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RESEARCH ARTICLE

Fractional Hermite-Hadamard type inequalities for functions whose derivatives are extended s-(α, m)-preinvex

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ABSTRACT

In this paper, we introduce the class of extended s-(α, m)-preinvex functions. We establish a new fractional integral identity and derive some new fractional Hermite-Hadamard type inequalities for functions whose derivatives are in this novel class of function.

1. Introduction

It is well known that convexity plays an important and central role in many areas, such as economic, finance, optimization, and game theory. Due to its diverse applications this concept has been extended and generalized in several directions.

One of the most well-known inequalities in mathematics for convex functions is the so called Hermite-Hadamard integral inequality

$$f\left(\tfrac{a+b}{2}\right) \leq \tfrac{1}{b-a}\int_a^b f(x)dx \leq \tfrac{f(a)+f(b)}{2}, \qquad (1)$$

where f is a real continuous convex function on the finite interval $[a, b]$. If the function f is concave, then (1) holds in the reverse direction (see [1]).

The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants have appeared in the literature, see [2–9] and references cited therein.

Kirmaci et al. [10] presented some results connected with inequality (1)

$$\left|\tfrac{1}{b-a}\int_a^b f(x)dx - f\left(\tfrac{a+b}{2}\right)\right| \leq \tfrac{b-a}{8}\left(\left|f'(a)\right| + \left|f'(b)\right|\right).$$

Recently, Sarikaya et al [11], gave the fractional analogue of (1)

$$\begin{aligned} f\left(\tfrac{a+b}{2}\right) &\leq \tfrac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\left(J^\alpha_{a^+}f\right)(b) + \left(J^\alpha_{b^-}f\right)(a)\right] \\ &\leq \tfrac{f(a)+f(b)}{2}. \end{aligned} \qquad (2)$$

Zhu et al [12] established the following result connected with inequality (2).

$$\begin{aligned} &\left|\tfrac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[\left(J^\alpha_{a^+}f\right)(b) + \left(J^\alpha_{b^-}f\right)(a)\right] - f\left(\tfrac{a+b}{2}\right)\right| \\ \leq\ &\tfrac{b-a}{4(1+\alpha)}\left(\left|f'(a)\right| + \left|f'(b)\right|\right)\left(\alpha + 3 - \frac{1}{2^{\alpha-1}}\right). \end{aligned}$$

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Motivated by the above results, in this paper, we introduce the class of extended s-(α, m)-preinvex functions. We establish a new fractional integral identity and derive some new fractional Hermite-Hadamard type inequalities for functions whose derivatives are in this novel class of functions.

2. Preliminaries

In this section we recall some definitions and lemmas

Definition 1. *[13] A function $f : I \to \mathbb{R}$ is said to be convex, if*

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 2. *[14] A nonnegative function f : $I \to \mathbb{R}$ is said to be P-convex, if*

$$f\left(tx + (1-t)\,y\right) \leq f\left(x\right) + f(y)$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

Definition 3. *[15] A nonnegative function f : $I \to \mathbb{R}$ is said to be Godunova-Levin function, if*

$$f\left(tx + (1-t)\,y\right) \leq \frac{f\left(x\right)}{t} + \frac{f(y)}{1-t}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 4. *[16] A nonnegative function f : $I \to \mathbb{R}$ is said to be s-Godunova-Levin function, where $s \in [0,1]$, if*

$$f\left(tx + (1-t)\,y\right) \leq \frac{f\left(x\right)}{t^s} + \frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 5. *[17] A nonnegative function f : $I \to \mathbb{R}$ is said to be α-Godunova-Levin function, where $\alpha \in (0,1]$, if*

$$f\left(tx + (1-t)\,y\right) \leq \frac{f\left(x\right)}{t^\alpha} + \frac{f(y)}{1-t^\alpha}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 6. *[18] A nonnegative function f : $I \subset [0,\infty) \to \mathbb{R}$ is said to be α-convex in the first sense for some fixed $\alpha \in (0,1]$, if*

$$f(tx + (1-t)y) \ \leq t^\alpha f(x) + (1-t^\alpha)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 7. *[19] A nonnegative function f : $I \subset [0,\infty) \to \mathbb{R}$ is said to be s-convex in the second sense for some fixed $s \in (0,1]$, if*

$$f(tx + (1-t)y) \ \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 8. *[20] A nonnegative function f : $I \subset [0,\infty) \to \mathbb{R}$ is said to be extended s-convex for some fixed $s \in [-1,1]$, if*

$$f(tx + (1-t)y) \ \leq t^s f(x) + (1-t)^s f(y)$$

holds for all $x, y \in I$ and $t \in (0,1)$.

