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RESEARCH ARTICLE

On the numerical investigations to the Cahn-Allen equation by using finite difference method

Asıf Yokuş [a] *, Hasan Bulut [b]

[a] *Department of Actuary, Faculty of Science, Firat University, Elazig, Turkey*
[b] *Department of Mathematics, Faculty of Science, Firat University, Elazig, Turkey*
asfyokus@yahoo.com, hbulut@firat.edu.tr

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ABSTRACT

In this study, by using the finite difference method (FDM for short) and operators, the discretized Cahn-Allen equation is obtained. New initial condition for the Cahn-Allen equation is introduced, considering the analytical solution given in *Application of the modified exponential function method to the Cahn-Allen equation, AIP Conference Proceedings 1798, 020033 [1]*. It is shown that the FDM is stable for the usage of the Fourier-Von Neumann technique. Accuracy of the method is analyzed in terms of the errors in L_2 and L_∞. Furthermore, the FDM is treated in order to obtain the numerical results and to construct a table including numerical and exact solutions as well as absolute measuring error. A comparison between the numerical and the exact solutions is supported with two and three dimensional graphics via Wolfram Mathematica 11.

1. Introduction

Russel has firstly studied the solitary wave [2,4] by following the water wave travelling through a tube. Investigation of the analytical and numerical solutions as well as other studies to the various class of nonlinear partial differential equations play an important role in the field of nonlinear sciences.

Most recently, some serious methods have been developed in order to solve nonlinear differential equation. For example, (G'/G)-expansion method [5,6], the improved (G'/G)-expansion method [7-9], the modified simple equation method [10], the Sumudu transform method [11-14], the Bäcklund transform method [15], the homotopy analysis method [16,17], the exponential function method [18-20], the modified exponential function method [21], generalized Bernoulli sub-ODE method [22], improved Bernoulli sub-ODE method [24-26], weak solutions[27] and galerkin method [28].

In the current work, we consider the Cahn-Allen equation given as:

$$u_t = u_{xx} - u^3 + u. \tag{1}$$

By using first integral method, Bulut et al. [23] have obtained some soliton to Eq. (1).

The discretize equation to the Cahn-Allen equation is derived by using the finite difference method (FDM) and its operators. We observe that the numerical method is stable with the Eq. (1) is stable when the Fourier-Von Neumann technique is utilzed. Furthermore, the accuracy in terms of the errors in and is analyzed. We then utilized the FDM in approximating exact and numerical solutions to Eq. (1). We present the computed exact and numerical approximations as well as the absolute error in tables. We compare the exact and numerical approximations calculated and support the comparison with some graphics plots, which are sketched by using the Wolfram Mathematica 11.

2. Fundamental properties of methods

2.1 Analysis of FDM

Some important notations are needed in order to describe the finite forward difference method, these are:

- Δx, which is the spatial step
- Δt, which is the time step

*Corresponding author

- $x_i = a + i\Delta x,\ i = 0,1,2,\ldots,N$ points, which are the coordinates of mesh and $N = \frac{b-a}{\Delta x},\ t_j = j\Delta t,\ j = 0,1,2,\ldots,M$ and $M = \frac{T}{\Delta t}$.
- The function $u(x,t)$ is the value of the solution at $u(x_i,t_j) \cong u_{i,j}$ (grid points), where $u_{i,j}$ will is the numerical approximations of the exact value of $u(x,t)$ at the points (x_i,t_j).

The difference operators are given as follows:

$$H_t u_{i,j} = u_{i,j+1} - u_{i,j}, \tag{2}$$

$$H_{xx} u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}. \tag{3}$$

Thus, the derivatives involve in Eq. (1) can be given in finite difference operators form as

$$\left.\frac{\partial u}{\partial t}\right|_{i,j} = \frac{H_t u_{i,j}}{\Delta t} + O(\Delta t^2), \tag{4}$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{i,j} = \frac{H_{xx} u_{i,j}}{(\Delta x)^2} + O(\Delta x^2). \tag{5}$$

The difference operator form to Eq. (1) is given as

$$\frac{H_t u_{i,j}}{\Delta t} = \frac{H_{xx} u_{i,j}}{(\Delta x)^2} - \left(u_{i,j}\right)^3 + u_{i,j}. \tag{6}$$

Inserting Eq. (4) and (5) into Eq. (1), one can be written as indexed

$$\begin{aligned} u_{i+1,j} &= -u_{i-1,j} + u_{i,j}\left(2 - (\Delta x)^2 - \frac{(\Delta x)^2}{\Delta t}\right) \\ &\quad + (\Delta x)^2 (u_{i,j})^3 + \frac{(\Delta x)^2}{\Delta t} u_{i,j+1}, \end{aligned} \tag{7}$$

where the initial values $u_{i,0} = u_0(x_i)$.

