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RESEARCH ARTICLE

# Stability of delay differential equations in the sense of Ulam on unbounded intervals

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ABSTRACT

In this paper, we consider the stability problem of delay differential equations in the sense of Hyers-Ulam-Rassias. Recently this problem has been solved for bounded intervals, our result extends and improve the literature by obtaining stability in unbounded intervals. An illustrative example is also given to compare these results and visualize the improvement.



## 1. Introduction

In 1940, Ulam [1] raised the following stability problem of functional equations: Assume one has a function $f(t)$ which is very close to solve an equation. Is there an exact solution $h(t)$ which is relatively close to $f(t)$? More precisely, Ulam raised the question: Given a group $G_1$ and a metric group $(G_2, \rho)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < k\varepsilon$ for all $x \in G_1$ and some $k > 0$? If the answer is affirmative, the equation $h(xy) = h(x)h(y)$ is called *stable* in the sense of Ulam. One year later, Hyers [2] gave an answer to this problem for linear functional equations on Banach spaces: Let $G_1, G_2$ be real Banach spaces and $\varepsilon > 0$. Then, for each mapping $f : G_1 \to G_2$ satisfying $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in G_1$, there exists a unique additive mapping $g : G_1 \to G_2$ such that $\|f(x) - h(x)\| \leq \varepsilon$ holds for all $x \in G_1$. The above result of Hyers [2] was extended by Aoki [3] and Bourgin [4]. In 1978, Rassias [5] provided a remarkable generalization, which known as Hyers-Ulam-Rassias stability today, by considering the constant $\varepsilon$ as a variable in Ulam's problem (see for example [3, 6–8]). After Hyers' answer, a new concept of stability for functional equations established, called today Hyers-Ulam stability, and is one of the central topics in mathematical analysis (see for example [9–12]).

The first result on Hyers-Ulam stability of differential equations was given by Obloza [13, 14]. Thereafter, in 1998, Alsina and Ger [15] investigated the Hyers-Ulam stability for the linear differential equation $y' = y$. They proved that if a differentiable function $y : I \to R$ satisfies

$$\left|y'(t) - y(t)\right| \leq \varepsilon$$

for all $t \in I$, then there exists a differentiable function $f : I \to R$ satisfying $f'(t) = f(t)$ for any $t \in I$ such that

$$|y(t) - f(t)| \leq 3\varepsilon$$

for all $t \in I$. Here, $I$ is an open interval and $\varepsilon > 0$.

Furthermore, Miura et al. [16], Miura [17] and Takahasi et al. [18] generalized the above result of Alsina and Ger [15]. Indeed, they proved the Hyers-Ulam stability of the dynamic equation $y' = \lambda y$.

In 2004, Jung [19] obtained a similar result for the differential equation $\varphi(t)\, y = y$. More later, the result of the Hyers-Ulam stability for first-order linear differential equations has been generalized by Miura et al. [20], Takahasi et al. [21] and Jung [22]. They studied the nonhomogeneous linear differential equation of first-order

$$y' + p(t)\, y + q(t) = 0. \tag{1}$$

In 2006, Jung [22] proved the Hyers-Ulam-Rassias stability of Eq. (1). Also, Jung [23] studied the generalized Hyers-Ulam stability of the differential equation of the form

$$ty'(t) + \alpha y(t) + \beta t^r x_0 = 0.$$

In 2008, Wang et al. [24] studied the first-order nonhomogeneous linear differential equation

$$p(t)\, y' - q(t)\, y - r(t) = 0. \tag{2}$$

Using the method of the integral factor, they proved the Hyers-Ulam stability of Eq. (2) and extend the existing results. In 2008, Jung and Rassias [25] generalized the Hyers-Ulam stability of the Riccati equation of the form

$$y' + g(t)\, y + h(t)\, y^2 = k(t)$$

under the some additional conditions. In 2009 and 2010, Rus [26, 27] gave four types of Ulam stability: Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the ordinary differential equations