Definition 9. *[21] A function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in (0,1]$,if*

$$f\left(tx + m\,(1-t)\,y\right) \leq tf\left(x\right) + m\,(1-t)\,f(y)$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 10. *[22] A function $f : [0,b] \to \mathbb{R}$ is said to be (α, m)-convex, where $\alpha, m \in (0,1]$,if*

$$f\left(tx + m\,(1-t)\,y\right) \leq t^\alpha f\left(x\right) + m\left(1-t^\alpha\right) f(y)$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 11. *[23] A function $f : [0,b] \to \mathbb{R}$ is said to be (s, m)-convex, where $\alpha, m \in (0,1]$,if*

$$f\left(tx + m\,(1-t)\,y\right) \leq t^s f\left(x\right) + m\,(1-t)^s\, f(y)$$

holds for all $x, y \in I$, and $t \in [0,1]$.

Definition 12. *[24] A function $f : I \to \mathbb{R}$ is said to be (α, m)-Godunova-Levin functions of first kind, where $\alpha, m \in (0,1]$, if*

$$f\left(tx + m\,(1-t)\,y\right) \leq \frac{f\left(x\right)}{t^\alpha} + m\frac{f(y)}{1-t^\alpha}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 13. *[24] A function $f : I \to \mathbb{R}$ is said to be (s, m)-Godunova-Levin functions of first kind, where $s \in [0,1]$ and $m \in (0,1]$, if*

$$f\left(tx + m\,(1-t)\,y\right) \leq \frac{f\left(x\right)}{t^s} + m\frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in I$ and all $t \in (0,1)$.

Definition 14. *[25] A nonnegative function $f : I \subset [0,\infty) \to [0,\infty)$ is said to be s-(α,m)-convex in the second sense where $\alpha, m \in [0,1]$ and $s \in (0,1]$, if the following inequality*

$$f(tx + (1-t)y) \leq (1-t^{\alpha})^s f(x) + m(t^{\alpha})^s f(\frac{y}{m})$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Definition 15. *[26] A set $K \subseteq \mathbb{R}^n$ is said an invex with respect to the bifunction $\eta : K \times K \to \mathbb{R}^n$, if for all $x, y \in K$, we have*

$$x + t\eta(y,x) \in K.$$

In what follows we assume that $K \subseteq \mathbb{R}$ be an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 16. *[26] A function $f : K \to \mathbb{R}$ is said to be preinvex with respect to η, if*

$$f(x + t\eta(y,x)) \leq (1-t) f(x) + t f(y)$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

Definition 17. *[27] A nonnegative function $f : K \to \mathbb{R}$ is said to be P-preinvex function with respect to η, if*

$$f(x + t\eta(y,x)) \leq f(x) + f(y)$$

holds for all $x, y \in K$ and all $t \in [0,1]$.

Definition 18. *[27] A nonnegative function $f : K \to \mathbb{R}$ is said to be Godunova-Levin preinvex function with respect to η, if*

$$f(x + t\eta(y,x)) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}$$

holds for all $x, y \in K$ and all $t \in (0,1)$.

Definition 19. *[28] A nonnegative function $f : K \to \mathbb{R}$ is said to be s-Godunova-Levin preinvex function with respect to η, where $s \in [0,1]$, if*

$$f(x + t\eta(y,x)) \leq \frac{f(x)}{t^s} + \frac{f(y)}{(1-t)^s}$$

holds for all $x, y \in K$ and all $t \in (0,1)$.

Definition 20. *[29] A nonnegative function $f : K \subset [0,\infty) \to \mathbb{R}$ is said to be α-preivex in the first sense with respect to η for some fixed $\alpha \in (0,1]$, if*

$$f(x + t\eta(y,x)) \leq (1-t^{\alpha}) f(x) + t^{\alpha} f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 21. *[30] A nonnegative function $f : K \subset [0,\infty) \to \mathbb{R}$ is said to be s-preinvex in the second sense with respect to η for some fixed $s \in (0,1]$, if*

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + t^s f(y)$$

holds for all $x, y \in K$ and $t \in [0,1]$.

