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RESEARCH ARTICLE

On refinements of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators

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ABSTRACT

In this paper, we first establish weighted versions of Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators utilizing weighted function. Then we obtain some refinements of these inequalities. The results obtained in this study would provide generalization of inequalities proved in earlier works.

1. Introduction

The Hermite-Hadamard inequality, which is the first fundamental result for convex mappings with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics.

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see, e.g., [17, p.137], [2]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \quad (1)$$

Both inequalities hold in the reversed direction if f is concave.

In [6], Fejér obtained the following inequality which is the weighted generalization of Hermite-Hadamard inequality (1):

Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x) &\le \int_a^b f(x)g(x)dx \\ &\le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \end{aligned}$$

holds, where $g : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a + b)/2$.

A number of mathematicians have devoted their efforts to generalise, refine, counterpart and extend these two inequalities for different classes of functions, (see, for example, [1]- [5], [8]- [11], [13], [14], [16], [19]- [26]) and the references cited therein.

The remainder of this work is organized as follows: we first give the definitions of Riemann-Liouville fractional integrals and present some Hermite-Hadamard type inequalities for Riemann-Liouville fractional integral operators in Section 2. In the main section, we first establish a new weighted version of Hermite-Hadamard inequality for Riemann-Liouville fractional integrals. Moreover, we obtain some refinements of this result using the symmetric weighted function. We give also some special cases of these inequalities. In the last section, we give some conclusions and future directions of research.

2. Preliminaries

In the following we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

Definition 1. *Let $f \in L_1[a,b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J_{a+}^{0}f(x) = J_{b-}^{0}f(x) = f(x)$.

It is remarkable that Sarikaya et al. [20] first give the following interesting integral inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[J_{a+}^{\alpha}f(b) + J_{b-}^{\alpha}f(a)\right] \qquad (2) \\ \leq\ & \frac{f(a)+f(b)}{2} \end{aligned}$$

with $\alpha > 0$.

Hermite-Hadamard-Fejér inequality for Riemann-Liouville fractional integral operators was given by İşcan in [11], as follows:

Let $f : [a,b] \to \mathbb{R}$ be convex function with with $a < b$ and $f \in L[a,b]$. If $g : [a,b] \to \mathbb{R}$ is non-negative, integrable and symmetric with respect to $\frac{a+b}{2}$ i.e. $g(a+b-x) = g(x)$, then the following inequalities hold

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \left[J_{a+}^{\alpha}(g)(b) + J_{b-}^{\alpha}(g)(a)\right] \\ \leq\ & \left[J_{a+}^{\alpha}(fg)(b) + J_{b-}^{\alpha}(fg)(a)\right] \\ \leq\ & \frac{f(a)+f(b)}{2} \left[J_{a+}^{\alpha}(g)(b) + J_{b-}^{\alpha}(g)(a)\right]. \end{aligned}$$

For more information for fractional calculus, please refer to ([7], [12], [15], [18]).

Now we give the following lemma:

Lemma 1. *[22, 25] Let $f : [a,b] \to \mathbb{R}$ be a convex function and h be defined by*

$$h(t) = \frac{1}{2}\left[f\left(\frac{a+b}{2} - \frac{t}{2}\right) + f\left(\frac{a+b}{2} + \frac{t}{2}\right)\right].$$

Then h is convex, increasing on $[0, b-a]$ and for all $t \in [0, b-a]$,

$$f\left(\frac{a+b}{2}\right) \leq h(t) \leq \frac{f(a)+f(b)}{2}.$$

In [22], Xiang obtained following important inequalities for the Riemann-Liouville fractional integrals utilizing the Lemma 1:

Theorem 2. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WH is convex and monotonically increasing on $[0,1]$ and*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= WH(0) \leq WH(t) \leq WH(1) \qquad (3) \\ &= \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}} \left[\left(J_{a+}^{\alpha}f\right)(b) + \left(J_{b-}^{\alpha}f\right)(a)\right] \end{aligned}$$

with $\alpha > 0$ where

$$\begin{aligned} WH(t) = \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b & f\left(tx + (1-t)\frac{a+b}{2}\right) \\ & \times \left((b-x)^{\alpha-1} + (x-a)^{\alpha-1}\right) dx. \end{aligned}$$

Theorem 3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then WP is convex and monotonically increasing on $[0,1]$ and*

$$\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[\left(J_{a^{+}}^{\alpha}f\right)(b)+\left(J_{b^{-}}^{\alpha}f\right)(a)\right] \tag{4}$$
$$= WP(0)\leq WP(t)\leq WP(1)=\frac{f(a)+f(b)}{2}$$

with $\alpha>0$ *where*

$$\begin{aligned} & WP(t) \\ = & \frac{\alpha}{4(b-a)^{\alpha}}\int_{a}^{b}\left[f\left(\left(\frac{1+t}{2}\right)a+\left(\frac{1-t}{2}\right)x\right)\right. \\ \times & \left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right) \\ + & f\left(\left(\frac{1+t}{2}\right)b+\left(\frac{1-t}{2}\right)x\right) \\ \times & \left.\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)\right]dx. \end{aligned}$$

In this study, we establish some refinements of Hermite-Hadamard type inequalities utilizing fractional integrals which generalize the inequalities (2), (3) and (4).