$$y' = f(t, y(t)) \tag{3}$$

and

$$y'(t) = p(t) + f(t, y(t)),$$

respectively. Also, in 2010, by using the fixed point method and adopting the idea used in Cădariu and Radu [9], Jung [28] proved the Hyers-Ulam sability for Eq. (3) defined on a finite and closed interval, and he also investigated the Hyers-Ulam-Rassias for Eq. (3). In 2013, Li and Wang [29] obtained Hyers-Ulm-Rassias and Ulam-Hyers stability results for the following semilinear differential equations with impulses on a compact interval:

$$y'(t) = \lambda y(t) + f(t, y(t)).$$

In 2014, Qarawani [30] established the stability of linear and nonlinear differential equations of first-order in the sense of Hyers-Ulam-Rassia. Also, he investigated stability and asymptotic stability in the sense of Hyers-Ulam-Rassias for a Bernoulli's differential equation. Same year, Alqifiary [31] gave a necessary and sufficient condition in order that the first order linear system of differential equations

$$y'(t) + A(t)\, y(t) + B(t) = 0$$

has the Hyers-Ulam-Rassias stability and find Hyers-Ulam stability constant under those conditions. In 2017, Onitsuka and Shoji [32] studied the Hyers-Ulam stability of the first-order linear differential equation

$$y' - ay = 0, \tag{4}$$

where $a$ is a nonzero reel number. They find an explicit solution $y(t)$ of Eq. (4) satisfying $|\phi(t) - y(t)| \le \varepsilon / |a|$ for all $t \in R$ under the assumption that a differential function $\phi(t)$ satisfies $|\phi'(t) - a\phi(t)| \le \varepsilon$ for all $t \in R$.

Serious studies on the stability problem of differential equations have been started since 2000s. Stability has been investigated for the different classes of differential equations with different approaches. For example, delay differential equations are a special type of ordinary differential equations. To our knowledge, in 2010, the first mathematicians who investigated the stability of delay differential equations are Jung and J.Brzdek [33]. Motivated by the above mentioned outcomes on Hyers-Ulam stability, they investigated the Hyers-Ulam stability of $y'(t) = \lambda y(t-\tau)$ for $[-\tau, \infty)$ with an initial condition, where $\lambda > 0$ and $\tau > 0$ are real constants. Thereafter, Otrocol and Ilea [34] investigated Ulam-Hyers stability and generalized Ulam-Hyers-Rassias for the following functional differential equation

$$y'(t) = f(t, y(t), y(g(t))).$$

In 2015, by using the fixed point method, Tunç and Biçer [35] proved two new results on the Hyers-Ulam-Rassias and the Hyers-Ulam stability for the first-order delay differential equation

$$y'(t) = F(t, y(t), y(t-\tau)).$$

Recently, in the last two decades, the theory time scale and related dynamic equations have been systematically studied. To our knowledge, only in 2013, András and Mészáros [36] studied the Ulam-Hyers stability of some linear and nonlinear dynamic equations and integral equations on time scales. They used both direct and operational methods. In 2013, Shen [37] established the Ulam stability of the first-order linear dynamic equation

$$y^{\Delta} = p\,(t)\, y + f\,(t)$$

and its adjoint equation

$$x^{\Delta} = -p\,(t)\, x^{\sigma} + f\,(t)$$

on a finite interval in the time scale by using the integrating factor method. Same year, by using the idea of time scale Zada et al. [38] studied a relationship between the Hyers-Ulam stability and dichotomy of the first-order linear dynamic system

$$x^{\Delta} = Gx\,(t)\,.$$

In the last decade, there has been a significant development in the theory of fractional differential equations. We refer to the papers [39–43] for qualitative study of fractional equations, including stability theory.