Definition 22. *[31] A function $f : K \subset [0,b^*] \to \mathbb{R}$ is said to be m-preinvex with respect to η where $b^* > 0$ and $m \in (0,1]$, if*

$$f(x + t\eta(y,x)) \leq (1-t) f(x) + mtf(\frac{y}{m})$$

holds for all $x, y \in K$, and $t \in [0,1]$.

Definition 23. *[31] A function $f : K \to \mathbb{R}$ is said to be (α,m)-preinvex with respect to η for some fixed $\alpha \in (0,1]$, and $m \in (0,1]$, if*

$$f(x + t\eta(y,x)) \leq (1-t^{\alpha}) f(x) + mt^{\alpha} f(\frac{y}{m})$$

holds for all $x, y \in K$, and $t \in [0,1]$.

Definition 24. *[32] A function $f : K \subset [0,b^*] \to \mathbb{R}$ is said to be (s,m)-preinvex with respect to η for some fixed $\alpha \in (0,1]$where $b^* > 0$ and $m \in (0,1]$, if*

$$f(x + t\eta(y,x)) \leq (1-t)^s f(x) + mt^s f(\frac{y}{m})$$

holds for all $x, y \in K$, and $t \in [0,1]$.

Lemma 1. *[33] For $t, n \in [0,1]$, we have*

$$(1-t)^n \leq 2^{1-n} - t^n.$$

Lemma 2. *[34] For any $0 \leq a < b$ and fixed $p \geq 1$, we have*

$$(b-a)^p \leq b^p - a^p.$$

We also recall that the incomplete beta function is defined as follows:

$$B_x(\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dx$$

for $x \in [0,1]$ and $\alpha,\beta > 0$, where $B_1(\alpha,\beta) = B(\alpha,\beta)$ is the beta function.

3. Main results

In what follows we assume that $[a, a+\eta(b,a)] \subset K \subset [0, b^*]$ where $b^* > 0$ such that K is an invex set with respect to the bifunction $\eta : K \times K \to \mathbb{R}$.

Definition 25. *A nonnegative function $f : K \to [0,\infty)$ is said to be extended s-(α,m)-preinvex in the second sense where $\alpha, m \in (0,1]$ and $s \in [-1,1]$, if the following inequality*

$$f(x + t\eta(y,x)) \le (1-t^\alpha)^s f(x) + m(t^\alpha)^s f(\frac{y}{m})$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Remark 1. *Definition 25 includes all the definitions cited above, except for Definition 15.*

Lemma 3. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$, and assume that $f' \in L([a, a+\eta(b,a)])$, then the following equality holds*

$$\begin{aligned}&\frac{\Gamma(\delta+1)}{2\eta^\delta(b,a)}\Big[\left(J^\delta_{a^+}f\right)(a+\eta(b,a)) \\ &+\left(J^\delta_{(a+\eta(b,a))^-}f\right)(a)\Big] - f\left(\tfrac{2a+\eta(b,a)}{2}\right) \quad (3)\\ =& \frac{\eta(b,a)}{2}\left(\int_0^1 kf'(a+t\eta(b,a))\,dt\right.\\ &\left.-\int_0^1\left(t^\delta-(1-t)^\delta\right)f'(a+t\eta(b,a))\,dt\right),\end{aligned}$$

where

$$k = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{2}, \\ -1 & \text{if } \frac{1}{2} \le t < 1. \end{cases} \quad (4)$$

Proof. Let

$$\begin{aligned} I &= \int_0^1 kf'(a+t\eta(b,a))\,dt \\ &\quad -\int_0^1\left(t^\delta-(1-t)^\delta\right)f'(a+t\eta(b,a))\,dt \\ &= I_1 - I_2, \end{aligned} \quad (5)$$

where

$$I_1 = \int_0^1 kf'(a+t\eta(b,a))\,dt, \quad (6)$$

and

$$I_2 = \int_0^1\left(t^\delta-(1-t)^\delta\right)f'(a+t\eta(b,a))\,dt, \quad (7)$$

k is defined by (3).
Clearly,

$$\begin{aligned} I_1 &= \frac{2}{\eta(b,a)}\Big[f\left(\tfrac{2a+\eta(b,a)}{2}\right) \\ &\quad -(f(a)+f(a+\eta(b,a)))\Big]. \end{aligned} \quad (8)$$

