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RESEARCH ARTICLE

Hermite-Hadamard's inequalities for conformable fractional integrals

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ABSTRACT

In this paper, we establish the Hermite-Hadamard type inequalities for conformable fractional integral and we will investigate some integral inequalities connected with the left and right-hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works and we show that some of our results are better than the other results with respect to midpoint inequalities.

1. Introduction

The convexity property of a given function plays an important role in obtaining integral inequalities. Proving inequalities for convex functions has a long and rich history in mathamatics. In [1], Beckenbach, a leading expert on the theory of convex functions, wrote that the inequality (1) was proved by Hadamard in 1893 [2]. In 1974, Mitrinovič found Hermite and Hadamard's note in Mathesis .

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function define on an interval I of real numbers, and $a, b \in I$ with $a < b$. Then, the following inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Inequality (1) is known in the literature as Hermite-Hadamard inequality for convex mappings. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

Over the last decade, classical inequalities have been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, [3–8]

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Definition 1. *The function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$, is said to be convex if the following inequality holds*

$$f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y) \quad (2)$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$.

In [7], Dragomir and Agarwal proved the following results connected with the right part of (1).

Lemma 1. *([7]) Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\\&=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.\end{aligned} \quad (3)$$

Theorem 1. *([7]) Let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)\,dx\right|\\&\leq\frac{(b-a)}{4}\left(\frac{|f'(a)|+|f'(b)|}{2}\right).\end{aligned} \quad (4)$$

In [6], Kırmacı gave the following results.

Lemma 2. *([6]) Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ (I° is the interior of I) with $a<b$. If $f'\in L[a,b]$, then the following equality holds:*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\\&=(b-a)\left[\int_0^{1/2}tf'(ta+(1-t)b)\,dt\right.\\&\left.+\int_{1/2}^1(t-1)f'(ta+(1-t)b)\,dt\right].\end{aligned} \quad (5)$$

Theorem 2. *([6]) Let $f:I^{\circ}\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I°, $a,b\in I^{\circ}$ (I° is the interior of I) with $a<b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\begin{aligned}&\left|\frac{\alpha}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{b-a}{8}\left(|f'(a)|+|f'(b)|\right).\end{aligned} \quad (6)$$

2. Definitions and Properties of Conformable Fractional Derivative and Integral

The following definitions and theorems with respect to conformable fractional derivative and integral were referred in [9–14].

Definition 2. ***(Conformable fractional derivative)*** *Given a function $f:[0,\infty)\to\mathbb{R}$. Then the "conformable fractional derivative" of f of order α is defined by*

$$D_\alpha(f)(t)=\lim_{\varepsilon\to0}\frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon} \quad (7)$$

for all $t>0$, $\alpha\in(0,1]$. If f is $\alpha-$differentiable in some $(0,a)$, $\alpha>0$, $\lim_{t\to0^+}f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0)=\lim_{t\to0^+}f^{(\alpha)}(t). \quad (8)$$

We can write $f^{(\alpha)}(t)$ for $D_\alpha(f)(t)$ to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is $\alpha-$differentiable.

Theorem 3. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then*

i. $D_\alpha(af+bg)=aD_\alpha(f)+bD_\alpha(g)$, for all $a,b\in\mathbb{R}$,

ii. $D_\alpha(\lambda)=0$, for all constant functions $f(t)=\lambda$,

iii. $D_\alpha(fg)=fD_\alpha(g)+gD_\alpha(f)$,

iv. $D_\alpha\left(\frac{f}{g}\right)=\frac{D_\alpha(f)g-D_\alpha(g)f}{g^2}$.

If f is differentiable, then

$$D_\alpha(f)(t)=t^{1-\alpha}\frac{df}{dt}(t). \quad (9)$$

Also:
1. $D_\alpha(1)=0$
2. $D_\alpha(e^{ax})=ax^{1-\alpha}e^{ax},\ a\in\mathbb{R}$
3. $D_\alpha(\sin(ax))=ax^{1-\alpha}\cos(ax),\ a\in\mathbb{R}$
4. $D_\alpha(\cos(ax))=-ax^{1-\alpha}\sin(ax),\ a\in\mathbb{R}$
5. $D_\alpha\left(\frac{1}{\alpha}t^\alpha\right)=1$
6. $D_\alpha\left(\sin(\frac{t^\alpha}{\alpha})\right)=\cos(\frac{t^\alpha}{\alpha})$

7. $D_\alpha\left(\cos(\frac{t^\alpha}{\alpha})\right) = -\sin(\frac{t^\alpha}{\alpha})$

8. $D_\alpha\left(e^{(\frac{t^\alpha}{\alpha})}\right) = e^{(\frac{t^\alpha}{\alpha})}$.