## 2. Preliminaries

As it is outlined in Introduction section, stability problem of differential equations in the sense of Hyers-Ulam was initiated by the papers of Obloza [13,14]. Later Alsina and Ger [15] proved that, with assuming $I$ is an open interval of reals, every differentiable mapping $y : I \to \mathbb{R}$ satisfying $|y'(x) - y(x)| \le \varepsilon$ for all $x \in I$ and for a given $\varepsilon > 0$, there exists a solution $y_0$ of the differential equation $y'(x) = y(x)$ such that $|y(x) - y_0(x)| \le 3\varepsilon$ for all $x \in I$. This result was later extended by Takahasi, Miura and Miyajima [18] to the equation $y'(x) = \lambda y(x)$ in Banach spaces, and [20, 44] to higher order linear differential equations with constant coefficients.

Recently Jung [28] proved Hyers-Ulam stability as well as Hyers-Ulam-Rassias stability of the equation

$$y' = f(t, y)$$

which extends the above mentioned results to nonlinear differential equations. Jung also shoved that some of his results are valid also on unbounded intervals. Jung's technique has been modified also for functional equations in the form

$$y'(t) = F\left(t, y(t), y(t-\tau)\right) \tag{5}$$

by Tun and Bier [35]. They obtained the following significant result for delay differential equations.

**Theorem 1.** *Let $I_0 := [t_0 - \tau, T]$ for given real numbers $t_0$, $T$ and $\tau$ with $T > t_0$. Suppose that the continous function $F : I_0 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition*

$$\begin{aligned} |F\,(t, x_1, y_1)\;\; F\,(t, x_2, y_2)| &\le L_1\,|x_1 - x_2| \\ &+ L_2\,|y_1 - y_2| \end{aligned}$$

*for all $(t, x_1, y_1), (t, x_2, y_2) \in I_0 \times \mathbb{R} \times \mathbb{R}$ and some $L_1, L_2 > 0$. Suppose also that $\Psi : [t_0 - \tau, t_0] \to \mathbb{R}$ is a continuous function. Let $\varphi : I_0 \to \mathbb{R}$ be a continuous and nondecreasing function satisfying*

$$\left| \int_{t_0}^{t} \varphi(s)\underset{\cdot}{s} \right| \le K\varphi(t) \tag{6}$$

*for all $t \in I_0$ and some $K > 0$ satisfying $0 < K\,(L_1 + L_2) < 1$. If a continuous function $y : I_0 \to \mathbb{R}$ satisfies*

$$\begin{cases} |y'(t) - F\,(t, y(t), y(t-\tau))| < \varphi(t), & t \in [t_0, T], \\ |y(t) - \Psi(t)| < \varphi(t), & t \in [t_0 - \tau, t_0], \end{cases}$$

*then there exists a unique continuous function $y_0 : I_0 \to \mathbb{R}$ satisfying Eq.*

$$\begin{cases} y_0'(t) = F\,(t, y_0(t), y_0(t-\tau))\,, & t \in [t_0, T], \\ y_0(t) = \Psi(t), & t \in [t_0 - \tau, t_0] \end{cases}$$

*and*

$$|y(t) - y_0(t)| \le \frac{K}{1 - K(L_1 + L_2)}\varphi(t)$$

*for all $t \in I_0$ and any number $L$ with $L > L_1 + L_2$.*

In this paper, we will extend and improve these result by proving the stability results for delay differential equations for unbounded intervals. To achive stability results on unbounded intervals, we will use the inspiring techniques used in the above mentioned papers [7] and [35].

## 3. Main result

Before stating our main result, let us define the Ulam-Hyers-Rassias stability precisely for the differential equation (5).

For some $\varepsilon \geq 0$, $\Psi \in C[t_0-\tau, t_0]$ and $t_0, T \in \mathbb{R}$ with $T > t_0$, assume that for any continous function $f : [t_0-\tau, T] \to \mathbb{R}$ satisfying

$$\begin{cases} |f'(t) - F(t, f(t), f(t-\tau))| < \varepsilon, & t \in [t_0, T], \\ |f(t) - \Psi(t)| < \varepsilon, & t \in [t_0-\tau, t_0]. \end{cases}$$