Now, by integration by parts, I_2 gives

$$\begin{aligned} I_2 &= \tfrac{1}{\eta(b,a)}f(a+\eta(b,a)) + \tfrac{1}{\eta(b,a)}f(a) \\ &\quad -\tfrac{\delta}{\eta(b,a)}\left(\int_0^1 t^{\delta-1}f(a+t\eta(b,a))\,dt\right. \\ &\quad \left.+\int_0^1(1-t)^{\delta-1}f(a+t\eta(b,a))\,dt\right) \\ &= \tfrac{1}{\eta(b,a)}f(a+\eta(b,a)) + \tfrac{1}{\eta(b,a)}f(a) \\ &\quad -\tfrac{\alpha}{\eta^{\delta+1}(b,a)}\left(\int_a^{a+\eta(a,b)}(u-a)^{\delta-1}f(u)\,du\right. \\ &\quad \left.+\int_a^{a+\eta(a,b)}(\eta(b,a)+a-u)^{\delta-1}f(u)\,du\right) \\ &= \tfrac{1}{\eta(b,a)}f(a+\eta(b,a)) + \tfrac{1}{\eta(b,a)}f(a) \\ &\quad -\tfrac{\Gamma(\delta+1)}{\eta^{\delta+1}(b,a)}\left(\left(I^\delta_{a^+}f\right)(a+\eta(b,a))\right. \\ &\quad \left.+\left(I^\delta_{(a+\eta(b,a))^-}f\right)(a)\right). \end{aligned} \quad (9)$$

Combining (8), (9) and (5), we obtain the desired equality in (3). □

Theorem 1. *Let $f : [a, a+\eta(b,a)] \to \mathbb{R}$ be a positive differentiable mapping on $(a, a+\eta(b,a))$ with $\eta(b,a) > 0$ and $f' \in L([a, a+\eta(b,a)])$. If $|f'|$ is extended s-(α,m)-preinvex function where $\alpha, m \in (0,1]$ and $s \in (-1,1]$, then the following fractional inequality holds for $\alpha s + \delta \ne -1$*

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\Big(2^{1-s}-\tfrac{1}{\alpha s+1}+\tfrac{2^{2-s}}{\delta+1}\left(1-\left(\tfrac{1}{2}\right)^{\delta}\right)-\tfrac{1}{\alpha s+\delta+1}-B(\alpha s+1,\delta+1)\Big)|f'(a)|$$
$$+m\Big(\tfrac{1}{\alpha s+1}+2B_{\frac{1}{2}}(\alpha s+1,\delta+1)-B(\alpha s+1,\delta+1)+\tfrac{1}{\alpha s+\delta+1}\left(1-\tfrac{1}{2^{\alpha s+\delta}}\right)\Big)\left|f'\left(\tfrac{b}{m}\right)\right|,$$

where $B(.,.)$ and $B_{\frac{1}{2}}(.,.)$ are the beta and the incomplete beta functions respectively.

Proof. From Lemma 3, and properties of modulus we have

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1|f'(ta+(1-t)b)|\,dt+\int_0^{\frac{1}{2}}\left((1-t)^{\delta}-t^{\delta}\right)|f'(ta+(1-t)b)|\,dt+\int_{\frac{1}{2}}^1\left(t^{\delta}-(1-t)^{\delta}\right)|f'(ta+(1-t)b)|\,dt\right). \tag{10}$$

Since $|f'|$ is extended s-(α,m)-preinvex function, (10) gives

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\left(\int_0^1(1-t^{\alpha})^s\times|f'(a)|+m(t^{\alpha})^s\left|f'(\frac{b}{m})\right|dt\right.$$
$$+\int_0^{\frac{1}{2}}\left((1-t)^{\delta}-t^{\delta}\right)\left((1-t^{\alpha})^s|f'(a)|+mt^{\alpha s}\left|f'(\frac{b}{m})\right|\right)dt$$
$$\left.+\int_{\frac{1}{2}}^1\left(t^{\delta}-(1-t)^{\delta}\right)\left((1-t^{\alpha})^s|f'(a)|+mt^{\alpha s}\left|f'(\frac{b}{m})\right|\right)dt\right). \tag{11}$$