Theorem 4 (Mean value theorem for conformable fractional differentiable functions)**.** *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ and an α-fractional differentiable mapping on (a,b) with $0 \le a < b$. Then, there exists $c \in (a,b)$, such that*

$$D_\alpha\left(f\right)\left(c\right) = \frac{f(b) - f(a)}{\frac{b^\alpha}{\alpha} - \frac{a^\alpha}{\alpha}}.$$

Definition 3 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \le a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f\left(x\right) d_\alpha x := \int_a^b f\left(x\right) x^{\alpha-1} dx \tag{10}$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L^1_\alpha\left([a,b]\right)$

Remark 1.

$$I^a_\alpha\left(f\right)\left(t\right) = I^a_1\left(t^{\alpha-1}f\right) = \int_a^t \frac{f\left(x\right)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

Theorem 5. *Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \le 1$. Then, for all $t > a$ we have*

$$I^a_\alpha D^a_\alpha f\left(t\right) = f\left(t\right) - f\left(a\right). \tag{11}$$

Theorem 6. ***(Integration by parts)*** *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is differentiable. Then*

$$\begin{aligned} &\textstyle\int_a^b f\left(x\right) D^a_\alpha\left(g\right)\left(x\right) d_\alpha x \\ &= fg|_a^b - \textstyle\int_a^b g\left(x\right) D^a_\alpha\left(f\right)\left(x\right) d_\alpha x. \end{aligned} \tag{12}$$

Theorem 7. *Assume that $f : [a,\infty) \to \mathbb{R}$ such that $f^{(n)}(t)$ is continuous and $\alpha \in (n, n+1]$. Then, for all $t > a$ we have*

$$D^a_\alpha f\left(t\right) I^a_\alpha = f\left(t\right).$$

Theorem 8. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be a continuous on $[a,b]$ with $0 \le a < b$. Then,*

$$\left|I^a_\alpha\left(f\right)\left(x\right)\right| \le I^a_\alpha\left|f\right|\left(x\right).$$

For more details and properties concerning the conformable integral operators, we refer, for example, to the works [15–18].

In this paper, we establish the Hermite-Hadamard type inequalities for conformable fractional integral and we will investigate some integral inequalities connected with the left and right hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works.

3. Hermite-Hadamard's Inequalities for Conformable Fractional Integral

We will start the following important result for α-fractional differentiable mapping;

Theorem 9. *Let $\alpha \in (0,1]$ and $f : [a,b] \to \mathbb{R}$ be an α-fractional differentiable mapping on (a,b) with $0 \le a < b$. Then, the following conditions are equivalent:*

i) f is a convex functions on $[a,b]$

ii) $D_\alpha f\left(t\right)$ is an increasing function on $[a,b]$

iii) for any $x_1, x_2 \in [a,b]$

$$f(x_2) \ge f(x_1) + \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha\left(f\right)\left(x_1\right). \tag{13}$$

Proof. *i) $\to$ ii)* Let $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ and we take $h > 0$ which is small enough such that $x_1 - h, x_2 + h \in [a,b]$. Since $x_1 - h < x_1 < x_2 < x_2 + h$, then we know that

$$\begin{aligned} &\frac{f(x_1) - f(x_1 - h)}{h} \\ &\le \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &\le \frac{f(x_2 + h) - f(x_2)}{h}. \end{aligned} \tag{14}$$

Multipling the inequality (14) with $x_1^{1-\alpha} \le x_2^{1-\alpha}$, for $x_1 < x_2$, $\alpha \in (0,1]$, we get

$$\begin{aligned} &x_1^{1-\alpha}\frac{f(x_1) - f(x_1 - h)}{h} \\ &\le x_2^{1-\alpha}\frac{f(x_2 + h) - f(x_2)}{h}. \end{aligned} \tag{15}$$

Let us put $h = \varepsilon x_1^{\alpha-1}$ (and $h = \varepsilon x_2^{\alpha-1}$) such that $h \to 0$, $\varepsilon \to 0$, then the inequality (14) can be converted to

$$\frac{f(x_1) - f(x_1 - \varepsilon x_1^{\alpha-1})}{\varepsilon} \le \frac{f(x_2 + \varepsilon x_2^{\alpha-1}) - f(x_2)}{\varepsilon}.$$

Since f is α-fractional differentiable mapping on (a,b), then let $\varepsilon \to 0^+$, we obtain

$$D_\alpha f(x_1) \leq D_\alpha f(x_2) \tag{16}$$

this show that $D_\alpha f$ is increasing in $[a,b]$.