3. Refinements of Hermite Hadamard Type Inequalities

In this section, we will present refinements of Hermite-Hadamard type inequalities via Riemann-Liouville fractional integral operators .

The following Lemma will be frequently used to prove our results.

Lemma 2. *[9] Let* $f:[a,b]\to\mathbb{R}$ *be convex function with* $a<b$ *and* $f\in L[a,b]$. *Let* $A,B,C,D\in[a,b]$ *with* $A+B=C+D$ *and* $|C-D|\leq|A-B|$. *Then,*

$$f(C)+f(D)\leq f(A)+f(B).$$

Theorem 4. *Let* $f:[a,b]\to\mathbb{R}$ *be convex function with* $a<b$ *and* $f\in L[a,b]$. *Let the weight function* $w:[a,b]\to\mathbb{R}$ *be continuous and symmetric about the point* $\left(\frac{a+b}{2},w\left(\frac{a+b}{2}\right)\right)$, *i.e.* $\frac{1}{2}[w(s)+w(a+b-s)]=w\left(\frac{a+b}{2}\right)$. *Then, we have the following inequality*

$$\begin{aligned} & f\left(w\left(\frac{a+b}{2}\right)\right) && (5) \\ \leq & \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(w(b))+J_{b^{-}}^{\alpha}f(w(a))\right] \end{aligned}$$

and if the function w *is monotonic on* $[a,b]$, *then we have*

$$\begin{aligned} & \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(w(b))+J_{b^{-}}^{\alpha}f(w(a))\right] \\ \leq & \frac{f(w(a))+f(w(b))}{2} && (6) \end{aligned}$$

with $\alpha>0$.

Proof. By the hypothesis of symmetricity of the function w, we have

$$2w\left(\frac{a+b}{2}\right)=w(s)+w(a+b-s)$$

and we also have

$$\left|w\left(\frac{a+b}{2}\right)-w\left(\frac{a+b}{2}\right)\right|\leq|w(s)-w(a+b-s)|$$

for $s\in[a,b]$. Applying Lemma 2, we obtain

$$\begin{aligned} & 2f\left(w\left(\frac{a+b}{2}\right)\right) && (7) \\ \leq & f(w(s))+f(w(a+b-s)). \end{aligned}$$

Multiplying by $\frac{(s-a)^{\alpha-1}}{\Gamma(\alpha)}$ both sides of (7) and integrating with respect to s on $[a,b]$, we deduce that

$$\begin{aligned} & \frac{2(b-a)^{\alpha}}{\Gamma(1+\alpha)}f\left(w\left(\frac{a+b}{2}\right)\right) \\ \leq & J_{a^{+}}^{\alpha}f(w(b))+J_{b^{-}}^{\alpha}f(w(a)) \end{aligned}$$

which completes the proof of the inequality (5). By the monotonicity w, we have

$$|w(s)-w(a+b-s)|\leq|w(a)-w(b)|$$

for $s\in[a,b]$ and by symmetricity of the function w, we have

$$w(s)+w(a+b-s)=w(a)+w(b)$$

for $s\in[a,b]$. Applying Lemma 2, we get

$$\begin{aligned} & f(w(s))+f(w(a+b-s)) && (8) \\ \leq & f(w(a))+f(w(b)). \end{aligned}$$

Multiplying both sides of (8) by $\frac{(s-a)^{\alpha-1}}{\Gamma(\alpha)}$ and integrating with respect to s on $[a,b]$ and dividing both sides by $\frac{2(b-a)^{\alpha}}{\Gamma(1+\alpha)}$, we obtain the desired inequality (6). □

Remark 1. *If we choose $w(t)=t$ in Theorem 4, then the inequalities (5) and (6) reduce to left and right hand sides of the inequality (2), respectively.*

Remark 2. *If we choose $\alpha=1$ in Theorem 4, then Theorem 4 reduces to Theorem 1 proved in [9].*