Now, applying Lemma 1 for (11), we get

$$\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\tfrac{2a+\eta(b,a)}{2}\right)\right|$$
$$\leq \frac{\eta(b,a)}{2}\Bigg(\Bigg(\int_0^1\left(2^{1-s}-t^{\alpha s}\right)dt+\int_0^{\frac{1}{2}}\Big(2^{1-s}\left((1-t)^{\delta}-t^{\delta}\right)\times\left(t^{\alpha s+\delta}-t^{\alpha s}(1-t)^{\delta}\right)\Big)dt$$
$$+\int_{\frac{1}{2}}^1\Big(2^{1-s}\left(t^{\delta}-(1-t)^{\delta}\right)-t^{\alpha s+\delta}-t^{\alpha s}(1-t)^{\delta}\Big)dt\Bigg)|f'(a)|$$
$$+m\left(\int_0^{\frac{1}{2}}\left(t^{\alpha s}(1-t)^{\delta}-t^{\alpha s+\delta}\right)dt+\int_{\frac{1}{2}}^1\left(t^{\alpha s+\delta}-t^{\alpha s}(1-t)^{\delta}\right)dt+\int_0^1 t^{\alpha s}dt\right)\Bigg)\left|f'\left(\tfrac{b}{m}\right)\right|$$
$$=\frac{\eta(b,a)}{2}\Big(\Big(2^{1-s}-\tfrac{1}{\alpha s+1}+\tfrac{2^{2-s}}{\delta+1}\left(1-\left(\tfrac{1}{2}\right)^{\delta}\right)-\tfrac{1}{\alpha s+\delta+1}-B(\alpha s+1,\delta+1)\Big)|f'(a)|$$
$$+m\Big(\tfrac{1}{\alpha s+1}+2B_{\frac{1}{2}}(\alpha s+1,\delta+1)-B(\alpha s+1,\delta+1)\times\tfrac{1}{\alpha s+\delta+1}\left(1-\tfrac{1}{2^{\alpha s+\delta}}\right)\Big)\left|f'\left(\tfrac{b}{m}\right)\right|\Big),$$

which is the desired result. □

Remark 2. *Theorem 1 will be reduces to Theorem 2.3 from [12], if we choose $s=\alpha=m=1$ and $\eta(b,a)=b-a$.*

Theorem 2. *Let $f:[a,a+\eta(b,a)]\to\mathbb{R}$ be a positive differentiable mapping on $(a,a+\eta(b,a))$ with $\eta(b,a)>0$ and $f'\in L([a,a+\eta(b,a)])$. If $|f'|^q$ $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, is extended s-(α,m)-preinvex function, where $\alpha,m\in(0,1]$ and $s\in[-1,1]$, and $q>1$, then the following fractional inequality holds for $s\alpha\neq-1$*