$ii) \rightarrow iii)$ Take $x_1, x_2 \in [a,b]$ with $x_1 < x_2$. Since $D_\alpha f$ is increasing in $[a,b]$, then by mean value theorem for conformable fractional differentiable we get

$$\begin{aligned} f(x_2) - f(x_1) &= \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(c) \\ &\geq \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(x_1) \end{aligned} \tag{17}$$

where $c \in (x_1, x_2)$. It is follow that

$$f(x_2) \geq f(x_1) + \frac{(x_2^\alpha - x_1^\alpha)}{\alpha} D_\alpha(f)(x_1).$$

$iii) \rightarrow i)$ For any $x_1, x_2 \in [a,b]$, we take $x_3 = \lambda x_1 + (1-\lambda)x_2$ and $x_3^\alpha = \lambda x_1^\alpha + (1-\lambda)x_2^\alpha$ for $\lambda \in (0,1)$. It is easy to show that $x_1^\alpha - x_3^\alpha = (1-\lambda)(x_1^\alpha - x_2^\alpha)$ and $x_2^\alpha - x_3^\alpha = -\lambda(x_1^\alpha - x_2^\alpha)$. Thus, by using (13), we obtain that

$$\begin{aligned} f(x_1) &\geq f(x_3) + \frac{(x_1^\alpha - x_3^\alpha)}{\alpha} D_\alpha(f)(x_3) \\ &= f(x_3) + (1-\lambda)\frac{(x_1^\alpha - x_2^\alpha)}{\alpha} D_\alpha(f)(x_3) \end{aligned}$$

and

$$\begin{aligned} f(x_2) &\geq f(x_3) + \frac{(x_2^\alpha - x_3^\alpha)}{\alpha} D_\alpha(f)(x_3) \\ &= f(x_3) - \lambda\frac{(x_1^\alpha - x_2^\alpha)}{\alpha} D_\alpha(f)(x_3). \end{aligned}$$

Both sides of the above two expressions, multiply by λ and $(1-\lambda)$, repectively, and add side to side, then we have

$$\begin{aligned} &\lambda f(x_1) + (1-\lambda) f(x_2) \\ &\geq f(x_3) \\ &= f(\lambda x_1 + (1-\lambda)x_2) \end{aligned}$$

which is show that f is a convex function. The proof is completed. □

Theorem 10. *Let $\alpha \in (0,1]$, $a \geq 0$, and $f : [a,b] \rightarrow \mathbb{R}$ is a continuous function and $\varphi : [0,\infty) \rightarrow \mathbb{R}$ be continuous and convex function. Then,*

$$\begin{aligned} &\varphi\left(\frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x)\, d_\alpha x\right) \\ &\leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \varphi(f(x))\, d_\alpha x. \end{aligned} \tag{18}$$

Proof. Let $\varphi : [0,\infty) \rightarrow \mathbb{R}$ be a convex function and $x_0 \in [0,\infty)$. From the definition of convexity, there exists $m \in \mathbb{R}$ such that,

$$\varphi(y) - \varphi(x_0) \geq m(y - x_0). \tag{19}$$

Since f is a continuous function

$$x_0 = \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x)\, d_\alpha x \tag{20}$$

is well defined. The function $\varphi \circ f$ is also continuous , thus we may apply (19) with $y = f(t)$ and (20) to obtain

$$\varphi(f(t)) - \varphi(x_0) \geq m(f(t) - x_0).$$

Integrating above inequality from a to b, we get

$$\begin{aligned} &\int_a^b \varphi(f(t)) d_\alpha t - \varphi(x_0)\int_a^b d_\alpha t \\ \geq\ & m\left(\int_a^b f(t) d_\alpha t - x_0\int_a^b d_\alpha t\right) \\ =\ & m\left(\int_a^b f(t) d_\alpha t - x_0^\alpha\int_a^b d_\alpha t\right) = 0. \end{aligned}$$