Theorem 5. *Let the weight function $w:[a,b]\to\mathbb{R}$ be continuous and symmetric about the point $\left(\frac{a+b}{2}, w\left(\frac{a+b}{2}\right)\right)$, i.e. $\frac{1}{2}[w(s)+w(a+b-s)] = w\left(\frac{a+b}{2}\right)$. If $f:[a,b]\to\mathbb{R}$ is a convex function on $[a,b]$, then WH_w is convex and monotonically increasing on $[0,1]$ and we have the following inequalities*

$$\begin{aligned} & f\left(w\left(\frac{a+b}{2}\right)\right) \qquad (9)\\ =\ & WH_w(0)\le WH_w(t)\le WH_w(1)\\ =\ & \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(w(b))+J_{b-}^{\alpha}f(w(a))\right] \end{aligned}$$

with $\alpha>0$ where

$$\begin{aligned} & WH_w(t)\\ =\ & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right)\\ & \times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx. \end{aligned}$$

Proof. Firstly, for $t_1,\ t_2,\ \beta\in[0,1]$, we have

$$\begin{aligned} & WH_w((1-\beta)t_1+\beta t_2)\\ =\ & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left(\left(w(x)-w\left(\frac{a+b}{2}\right)\right)\right.\\ & \left.\times[(1-\beta)t_1+\beta t_2]+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\\ =\ & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^b f\left((1-\beta)\right.\\ & \times\left[\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_1+w\left(\frac{a+b}{2}\right)\right]\\ & \left.+\beta\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_2+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx. \end{aligned}$$

Since f is convex, we have

$$\begin{aligned} & WH_w((1-\beta)t_1+\beta t_2)\\ \le\ & \frac{\alpha(1-\beta)}{2(b-a)^{\alpha}}\\ \times\ & \int_a^b f\left(\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_1+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\\ +\ & \frac{\alpha\beta}{2(b-a)^{\alpha}}\\ \times\ & \int_a^b f\left(\left(w(x)-w\left(\frac{a+b}{2}\right)\right)t_2+w\left(\frac{a+b}{2}\right)\right)\\ & \times\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]dx\\ =\ & (1-\beta)WH_w(t_1)+\beta WH_w(t_2). \end{aligned}$$

Hence, we get WH_w is convex on $[0,1]$. On the other hand, we have

$$\begin{aligned} & WH_w(t)\\ =\ & \frac{\alpha}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right)\\ & \times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx\\ & +\frac{\alpha}{2(b-a)^{\alpha}}\int_{\frac{a+b}{2}}^b f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right)\\ & \times\left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx \end{aligned}$$

$$= \frac{\alpha}{2(b-a)^{\alpha}} \times \int_a^{\frac{a+b}{2}} f\left(tw(x)+(1-t)w\left(\frac{a+b}{2}\right)\right) \times \left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx + \frac{\alpha}{2(b-a)^{\alpha}} \times \int_a^{\frac{a+b}{2}} f\left(tw(a+b-x)+(1-t)w\left(\frac{a+b}{2}\right)\right) \times \left((b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right)dx. \tag{10}$$

Let $t_1 < t_2, t_1, t_2, \in [0,1]$. By the symmetricity of the function w, we have

$$\left[t_1 w(x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right] + \left[t_1 w(a+b-x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right] = \left[t_2 w(x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right] + \left[t_2 w(a+b-x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right]$$

and

$$\left|\left[t_1 w(x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right] - \left[t_1 w(a+b-x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right]\right| = t_1\left|w(x)-w(a+b-x)\right| \leq t_2\left|w(x)-w(a+b-x)\right| = \left|\left[t_2 w(x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right] - \left[t_2 w(a+b-x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right]\right|$$

for $x \in [a,b]$. Hence, applying Lemma 2, we have

$$f\left(t_1 w(x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right) + f\left(t_1 w(a+b-x)+(1-t_1)w\left(\frac{a+b}{2}\right)\right) \leq f\left(t_2 w(x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right) + f\left(t_2 w(a+b-x)+(1-t_2)w\left(\frac{a+b}{2}\right)\right). \tag{11}$$

Multiplying both sides of (11) by

$$\frac{\alpha}{2(b-a)^{\alpha}}\left[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}\right]$$

and integrating with respect to s on $\left[a, \frac{a+b}{2}\right]$, then by considering the equality (10), we deduce that $WH_w(t_1) \leq WH_w(t_2)$. Thus, WH_w is monotonically increasing on $[0,1]$. Using the facts that

$$WH_w(0) = f\left(w\left(\frac{a+b}{2}\right)\right)$$

and

$$WH_w(1) = \frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(w(b)) + J_{b^-}^{\alpha} f(w(a))\right]$$

then we obtain the desired result. Thus, the proof is completed. □

Remark 3. *If we choose $w(t) = t$ in Theorem 5, then the inequality (9) reduces to the inequality (3).*

Remark 4. *If we choose $\alpha = 1$ in Theorem 5, then Theorem 5 reduces to Theorem 2 proved in [9].*