2.2. Consistency analysis

In this subsection, the consistency of Eq. (1) with difference method is discussed. Firstly, the Taylor series expansions as taking the following form [11-13],

$$u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u}{\partial t} + O(\Delta t)^2, \tag{8}$$

$$u_{i-1,j} = u_{i,j} - \Delta x \frac{\partial u}{\partial x} + (\Delta x)^2 \frac{\partial^2 u}{\partial x^2} - O(\Delta x^3). \tag{9}$$

One may define the operator L as

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}.$$

The indexed form of operator L takes the following form:

$$L_{i,j} = \frac{H_t u_{i,j}}{\Delta t} - \frac{H_{xx} u_{i,j}}{(\Delta x)^2}. \tag{10}$$

Inserting the indexed form (8) and (9) into the equality (10) and making some theoretical calculations, then the approach will be the $\Delta t \to 0$ and $\Delta x \to 0$. Therefore, the equality (10) will be same as left hand side of the Eq. (1). Thus, it can be seen that the Eq. (1) is consistent with FDM.

2.3 Truncation error and stability analysis

In this subsection, the stability and error analysis for the FDM are studied. For the stability, if there is a perturbation in the initial condition and then the small change would not cause the large error in the numerical solution. Simply, stability means that the scheme does not amplify errors and the error caused by a small perturbation in the numerical solution remains bound.

Theorem 1. The truncation error of the finite different method to the Eq. (1) is $(\Delta x)^2\left[O(\Delta t)^2 + O(\Delta x)^3\right]$.

Proof. Inserting Eq. (4) and (5) into Eq. (1) gives

$$\begin{aligned} \frac{H_t u_{i,j}}{\Delta t} + O(\Delta t)^2 &= \left(\frac{H_{xx} u_{i,j}}{(\Delta x)^2} + O(\Delta x)^3\right) \\ &\quad - \left(u_{i,j}\right)^3 + u_{i,j}. \end{aligned} \tag{11}$$

Inserting the equalities (2) and (3) into the Eq. (11) and do some necessary manipulations, then we obtain the following equality

$$\begin{aligned} u_{i+1,j} &= -u_{i-1,j} + u_{i,j}\left(2 - (\Delta x)^2 - \frac{(\Delta x)^2}{\Delta t}\right) \\ &\quad + (\Delta x)^2 (u_{i,j})^3 + \frac{(\Delta x)^2}{\Delta t} u_{i,j+1} \\ &\quad + (\Delta x)^2 (O(\Delta t)^2 + O(\Delta x)^3). \end{aligned} \tag{12}$$

Utilizing Eq. (12), one may write numerical solution $\hat{U}$ as

$$\begin{aligned} \hat{U} &= -u_{i-1,j} + u_{i,j}\left(2 - (\Delta x)^2 - \frac{(\Delta x)^2}{\Delta t}\right) \\ &\quad + (\Delta x)^2 (u_{i,j})^3 + \frac{(\Delta x)^2}{\Delta t} u_{i,j+1}, \end{aligned}$$

and the truncation error E as

$$E = (\Delta x)^2\left[O(\Delta t)^2 + O(\Delta x)^3\right].$$

Moreover, if Δt and Δx are considered as small as necessary, truncation error will be obviously very small. The limit of E can be written as

$$\lim_{\substack{\Delta x\to 0\\ \Delta t\to 0}} E = 0 \cdot$$

We can see that if Δt and Δx are configured for a value close to zero $\delta > 0$, the following inequality is gotten

$$|E| < \delta ,$$

which proves the stability of the FDM.

Theorem 2. The FDM in respect to the Cahn-Allen equation is unconditionally stable.