It is obvious that the inequality (18) holds. □

Hermite-Hadamard's inequalities can be represented in conformable fractional integral forms as follows:

Theorem 11. *Let $\alpha \in (0,1]$ and $f : I \subset \mathbb{R}^+ \rightarrow \mathbb{R}$ be a convex function and $f \in L_\alpha^1([a^\alpha, b^\alpha])$ with $0 \leq a < b$. Then, the following inequality for conformable fractional integral holds:*

$$\begin{aligned} &f\left(\frac{a^\alpha + b^\alpha}{2}\right) \\ &\leq \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x \\ &\leq \frac{f(a^\alpha) + f(b^\alpha)}{2}. \end{aligned} \tag{21}$$

Proof. Since f is a convex function on $I \subset \mathbb{R}^+$, for $x^\alpha, y^\alpha \in [a^\alpha, b^\alpha]$ with $\lambda = \frac{1}{2}$, we have

$$f\left(\frac{x^\alpha+y^\alpha}{2}\right)\leq\frac{f(x^\alpha)+f(y^\alpha)}{2} \tag{22}$$

i.e, with $x^\alpha = t^\alpha a^\alpha + (1-t^\alpha)b^\alpha$, $y^\alpha = (1-t^\alpha)a^\alpha + t^\alpha b^\alpha$, for $t\in[0,1]$, $\alpha\in(0,1]$

$$\begin{aligned}&2f\left(\frac{a^\alpha+b^\alpha}{2}\right)\\&\leq f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\\&+f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right).\end{aligned} \tag{23}$$

By integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&2\int_0^1 f\left(\tfrac{a^\alpha+b^\alpha}{2}\right)d_\alpha t\\&\leq\int_0^1 f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)d_\alpha t\\&+\int_0^1 f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)d_\alpha t\\&=\tfrac{2\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x,\end{aligned} \tag{24}$$

and the first inequality is proved. For the proof of the second inequality in (22) we first note that if f is a convex function, then, for $\lambda\in[0,1]$, it yields

$$f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\leq t^\alpha f(a^\alpha)+(1-t^\alpha)f(b^\alpha)$$

and

$$f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)\leq(1-t^\alpha)f(a^\alpha)+t^\alpha f(b^\alpha).$$

By adding these inequalities we have

$$\begin{aligned}&f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)+f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)\\&\leq f(a^\alpha)+f(b^\alpha).\end{aligned} \tag{25}$$

Integrating inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}&\int_0^1 f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)d_\alpha t\\&+\int_0^1 f\left((1-t^\alpha)a^\alpha+t^\alpha b^\alpha\right)d_\alpha t\\&\leq\left[f(a^\alpha)+f(b^\alpha)\right]\int_0^1 d_\alpha t\end{aligned}$$

i.e.

$$\frac{1}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x\leq\frac{f(a)+f(b)}{2\alpha}.$$

The proof is completed. □

Remark 2. *If we choose $\alpha=1$ in (21), then inequality (21) become inequality (1).*

Theorem 12. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be a convex function and $f\in L^1_\alpha([a^\alpha,b^\alpha])$ with $0\leq a<b$. Then, for $t\in[0,1]$, the following inequality for conformable fractional integral holds:*

$$\begin{aligned}&f\left(\frac{a^\alpha+b^\alpha}{2}\right)\leq h(t^\alpha)\\&\leq\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x\\&\leq H(t^\alpha)\leq\frac{f(a^\alpha)+f(b^\alpha)}{2}\end{aligned} \tag{26}$$

where

$$\begin{aligned}&h(t^\alpha)=(1-t^\alpha)f\left(\frac{(1+t^\alpha)a^\alpha+(1-t^\alpha)b^\alpha}{2}\right)\\&+t^\alpha f\left(\frac{a^\alpha t^\alpha+(2-t^\alpha)b^\alpha}{2}\right)\end{aligned}$$

and

$$\begin{aligned}&H(t^\alpha)=\tfrac{1}{2}\left[(1-t^\alpha)f(a^\alpha)\right.\\&\left.+f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)+t^\alpha f(b^\alpha)\right].\end{aligned}$$