$$\begin{aligned}
&\left|\frac{\Gamma(\delta+1)}{2\eta^{\alpha}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))\right.\right.\\
&\left.\left.+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\frac{2a+\eta(b,a)}{2}\right)\right|\\
\leq\ &\frac{\eta(b,a)}{2}\left(\left((2^{1-s}-\frac{1}{s\alpha+1})\left|f'(a)\right|^q\right.\right.\\
&\left.+\frac{m}{s\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{\delta p+1}\left(1-\left(\frac{1}{2}\right)^{\delta p}\right)\right)^{\frac{1}{p}}\\
&\times\left(\left(\left(\frac{1}{2^s}-\frac{1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(a)\right|^q\right.\right.\\
&\left.+\frac{m}{(s\alpha+1)2^{s\alpha+1}}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\\
&\times\left(\left(\frac{1}{2^s}-\frac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}}\right)\left|f'(a)\right|^q\right.\\
&\left.\left.\left.+\ m\frac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}}\left|f'\left(\frac{b}{m}\right)\right|^q\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Proof. From Lemma 3, properties of modulus, Hölder inequality, and Lemma 2, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\delta+1)}{2\eta^{\delta}(b,a)}\left[\left(J^{\delta}_{a^+}f\right)(a+\eta(b,a))\right.\right.\\
&\left.\left.+\left(J^{\delta}_{(a+\eta(b,a))^-}f\right)(a)\right]-f\left(\frac{2a+\eta(b,a)}{2}\right)\right|\\
\leq\ &\frac{\eta(b,a)}{2}\left(\left(\int_0^1 dt\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\delta}-t^{\delta}\right)^p dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left((1-t)^{\delta}-t^{\delta}\right)^p dt\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)\\
\leq\ &\frac{\eta(b,a)}{2}\left(\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^{\frac{1}{2}}\left((1-t)^{\delta p}-t^{\delta p}\right) dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^1\left(t^{\delta p}-(1-t)^{\delta p}\right) dt\right)^{\frac{1}{p}}\\
&\left.\times\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)\\
=\ &\frac{\eta(b,a)}{2}\left(\left(\int_0^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{\delta p+1}\left(1-\frac{1}{2^{\delta p}}\right)\right)^{\frac{1}{p}}\\
&\times\left(\left(\int_0^{\frac{1}{2}}\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^1\left|f'(a+t\eta(b,a))\right|^q dt\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Using the fact that $|f'|^q$ is extended s-preinvex function, and Lemma 1, (3) gives

$$
\begin{aligned}
& \left| \tfrac{\Gamma(\delta+1)}{2\eta^{\alpha}(b,a)} \left[\left(J^{\delta}_{a^+} f \right) (a + \eta(b,a)) \right.\right. \\
& \left.\left. + \left(J^{\delta}_{(a+\eta(b,a))^-} f \right)(a) \right] - f\left(\tfrac{2a+\eta(b,a)}{2} \right) \right| \\
\leq \quad & \tfrac{\eta(b,a)}{2} \left(\left(\int_0^1 (2^{1-s} - t^{s\alpha}) \left| f'(a) \right|^q \right.\right. \\
& \left. + m\left(t^{s\alpha}\right) \left| f'(\frac{b}{m}) \right|^q dt \right)^{\frac{1}{q}} \\
& + \left(\tfrac{1}{\delta p+1} \left(1 - \left(\frac{1}{2} \right)^{\delta p} \right) \right)^{\frac{1}{p}} \\
& \times \left(\left(\int_0^{\frac{1}{2}} (2^{1-s} - t^{s\alpha}) \left| f'(a) \right|^q \right.\right. \\
& \left. + m\left(t^{s\alpha}\right) \left| f'(\frac{b}{m}) \right|^q dt \right)^{\frac{1}{q}} \\
& \times \left(\int_{\frac{1}{2}}^1 (2^{1-s} - t^{s\alpha}) \left| f'(a) \right|^q \right. \\
& \left.\left.\left. + m\left(t^{s\alpha}\right) \left| f'(\frac{b}{m}) \right|^q dt \right)^{\frac{1}{q}} \right) \right) \\
= \quad & \tfrac{\eta(b,a)}{2} \left(\left((2^{1-s} - \frac{1}{s\alpha+1}) \left| f'(a) \right|^q \right.\right. \\
& \left. + m \frac{1}{s\alpha+1} \left| f'(\frac{b}{m}) \right|^q \right)^{\frac{1}{q}} \\
& + \left(\tfrac{1}{\delta p+1} \left(1 - \left(\frac{1}{2} \right)^{\delta p} \right) \right)^{\frac{1}{p}} \\
& \times \left(\left(\left(\frac{1}{2^s} - \tfrac{1}{(s\alpha+1)2^{s\alpha+1}} \right) \left| f'(a) \right|^q \right.\right. \\
& \left. + m\left(\tfrac{1}{(s\alpha+1)2^{s\alpha+1}} \right) \left| f'(\frac{b}{m}) \right|^q \right)^{\frac{1}{q}} \\
& \times \left(\left(\frac{1}{2^s} - \tfrac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}} \right) \left| f'(a) \right|^q \right. \\
& \left.\left.\left. + m\left(\tfrac{2^{s\alpha+1}-1}{(s\alpha+1)2^{s\alpha+1}} \right) \left| f'(\frac{b}{m}) \right|^q \right)^{\frac{1}{q}} \right) \right)
\end{aligned}
$$

which is the desired result. □

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