Proof. We consider the Von Neumann's Stability of the finite difference method for the Cahn-Allen. Let

$$u_{i,j} = u(i\Delta x,\ j\Delta t) = u(p,q) = \varepsilon^q e^{I\xi p}, \xi \in [-\pi, \pi], \tag{13}$$

where $p = i\Delta x,\ q = j\Delta t$ and $I = \sqrt{-1}$. Inserting Eq. (2), (3) and (13) into the equality (6), we can obtain

$$\varepsilon \to 0,$$

According to the Von Neumann's Stability analysis [29], the FDM is stable if $|\varepsilon| \le 1$. Hence, the FDM is unconditionally stable with the Cahn-Allen equation.

2.4. L_2 and L_∞ Error Norms

To show how close the numerical approximations are close to the exact approximations the L_2 and L_∞ error norms are utilized [30].

The L_2 error norm is defined as [30].

$$L_2 = \left\| u^{exact} - u^{numeric} \right\|_2 = \sqrt{h \sum_{j=0}^{N} \left| u_j^{exact} - u_j^{numeric} \right|^2},$$

and L_∞ error norm is defined as [30]

$$L_\infty = \left\| u^{exact} - u^{numeric} \right\|_\infty = \underset{j}{Max} \left| u_j^{exact} - u_j^{numeric} \right|.$$

3. Application

In this section, we apply Finite Difference Method for Eq. (1) and consider numerical experiments. Recall the following hyperbolic function solution for Eq. (1) given in [1]:

$$u_1(x,t) = -\frac{\left(3A_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)\left(-1 + Tanh[f(x,t)]\right)}{6A_1 + 2\sqrt{9A_1^2 + 24cA_0B_1} - 6B_1\left(1 + Tanh[f(x,t)]\right)}, \tag{14}$$

where $f(x,t) = \dfrac{3c_1 - 3ct + \sqrt{2}cx}{4c}$ and

$$\mu \ne 0,$$

$$\frac{\left(3A_1 - 3B_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)^2}{4c^2B_1^2} - \frac{3\left(4cA_0B_1 + (A_1 - B_1)\left(3A_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)\right)}{2c^2B_1^2} > 0.$$

If we put

$c = 0.6,\ A_0 = -3,\ B_1 = -5,\ A_1 = -1, c_1 = 0.1$,

$0 < x < 1$ and $0 < t < 1$ for Eq. (14), the initial condition is

$$u_0(x) = u(x,0) = -\frac{\left(3A_1 + \sqrt{9A_1^2 + 24cA_0B_1}\right)\left(-1 + \text{Tanh}\left[\frac{3c_1 + \sqrt{2}cx}{4c}\right]\right)}{6A_1 + 2\sqrt{9A_1^2 + 24cA_0B_1} - 6B_1\left(1 + \text{Tanh}\left[\frac{3c_1 + \sqrt{2}cx}{4c}\right]\right)}, \tag{15}$$

and under the above assumptions the exact solution of the Eq. (1) is as following

$$u(x,t) = -\frac{12\left(-1 + \text{Tanh}[0.416667(0.3 - 1.8t + 0.848528x)]\right)}{24 + 30\left(1 + \text{Tanh}[0.416667(0.3 - 1.8t + 0.848528x)]\right)}. \tag{16}$$

Eq. (1) can be written as indexed with the help of finite difference operators

$$u_{i+1,j} = -0.0001\left[10000\, u_{i-1,j} - 19999\, u_{i,j} - u_{i,j}^3 - 100\left(-u_{i,j} + u_{i,j+1}\right)\right].$$

A comparison of the obtained exact and numerical solutions are tabulate in Table 1.

Table 1. Numerical and exact solutions of equation (1) and absolute errors when $\Delta x = 0.01$.

x_i	t_j	*Numerical solution*	*Exact Solution*	*Absolute Error*
0.00	0.01	0.184247	0.184258	1.06626×10^{-5}
0.01	0.01	0.183187	0.183197	1.06500×10^{-5}
0.02	0.01	0.182131	0.182142	1.06370×10^{-5}
0.03	0.01	0.181080	0.181091	1.06236×10^{-5}
0.04	0.01	0.180034	0.180044	1.06098×10^{-5}
0.05	0.01	0.178992	0.170900	1.05956×10^{-5}
0.06	0.01	0.177955	0.177966	1.05810×10^{-5}

Table 2. L_2 and L_∞ error norm when $0 \le h \le 1$ and $0 \le x \le 1$

$\Delta x = \Delta t$	L_2	L_∞	
			0.2
2.01978×10^{-3}		4.317×10^{-3}	0.1
6.96142×10^{-4}		1.074×10^{-3}	
0.05	2.42301×10^{-4}	2.670×10^{-4}	
0.01	1.04962×10^{-5}	1.100×10^{-5}	

Table2 shows that when Δx and Δt are small, the L_2 and L_∞ error norm are decreasing. From Table 1-2 it is easily seen that results are in good agreement with the exact solution.