Proof. Since f is a convex function on I, by applying (21) on the subinterval $[a^\alpha, t^\alpha a^\alpha+(1-t^\alpha)b^\alpha]$, with $t\neq1$, we have

$$\begin{aligned}&f\left(\frac{(1+t^\alpha)a^\alpha+(1-t^\alpha)b^\alpha}{2}\right)\\&\leq\frac{\alpha}{(1-t^\alpha)(b^\alpha-a^\alpha)}\\&\times\int_a^{(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)^{\frac{1}{\alpha}}}f(x^\alpha)\,d_\alpha x\\&\leq\frac{f(a^\alpha)+f\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)}{2}.\end{aligned} \tag{27}$$

Now, by applying (21) on the subinterval $[t^\alpha a^\alpha+(1-t^\alpha)b^\alpha, b^\alpha]$, with $t\neq0$, we have

$$
\begin{aligned}
& f\left(\frac{a^{\alpha}t^{\alpha}+(2-t^{\alpha})b^{\alpha}}{2}\right) \\
\leq\ & \frac{\alpha}{t^{\alpha}(b^{\alpha}-a^{\alpha})}\int_{(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})^{\frac{1}{\alpha}}}^{b} f(x^{\alpha})\,d_{\alpha}x \qquad (28)\\
\leq\ & \frac{f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})+f(b^{\alpha})}{2}.
\end{aligned}
$$

Multiplying (27) by $(1-t^{\alpha})$, and (27) by t^{α}, and adding the resulting inequalities, we obtain the following inequalities

$$
h(t^{\alpha})\leq\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b} f(x^{\alpha})\,d_{\alpha}x\leq H(t^{\alpha}) \qquad (29)
$$

where $h(t^{\alpha})$ and $H(t^{\alpha})$ are defined as in Thereom 12. Using the fact that f is a convex function, we get

$$
\begin{aligned}
& f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right) \\
=\ & f\left((1-t^{\alpha})\frac{(1+t^{\alpha})a^{\alpha}+(1-t^{\alpha})b^{\alpha}}{2}\right. \\
& \left.+t^{\alpha}\frac{a^{\alpha}t^{\alpha}+(2-t^{\alpha})b^{\alpha}}{2}\right) \qquad (30)\\
\leq\ & (1-t^{\alpha})f\left(\frac{a^{\alpha}+[t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha}]}{2}\right) \\
& +t^{\alpha}f\left(\frac{[a^{\alpha}t^{\alpha}+(1-t^{\alpha})b^{\alpha}]+b^{\alpha}}{2}\right) \\
\leq\ & \frac{1}{2}\left[(1-t^{\alpha})f(a^{\alpha})\right. \\
& \left.+f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})+t^{\alpha}f(b^{\alpha})\right] \\
\leq\ & \frac{f(a^{\alpha})+f(b^{\alpha})}{2}.
\end{aligned}
$$

Therefore, by (29) and (30) we have (26). □

4. Trapezoid Type Inequalities for Conformable Fractional Integral

We need the following lemma. With the help of this, we give some integral inequalities connected with the right-side of Hermite–Hadamard-type inequalities for conformable fractional integral.

Lemma 3. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^{+}\to\mathbb{R}$ be an α-fractional differentiable function on (a,b) with $0\leq a<b$. If $D_{\alpha}(f)$ be an α-fractional integrable function on $[a^{\alpha},b^{\alpha}]$, then the following identity for conformable fractional integral holds:*

$$
\begin{aligned}
& \frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b} f(x^{\alpha})\,d_{\alpha}x-\frac{f(a^{\alpha})+f(b^{\alpha})}{2} \\
& =\frac{1}{2}\int_{0}^{1}(1-2t^{\alpha}) \qquad (31)\\
& \times D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\,d_{\alpha}t.
\end{aligned}
$$

Proof. Integrating by parts

$$
\begin{aligned}
& \int_{0}^{1}(1-2t^{\alpha})D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\,d_{\alpha}t \\
& =(1-2t^{\alpha})f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\big|_{0}^{1} \\
& +2\alpha\int_{0}^{1} f(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})\,d_{\alpha}t \\
& =-[f(a^{\alpha})+f(b^{\alpha})]+\frac{2\alpha}{(b^{\alpha}-a^{\alpha})}\int_{a}^{b} f(x^{\alpha})\,d_{\alpha}x.
\end{aligned}
$$

Thus, by multiplying both sides by $\frac{1}{2}$, we have conclusion (31). □

Remark 3. *If we choose $\alpha=1$ in (31), then equality (31) become equality (3).*