If there exists a continuous function $f_0 : [t_0 - \tau, T] \to \mathbb{R}$ satisfying

$$\begin{cases} f_0'(t) = F(t, f_0(t), f_0(t-\tau)), & t \in [t_0, T], \\ f_0(t) = \Psi(t), & t \in [t_0-\tau, t_0] \end{cases}$$

and

$$|f(t) - f_0(t)| < K(\varepsilon), \quad t \in [t_0-\tau, T],$$

where $K(\varepsilon)$ is an expression of $\varepsilon$ only, we say that Eq. (5) has the Hyers-Ulam stability. If the above statement is also true when we replace $\varepsilon$ and $K(\varepsilon)$ by $\varphi$ and $\Phi$, where $\varphi, \Phi \in C[t_0-\tau, T]$ are functions not depending $f$ and $f_0$ explicitly, then we say that Eq. (5) has the Hyers-Ulam-Rassias stability. These definitons may be applied to different classes of differential equations, we refer to Jung [28], Tun and Bier [35] and references cited therein for more detailed definitions of Hyers-Ulam stability and Hyers-Ulam-Rassias stability.

Our main result concerning the Ulam-Hyers-Rassias stability of delay differential equations on unbounded intervals reads as follows.

**Theorem 2.** *For a given real number $t_0$, let $I := [t_0-\tau, \infty)$. Let $K$, $L_1$ and $L_2$ be positive constants with $0 < K(L_1+L_2) < 1$. Assume that $F : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies the Lipschitz condition (6) for all $(t, x_1, y_1), (t, x_2, y_2) \in I \times \mathbb{R} \times \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies,*

$$\begin{cases} |y'(t) - F(t, y(t), y(t-\tau))| < \varphi(t), & t \in [t_0, \infty), \\ |y(t) - \Psi(t)| < \varphi(t), & t \in [t_0-\tau, t_0], \end{cases} \tag{7}$$

*where $\varphi : I \to (0, \infty)$ is a continuous function satisfying the condition (6) for all $t \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ which satisfies*

$$\begin{cases} y_0'(t) = F(t, y_0(t), y_0(t-\tau)), & t \in [t_0, \infty), \\ y_0(t) = \Psi(t), & t \in [t_0-\tau, t_0] \end{cases} \tag{8}$$

*and*

$$|y(t) - y_0(t)| \leq \frac{K}{1 - K(L_1+L_2)} \varphi(t) \tag{9}$$

*for all $t \in I$.*

***Proof.*** For any $n \in \mathbb{N}$, define the sets $I_n := [t_0, t_0+n]$. Then according to Theorem 1, for each $n$, there exists a unique continuous function $y_n : I_n \to \mathbb{R}$ such that

$$y_n(t) = y(t_0) + \int_{t_0}^{t} F(s, y_n(s), y_n(s-\tau))\, ṣ \tag{10}$$

and

$$|y(t) - y_n(t)| \leq \frac{K}{1 - K(L_1+L_2)} \varphi(t) \tag{11}$$

for all $t \in I_n$. Keep in mind that $y(t) = y_0(t) = \Psi(t)$ for $t \in [t_0-\tau, t_0]$. If $t \in I_n$, uniqueness of the functions $y_n$ implies that

$$y_n(t) = y_{n+1}(t) = y_{n+2}(t) = \cdots. \tag{12}$$

Now, for any $t \in \mathbb{R}$, define the number $n(t) \in \mathbb{N}$ as

$$n(t) := \min\{n \in \mathbb{N} \,:\, t \in I_n\}.$$

Moreover, we define the function $y_0 : \mathbb{R} \to \mathbb{R}$ with

$$y_0(t) = y_{n(t)}(t) \tag{13}$$

and we claim that $y_0$ is continuous. To prove this, for arbitrary $t_1 \in \mathbb{R}$, we choose the integer $n_1 := n(t_1)$. Then $n_1$ belongs to interior of $I_{n+1}$ and there exists an $\varepsilon > 0$ such that $y_0(t) = y_{n+1}(t)$ for all $t \in (t_1-\varepsilon, t_1+\varepsilon)$. Since $y_{n+1}$ is continuous at $t_1$, so is $y_0$. That is, $y_0$ is continuous at $t_1$ for any $t_1 \in \mathbb{R}$.