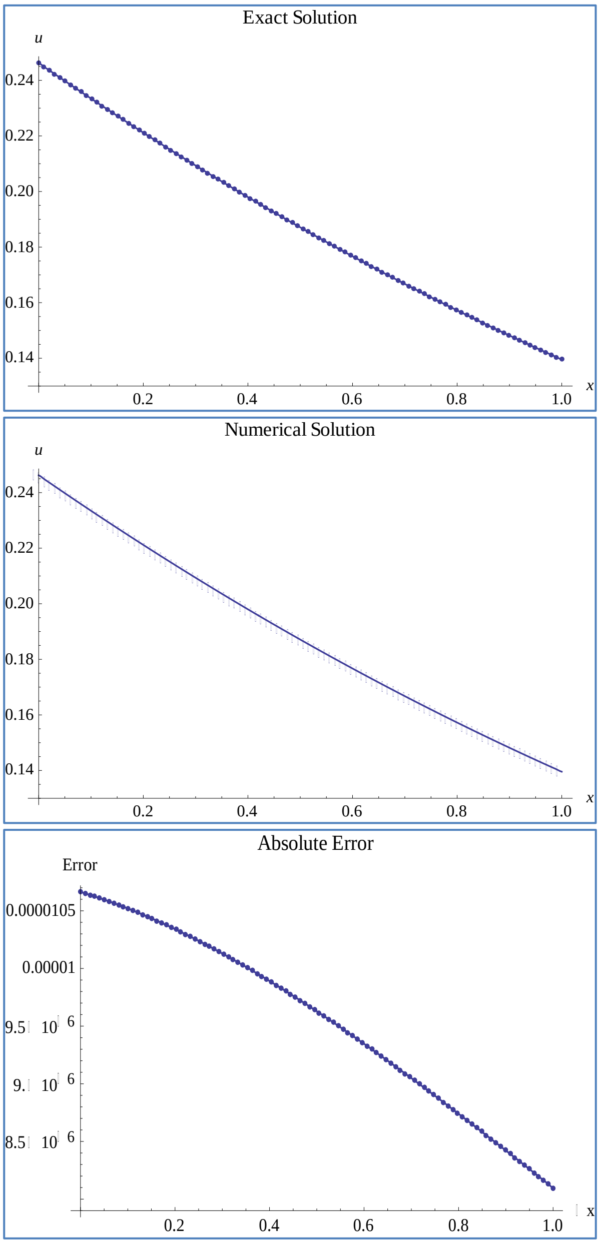

Figure 1. Numerical solution of Eq. (1) for finite difference method

Fig. 1 displays the physical behavior of the solution and shows that the exact approximations values are almost close to the numerically computed values. It is known that the truncation error depends on the choice of Δx and Δt. Choosing the values to be very small gives rise to very small truncation error. This behavior of the numerical and exact solutions can be seen in the graphs above when the values of $\Delta x = \Delta t = 0.01$.

4. Remark

The numerical results for example 1 have been obtained by using the programming language Wolfram Mathematica package. To the best of our knowledge, these numerical solutions have not been published previously, and these results are new numerical solutions for (1).

5. Conclusion

In this study, the FDM is used in approximating the numerical solutions to the Cahn-Allen equation. FDM is a useful numerical scheme for approximating the solutions of various nonlinear differential equations by defining suitable differential operators. The initial condition for the Cahn-Allen equation is obtained using the new analytical solution. The Cahn-Allen equation is written as indexed with the help of finite difference operators. Error analysis of the index equation was analyzed. Cahn-Allen equation is discussed with an example and error estimates obtained for the FDM. Furthermore, the behavior of potentials *u* and absolute error are examined graphically.

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Asıf Yokuş *is currently doctor of Actuary in Firat University. His research interests include finite different method, collocation method, analytical methods for nonlinear differential equations and numerical solutions of the partial differential equations.*

Hasan Bulut *is currently professor of Mathematics in Firat University. His research interests include stochastic differential equations, fluid and heat mechanics, finite element method, analytical methods for nonlinear differential equations and numerical solutions of the partial differential equations.*