Theorem 13. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^{+}\to\mathbb{R}$ be an α-fractional differentiable function on I° and $D_{\alpha}(f)$ be an α-fractional integrable function on I with $0\leq a<b$. If $|f'|$ be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$
\begin{aligned}
& \left|\frac{f(a^{\alpha})+f(b^{\alpha})}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b} f(x^{\alpha})\,d_{\alpha}x\right| \\
\leq\ & \frac{\alpha(b^{\alpha}-a^{\alpha})}{2}\left(\frac{2^{3\alpha^{2}}+\left(6\times 2^{\alpha^{2}}\right)-8}{3\alpha\times 2^{3\alpha^{2}}}\right) \qquad (32)\\
& \left[\frac{a^{\alpha(\alpha-1)}|D_{\alpha}(f)(a^{\alpha})|+b^{\alpha(\alpha-1)}|D_{\alpha}(f)(b^{\alpha})|}{2}\right].
\end{aligned}
$$

Proof. Using Lemma 3, it follows that

$$
\begin{aligned}
& \left|\frac{f(a^{\alpha})+f(b^{\alpha})}{2}-\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b} f(x^{\alpha})\,d_{\alpha}x\right| \\
& \leq\frac{1}{2}\int_{0}^{1}|1-2t^{\alpha}|\,|D_{\alpha}(f)(t^{\alpha}a^{\alpha}+(1-t^{\alpha})b^{\alpha})|\,d_{\alpha}t.
\end{aligned}
$$

Since $|f'|$ is a convex function, by using the properties $D_{\alpha}(f\circ g)(t)=f'(g(t))D_{\alpha}g(t)$ and $D_{\alpha}(f)(t)=t^{1-\alpha}f'(t)$, it follows that

$$
\begin{aligned}
& |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)| \\
\leq\ & \alpha(b^\alpha - a^\alpha)\left[t^\alpha a^{\alpha(\alpha-1)} |D_\alpha(f)(a^\alpha)| \right. \\
& \left. + (1-t^\alpha) b^{\alpha(\alpha-1)} |D_\alpha(f)(b^\alpha)|\right]
\end{aligned} \tag{33}
$$

Using (33), we have

$$
\begin{aligned}
& \left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right| \\
& \leq \frac{\alpha(b^\alpha - a^\alpha)}{2}\int_0^1 |1-2t^\alpha| \\
& \times \left[t^\alpha a^{\alpha(\alpha-1)} |D_\alpha(f)(a^\alpha)| \right. \\
& \left. + (1-t^\alpha) b^{\alpha(\alpha-1)} |D_\alpha(f)(b^\alpha)|\right] d_\alpha t \\
& = \frac{\alpha(b^\alpha - a^\alpha)}{2} \\
& \times \left\{a^{\alpha(\alpha-1)} |D_\alpha(f)(a^\alpha)| \int_0^1 |1-2t^\alpha| t^\alpha d_\alpha t \right. \\
& \left. + b^{\alpha(\alpha-1)} |D_\alpha(f)(b^\alpha)| \int_0^1 |1-2t^\alpha| (1-t^\alpha)\, d_\alpha t\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
& \int_0^1 |1-2t^\alpha| (1-t^\alpha)\, d_\alpha t \\
= & \int_0^1 |1-2t^\alpha| t^\alpha d_\alpha t = \frac{2^{3\alpha^2} + \left(6\times 2^{\alpha^2}\right) - 8}{3\alpha \times 2^{3\alpha^2}}
\end{aligned}
$$

Thus, the proof is completed. □

Remark 4. *If we choose $\alpha = 1$ in (32), then inequality (32) become inequality (4).*

Theorem 14. *Let $\alpha \in (0,1]$ and $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ be an α-fractional differentiable function on I° and $D_\alpha(f)$ be an α-fractional integrable function on I with $0 \leq a < b$. If $|f'|^q$, $q > 1$, be a convex function on I, then the following inequality for conformable fractional integral holds:*

$$
\begin{aligned}
& \left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right| \\
& \leq \frac{\alpha(b^\alpha - a^\alpha)}{2} (A(\alpha))^{\frac{1}{p}} \\
& \left(\frac{a^{q\alpha(\alpha-1)} |D_\alpha(f)(a)|^q + b^{q\alpha(\alpha-1)} |D_\alpha(f)(b)|^q}{2\alpha}\right)^{\frac{1}{q}}
\end{aligned} \tag{34}
$$

where $\frac{1}{p}+\frac{1}{q} = 1$, $A(\alpha)$ is given by

$$
\begin{aligned}
A(\alpha) = & \frac{1}{2\alpha(p+1)}\left\{2 - \left(1 - \frac{1}{2^{\alpha^2-1}}\right)^{p+1} \right. \\
& \left. - \left(\frac{1}{2^{\alpha^2-1}} - 1\right)^{p+1}\right\}.
\end{aligned}
$$

