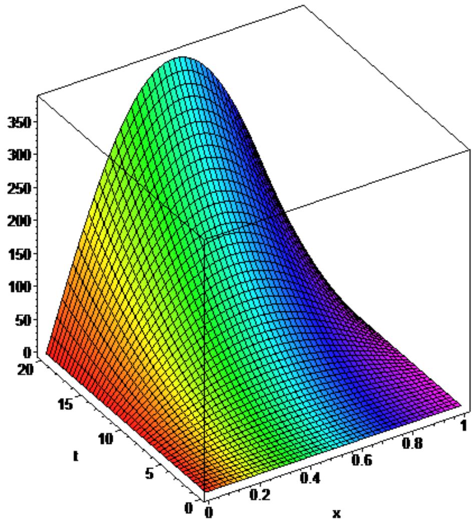


Figure 2. First one is the comparison of α values for the obtained solution; the second one is the comparison of exact solution with the analytical solution obtained via SGEM

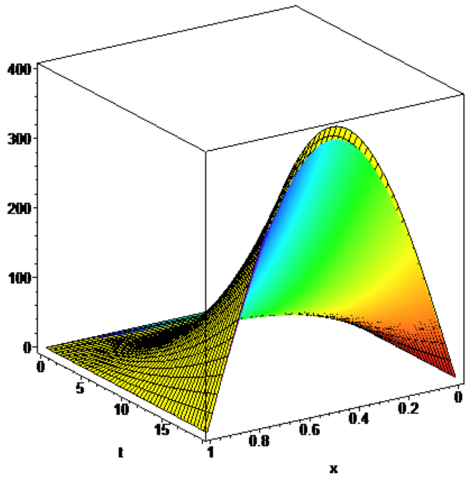
As a result, the solution of Eq. (6) with the obtained functions is

$$z(\zeta)=\pm\frac{3\left(e^{\beta\zeta}+c_1\beta\right)}{\sqrt{-24g_2\beta^2e^{2\beta\zeta}-48g_2c_1\beta^3e^{\beta\zeta}+9c_2}}.$$

Hence the solution of Eq. (5) is given by Figure 3 for the parameter values $c_1=-1.1, c_2=10^{-5}, \beta=0.8, g_1=10^{-4}$. In the following figures are obtained for $\alpha\in(0,1)$ and $x=0.6$, the comparison of α values, comparison of approximate and exact solutions.



i) The solution via Bernoulli approximation method



ii) the comparison between analytical solution (surface) with the exact solution(surfacewireframe)

Figure 3. The solutions obtained via Bernoulli approximation Method

4. Conclusion

In this work the Bagley-Torvik equation with the conformable derivative is considered and the solutions are obtained expected behavior via the Sine-Gordon expansion method and Bernoulli approximation method. Also the exact solution comparisons and the obtained analytical solutions are given by Figure 1 and Figure 3. These solutions are not have any sense in physics but in the future they will be useful for developing technology.

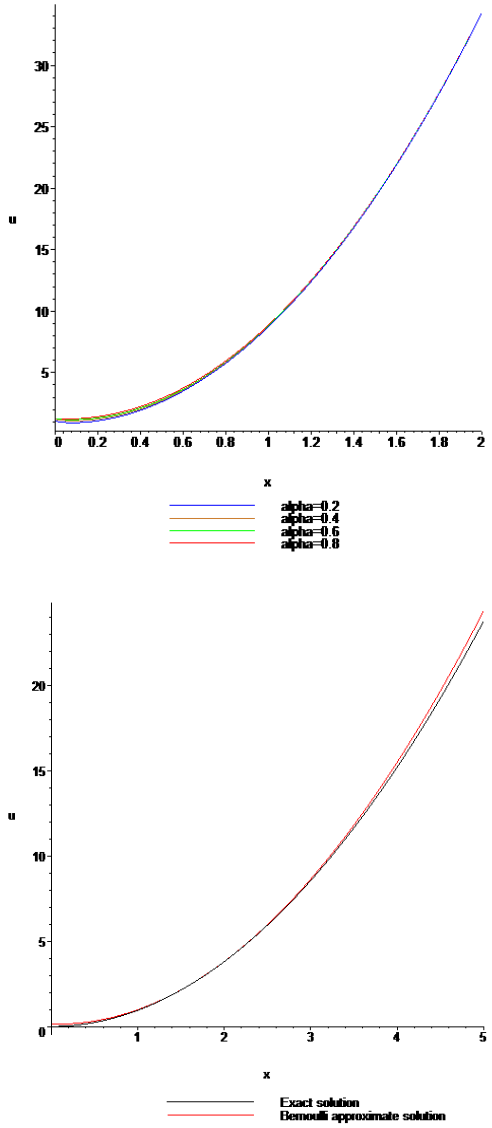


Figure 4. First one is the comparison of α values for the obtained solution; the second one is the comparison of exact solution with the analytical solution obtained via Bernoulli approximation method.

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