Now, for arbitrary $t \in I$, we choose the number $n(t)$. Then, we have $t \in I_{n(t)}$ and it follows from (10) and (13) that

$$\begin{aligned} y_0(t) &= y_{n(t)}(t) \\ &= y(t_0) + \int_{t_0}^{t} F\left(s, y_{n(t)}(s), y_{n(t)}(s-\tau)\right) ṣ \\ &= y(t_0) + \int_{t_0}^{t} F(s, y_0(s), y_0(s-\tau))\, ṣ. \end{aligned} \tag{14}$$

Here, the last equality is valid because $n(s) \leq n(t)$ for any $s \in I_{n(t)}$ and it follows from (12) and (13) that

$$y_{n(t)}(t)(s) = y_{n(s)}(s) = y_0(s).$$

The equality (14) implies that the function $y_0$ satisfies the equations (8).

Now we will show that the function $y_0$ satisfies the inequality (9). Since $t \in I_{n(t)}$ for all $t \in I$, from (11) and (13), we have

$$\begin{aligned} |y(t) - y_0(t)| &= \left|y(t) - y_{n(t)}(t)\right| \\ &\leq \frac{K}{1 - K(L_1 + L_2)} \varphi(t) \end{aligned}$$

for all $t \in I_n$.

Finally, we will now show that the function $y_0$ is unique. Let $u_0 : I \to \mathbb{R}$ be another continuous function satisfies (8) and (9), with $u_0$ in place of $y_0$, for all $t \in I$. For arbitrary $t \in I$, the restrictions $y_0|_{I_{n(t)}}$ $\left(= y_{n(t)}\right)$ and $u_0|_{I_{n(t)}}$ both satisfy (8) and (9) for all $t \in I_{n(t)}$. Then, it follows from the uniqueness of $y_{n(t)} = y_0|_{I_{n(t)}}$ that

$$y_0(t) = y_0|_{I_{n(t)}} = u_0|_{I_{n(t)}} = u_0(t),$$

which completes the proof. □

## 4. Example

**Example 1.** For any $\lambda_1, \lambda_2 > 0$, consider the following delay differential equation

$$y'(t) + \lambda_1 y(t) + \lambda_2 y(t - \tau) = q(t) \tag{15}$$

on the interval $I := [t_0 - \tau, \infty]$, where $t_0$ and $\tau$ are arbitrary real numbers. Since

$$F\left(t, y(t), y(t-\tau)\right) = y(t) + y(t-\tau) - q(t),$$

we have

$$\begin{aligned} |F\left(t, x_1, y_1\right) &- F\left(t, x_2, y_2\right)| \\ &= |\lambda_1 x_1 + \lambda_2 y_1 - q(t) \\ &\qquad - \lambda_1 x_2 - \lambda_2 y_2 + q(t)| \\ &= |\lambda_1 \left(x_1 - x_2\right) + \lambda_2 \left(y_1 - y_2\right)| \\ &\leq \lambda_1 |x_1 - x_2| + \lambda_2 |y_1 - y_2| \end{aligned}$$

for all $t \in I$. So all the conditions of Theorem 2 are satisfied and we obtain stability of the differential equation (15) in the sense of Hyers-Ulam.

Now, if we define the function $\varphi(t) := \mathrm{e}^{\lambda t}$ $(K > 0)$, we have

$$\begin{aligned} \left|\int_{t_0}^{t} \varphi(t) \text{ṣ}\right| &= \int_0^t \mathrm{e}^{\lambda t} \text{ṣ} \\ &= \frac{1}{\lambda}\left(\mathrm{e}^{\lambda t} - 1\right) \leq \frac{1}{\lambda} \mathrm{e}^{\lambda t} \\ &= \frac{1}{\lambda} \varphi(t) \end{aligned}$$

for all $t \in I$. Then, according to Theorem 2, the equation (1) is stable in the sense of Hyers-Ulam-Rassias.