Proof. Using Lemma 3 and Hölder's integral inequality, we find

$$
\begin{aligned}
& \left|\frac{f(a^\alpha)+f(b^\alpha)}{2} - \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b f(x^\alpha)\, d_\alpha x\right| \\
& \leq \frac{1}{2}\int_0^1 |1-2t^\alpha| \, |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)|\, d_\alpha t \\
& \leq \frac{1}{2}\left(\int_0^1 |1-2t^\alpha|^p d_\alpha t\right)^{\frac{1}{p}} \\
& \left(\int_0^1 |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)|^q d_\alpha t\right)^{\frac{1}{q}}.
\end{aligned}
$$

Since $|f'|^q$ is a convex function, by using the properties $D_\alpha(f \circ g)(t) = f'(g(t))\, D_\alpha g(t)$ and $D_\alpha(f)(t) = t^{1-\alpha} f'(t)$, it follows that

$$
\begin{aligned}
& |D_\alpha(f)(t^\alpha a^\alpha + (1-t^\alpha) b^\alpha)|^q \\
\leq\ & \alpha^q (b^\alpha - a^\alpha)^q \\
& \left[t^\alpha a^{q\alpha(\alpha-1)} |D_\alpha(f)(a^\alpha)|^q \right. \\
& \left. + (1-t^\alpha) b^{q\alpha(\alpha-1)} |D_\alpha(f)(b^\alpha)|^q\right].
\end{aligned} \tag{35}
$$

By using (35), we have

$$\left|\frac{f(a^\alpha)+f(b^\alpha)}{2}-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x\right|$$
$$\leq \frac{\alpha(b^\alpha-a^\alpha)}{2}\left(\int_0^1|1-2t^\alpha|^p\,d_\alpha t\right)^{\frac{1}{p}}$$
$$\left[\int_0^1\left(t^\alpha a^{q\alpha(\alpha-1)}|D_\alpha(f)(a^\alpha)|^q\right.\right.$$
$$\left.\left.+(1-t^\alpha)\,b^{q\alpha(\alpha-1)}|D_\alpha(f)(b^\alpha)|^q\right)d_\alpha t\right]^{\frac{1}{q}}$$
$$\leq \frac{\alpha(b^\alpha-a^\alpha)}{2}\left(\int_0^1|1-2t^\alpha|^p\,d_\alpha t\right)^{\frac{1}{p}}$$
$$\left(\frac{a^{q\alpha(\alpha-1)}|D_\alpha(f)(a)|^q+b^{q\alpha(\alpha-1)}|D_\alpha(f)(b)|^q}{2\alpha}\right)^{\frac{1}{q}}.$$

It follows that

$$\begin{aligned}&\int_0^1|1-2t^\alpha|^p\,d_\alpha t\\ =\;&\int_0^{\frac{1}{2^\alpha}}(1-2t^\alpha)^p\,d_\alpha t+\int_{\frac{1}{2^\alpha}}^1(2t^\alpha-1)^p\,d_\alpha t\\ =\;&\frac{1}{2\alpha(p+1)}\left\{2-\left(1-\frac{1}{2^{\alpha^2-1}}\right)^{p+1}\right.\\ &\left.-\left(\frac{1}{2^{\alpha^2-1}}-1\right)^{p+1}\right\}\end{aligned}$$

which is completed the proof. □

Remark 5. *If we choose $\alpha=1$ in (34), then inequality (34) become Theorem 2.3. in [7].*

5. Midpoint Type Inequalities for Conformable Fractional Integral

We need the following lemma. With the help of this, we give some integral inequalities connected with the left-side of Hermite–Hadamard-type inequalities for conformable fractional integral.