Theorem 6. *Let the weight function $w : [a,b] \to \mathbb{R}$ be continuous and monotonic on $[a,b]$ and let w be symmetric about the point $\left(\frac{a+b}{2}, w\left(\frac{a+b}{2}\right)\right)$, i.e. $\frac{1}{2}\left[w(s)+w(a+b-s)\right] = w\left(\frac{a+b}{2}\right)$. If $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$, then WP_w is convex and monotonically increasing on $[0,1]$ and we have the following inequalities*

$$\frac{\Gamma(1+\alpha)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(w(b)) + J_{b^-}^{\alpha} f(w(a))\right] = WP_w(0) \leq WP_w(t) \leq WP_w(1) = \frac{f(w(a))+f(w(b))}{2} \tag{12}$$

with $\alpha > 0$ where

$$\begin{aligned}&WP_w(t)\\=&\frac{\alpha}{4(b-a)^{\alpha}}\int_a^b f\left((1-t)w\left(\frac{a+x}{2}\right)+tw(a)\right)\\&\times\left(\left(\frac{2b-a-x}{2}\right)^{\alpha-1}+\left(\frac{x-a}{2}\right)^{\alpha-1}\right)dx\\&+\frac{\alpha}{4(b-a)^{\alpha}}\int_a^b f\left((1-t)w\left(\frac{x+b}{2}\right)+tw(b)\right)\\&\times\left(\left(\frac{b-x}{2}\right)^{\alpha-1}+\left(\frac{x+b-2a}{2}\right)^{\alpha-1}\right)dx.\end{aligned}$$

Proof. By the way similar to in Theorem, it can be easily proved by convexity of f that WP_w is convex on $[0,1]$. Using change of variable, we have

$$\begin{aligned}&WP_w(t)\qquad(13)\\=&\frac{\alpha}{2(b-a)^{\alpha}}\int_a^{\frac{a+b}{2}} f((1-t)w(s)+tw(a))\\&\times\left((b-s)^{\alpha-1}+(u-s)^{\alpha-1}\right)ds\\&+\frac{\alpha}{2(b-a)^{\alpha}}\\&\times\int_a^{\frac{a+b}{2}} f((1-t)w(a+b-s)+tw(b))\\&\times\left((b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right)ds.\end{aligned}$$

Let $t_1<t_2, t_1, t_2, \in[0,1]$. Since w is symmetric to $\frac{a+b}{2}$,

$$w(s)+w(a+b-s)=2w\left(\frac{a+b}{2}\right)\qquad(14)$$

and w is monotonic, we have

$$|w(s)-w(a+b-s)|\leq|w(a)-w(b)|\qquad(15)$$

for $s\in[a,b]$. By the equality (14) and the inequality (15), we have

$$\begin{aligned}&[(1-t_1)w(s)+t_1w(a)]\\&+[(1-t_1)w(a+b-s)+t_1w(b)]\\=&[(1-t_2)w(s)+t_2w(a)]\\&+[(1-t_2)w(a+b-s)+t_2w(b)]\end{aligned}$$

and

$$\begin{aligned}&|[(1-t_1)w(s)+t_1w(a)]\\&-[(1-t_1)w(a+b-s)+t_1w(b)]|\\=&|(1-t_1)[w(s)-w(a+b-s)]\\&+t_1[w(a)-w(b)]|\\\leq&(1-t_1)|w(s)-w(a+b-s)|\\&+t_1|w(a)-w(b)|\\\leq&(1-t_2)|w(s)-w(a+b-s)|\\&+t_2|w(a)-w(b)|\\=&|[(1-t_2)w(s)+t_2w(a)]\\&-[(1-t_2)w(a+b-s)+t_2w(b)]|\end{aligned}$$

for $s\in\left[a,\frac{a+b}{2}\right]$. Therefore, applying Lemma 2, we have

$$\begin{aligned}&f((1-t_1)w(s)+t_1w(a))\qquad(16)\\&+f((1-t_1)w(a+b-s)+t_1w(b))\\\leq&f((1-t_2)w(s)+t_2w(a))\\&+f((1-t_2)w(a+b-s)+t_2w(b)).\end{aligned}$$

Multiplying both sides of (16) by

$$\frac{\alpha}{2(b-a)^{\alpha}}\left[(b-s)^{\alpha-1}+(s-a)^{\alpha-1}\right]$$

and integrating with respect to s on $\left[a,\frac{a+b}{2}\right]$, then by considering the equality (13), we deduce that $WP_w(t_1)\leq WP_w(t_2)$. Hence, WP_w is monotonically increasing on $[0,1]$. This completes the proof. □

Remark 5. *If we choose $w(t)=t$ in Theorem 6, then the inequality (12) reduces to the inequality (4).*

Remark 6. *If we choose $\alpha=1$ in Theorem 6, then Theorem 6 reduces to Theorem 3 proved in [9].*

4. Conclusion

In this paper, we present some new weighted refinements of Hermite-Hadamard inequalities for Riemann-Liouville fractional integrals. For further studies we propose to consider the Hermite-Hadamard type inequalities for other fractional integral operators

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