It should be remarked that Theorem 2 guarantees the stability of (15) for any $T \leq \infty$, while the result of Tun and Bier [35] can guarantee stability in only a bounded subset of $I$. In this example, their result works only for $T < \infty$.

## References


[1] Ulam, S.M. (1960) A Collection of Mathematical Problems. Interscience, New York.

[2] Hyers, D. (1941) On the stability of the linear functional equation. Proceedings of National Academy of Sciences of the United States of America, 27(4), 222–224.

[3] Aoki, T. (1951) On the stability of the linear transformations in banach spaces. Bulletin of the American Mathematical Society, 57(1), 223–237.

[4] Bourgin, T.G. (1950) Classes of transformations and bordering transformations. Journal of the Mathematical Society of Japan, 2(1), 64–66.

[5] Rassias, T. (1978) On the stability of linear mappings in Banach spaces. Proceedings of the American Mathematical Society, 72(1), 297–300.

[6] Popa, D. (2005) Hyers-Ulam-Rassias stability of a linear recurrence. Journal of Mathematical Analysis and Applications, 309(2), 591–597.

[7] Jung, S.M. (2001) Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis. Hadronic Press, London.

[8] Rassias, T. (1953) Handbook of Functional Equations: Stability Theory. Springer, London.

[9] Cadariu, L., & Radu, V. (2008) Fixed point methods for the generalized stability of functional equations on a single variable. Fixed Point Theory and Applications, article ID749392, 15 pages.

[10] Forti, G.L. (2004) Comments on the core of the direct method for proving Hyers-Ulam stability of functional equations. Journal of Mathematical Analysis and Applications, 295(1), 127–133.

[11] Brzdek, J., Popa, D., & Xu, B. (2007) The Hyers-Ulam stability of nonlinear recurrences. Journal of Mathematical Analysis and Applications, 335(1), 443–449.

[12] Petru, T.P., Petruel, A., & Yao, Y.C. (2011) Ulam-Hyers stability for operatorial equations and inclusions via nonself operators.

Taiwanese Journal of Mathematics, 15(5), 2195–2212.

[13] Obloza, M. (1993) Hyers-Ulam stability of the linear differential equations. Rocznik Nauk-Dydakt Prace Matematik, 19(1), 259–270.

[14] Obloza, M. (1997) Connections between Hyers and Lyapunov stability of the ordinary differential equations. Rocznik Nauk-Dydakt Prace Matematik, 14(1), 141–146.

[15] Alsina, C., & Ger, R. (1998) On some inequalities and stability results related to the exponential function. Journal of Inequalities and Applications, 2(1), 373–380.

[16] Miura, T., Takahasi, S.E., & Choda, H. (2001) On the Hyers-Ulam stability of real continuous function valued differentiable map. Tokyo Journal of Mathematics, 24(2), 467–476.

[17] Miura, T. (2002) On the Hyers-Ulam stability of a differentiable map. Japanese Journal of Mathematics, 55(1), 17–24.

[18] Takahasi, S.H., Miura, T., & Miyajima, S. (2002) The Hyers-Ulam stability constants of first order linear differential operators. Bulletin of Korean Mathematical Society, 39(1), 309–315.

[19] Jung, S.M. (2004) Hyers-Ulam stability of linear differential equations of first order. Applied Mathematics Letters, 17(10), 1135–1140.

[20] Miura, T., Miyajima, S., & Takahasi, S.H. (2003) A characterization of Hyers-Ulam stability of first order linear differential operators. Journal of Mathematical Analysis and Applications, 286(1), 136–146.

[21] Takahasi, S.E., Takagi, H., Miura, T., & Miyajima, S. (2004) The Hyers-Ulam stability constants of first order linear differential operators. Journal of Mathematical Analysis and Applications, 296(2), 403–409.

[22] Jung, S.M. (2006) Hyers-Ulam stability of linear differential equations of first order. ii. Applied Mathematics Letters, 19(9), 854–858.

[23] Jung, S.M. (2005) Hyers-Ulam stability of linear differential equations of first order. iii. Applied Mathematics Letters, 18(2), 139–146.

[24] Wang, G., Zhou, M., & Sun, L. (2008) Hyers-Ulam stability of linear differential equations of first order. Applied Mathematics Letters, 21(10), 1024–1028.

[25] Jung, S.M., & Rassias, T.M. (2008) Generalized Hyers-Ulam stability of riccati differential equation. Mathematical Inequalities & Applications, 11(4), 777–782.

[26] Rus, I.A. (2009) Ulam stability of ordinary differential equations. Studia Universitatis Babe-Bolyai Mathematica, 54(4), 297–300.

[27] Rus, I.A. (2010) Ulam stabilities of ordinary differential equations in a Banach space. Carpathian Journal of Mathematics, 26(1), 103–107.

[28] Jung, S.M. (2010) A fixed point approach to the stability of differential equations $y' = f(x, y)$. Bulletin of the Malaysian Mathematical Sciences Society, 33(1), 47–56.

[29] Li, X & Wang, J. (2013) Ulam-Hyers-Rassias stability of semilinear differential equations with impulses. Electronic Journal of Differential Equations, 2013(172), 1–8.

[30] Qarawani, M.N. (2014) On Hyers-Ulam-Rassias stability for bernoulli's and first order linear and nonlinear differential equations. British Journal of Mathematics and Computer Science, 4(11), 1615–1628.

[31] Alqifiary, Q.H. (2014) Note on the stability for linear systems of differential equations. International Journal of Applied Mathematical Research, 3(1), 15–22.

[32] Onitsuka, M., & Shoji, T. (2017) Hyers-ulam stability of first-order homogeneous linear differential equations with a real-valued coefficient. Applied Mathematics Letters, 63(1), 102–108.

[33] Jung, S.M., & Brzdek, J. (2010) Hyers-Ulam stability of the delay equation $y'(t) = \lambda y(t - \tau)$. Abstract and Applied Analysis, 2010(3), 10 pages.

[34] Otrocol, D., & Ilea, V. (2013) Ulam stability for a delay differential equation. Central European Journal of Mathematics, 11(7), 1296–1303.

[35] Tun, C., & Bier, E. (2015) Hyers-Ulam-Rassias stability for a first order functional differential equation. Journal of Mathematical and Fundamental Sciences, 47(2), 143–153.

[36] András, S., & Mészáros, A.R. (2013) Ulam-Hyers stability of dynamic equations on time scales via Picard operators. Applied Mathematics and Computation, 219(4), 4853–4864.

[37] Shen, Y. (2017) The Ulam stability of first order linear dynamic equations on time scales. Results in Mathematics, Online first.

[38] Zada, A., Shah, S.O., Ismail, S., & Li, T. (2017) Hyers-Ulam stability in terms of dichotomy of first order linear dynamic system. Punjab University Journal of Mathematics, 49(3), 37-47.

[39] Trigeassou, J.C., Maamri, N., Sabatier, J., & Oustaloupa, A. (2011) A Lyapunov approach to the stability of fractional differential equations. Signal Processing, 91(3), 437–445.

[40] Li, C.P., & Zhang, F.R. (2011) A survey on the stability of fractional differential equations. The European Physical Journal Special Topics, 193(1), 27–47.

[41] Bayram, M,. Seer, A., & Adgzel, H. (2017) Oscillatory behavior of solutions of differential equations with fractional order. Applied Mathematics & Information Sciences, 11(3), 683–691.

[42] Bayram, M., Adgzel, H., & Seer, A. (2016) Oscillation criteria for nonlinear fractional differential equation with damping term. Open Physics, 14(1), 119–128.

[43] Seer, A., & Adgzel, H. (2016) Oscillation of solutions for a class of nonlinear fractional difference equations. The Journal of Nonlinear Science and Applications, 9(11), 5862–5869.

[44] Miura, T., Miyajima, S., & Takahasi, S.H. (2003) Hyers-Ulam stability of linear differential operator with constant coefficients. Mathematische Nachrichten, 258(2), 90–96.

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