Lemma 4. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be an α-fractional differentiable function on I° with $0\leq a<b$. If $D_\alpha(f)$ be an α-fractional integrable function on I, then the following identity for conformable fractional integral holds:*

$$f\left(\frac{a^\alpha+b^\alpha}{2}\right)-\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x \tag{36}$$
$$=\int_0^1 P(t)D_\alpha(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\,d_\alpha t$$

where

$$P(t)=\begin{cases}t^\alpha, & 0\leq t<\frac{1}{2^{1/\alpha}}\\ t^\alpha-1, & \frac{1}{2^{1/\alpha}}\leq t\leq 1.\end{cases}$$

Proof. Integrating by parts

$$\begin{aligned}&\int_0^1 P(t)D_\alpha(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\,d_\alpha t\\ =\;&\int_0^{\frac{1}{2^{1/\alpha}}}t^\alpha D_\alpha(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\,d_\alpha t\\ &+\int_{\frac{1}{2^{1/\alpha}}}^1(t^\alpha-1)D_\alpha(f)(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\,d_\alpha t\\ =\;&t^\alpha f(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\Big|_0^{\frac{1}{2^{1/\alpha}}}\\ &-\alpha\int_0^{\frac{1}{2^{1/\alpha}}}f(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\,d_\alpha t\\ &+(t^\alpha-1)f(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\Big|_{\frac{1}{2^{1/\alpha}}}^1\\ &-\alpha\int_{\frac{1}{2^{1/\alpha}}}^1 f(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha)\,d_\alpha t\\ =\;&f\left(\frac{a^\alpha+b^\alpha}{2}\right)-\frac{\alpha}{(b^\alpha-a^\alpha)}\int_a^b f(x^\alpha)\,d_\alpha x.\end{aligned}$$

Thus, we have conclusion (36). □

Remark 6. *If we choose $\alpha=1$ in (36), then equality (36) become equality (5).*

Theorem 15. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be an α-fractional differentiable function on I° and $D_\alpha(f)$ be an α-fractional integrable function on I. If $|f'|$ be a convex function on I, then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f(x^\alpha)\,d_\alpha x-f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$
$$\leq \frac{\alpha(b^\alpha-a^\alpha)}{8} \tag{37}$$
$$\left(\frac{a^{\alpha(\alpha-1)}|D_\alpha(f)(a^\alpha)|+b^{\alpha(\alpha-1)}|D_\alpha(f)(b^\alpha)|}{\alpha}\right).$$

Proof. Using Lemma 3, it follows that

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f\left(x^\alpha\right)d_\alpha x-f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$

$$\leq\left\{\int_0^{\frac{1}{2^{1/\alpha}}} t^\alpha\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|d_\alpha t\right.$$

$$\left.+\int_{\frac{1}{2^{1/\alpha}}}^1(1-t^\alpha)\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|d_\alpha t\right\}.$$

By using (33), we have

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f\left(x^\alpha\right)d_\alpha x-f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$

$$\leq\alpha\left(b^\alpha-a^\alpha\right)\left\{\int_0^{\frac{1}{2^{1/\alpha}}} t^\alpha\left[t^\alpha a^{\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|\right.\right.$$

$$\left.+(1-t^\alpha)b^{\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|\right]d_\alpha t$$

$$+\int_{\frac{1}{2^{1/\alpha}}}^1(1-t^\alpha)\left[t^\alpha a^{\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|\right.$$

$$\left.\left.+(1-t^\alpha)b^{\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|\right]d_\alpha t\right\}$$

$$=\frac{\alpha\left(b^\alpha-a^\alpha\right)}{8}\times\left(\frac{a^{\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|+b^{\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|}{\alpha}\right).$$

Thus, the proof is completed. □

Remark 7. *If we choose $\alpha=1$ in (37), then inequality (37) become the inequality (6).*

Theorem 16. *Let $\alpha\in(0,1]$ and $f:I\subset\mathbb{R}^+\to\mathbb{R}$ be an α-fractional differentiable function on I° and $D_\alpha(f)$ be an α-fractional integrable function on I. If $|f'|^q$, $q>1$,be a convex function on I, then the following inequality for conformable fractional integrals holds:*

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f\left(x^\alpha\right)d_\alpha x-f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right| \tag{38}$$

$$\leq\alpha\left(b^\alpha-a^\alpha\right)\left(\frac{1}{\alpha(p+1)2^{p+1}}\right)^{1/p}B(\alpha)$$

where $\frac{1}{p}+\frac{1}{q}=1$, $B(\alpha)$ is defined by

$$B(\alpha)=\left(\frac{a^{q\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|^q}{8\alpha}+\frac{3b^{q\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|}{8\alpha}\right)^{1/q}+\left(\frac{3a^{q\alpha(\alpha-1)}\left|D_\alpha(f)(a^\alpha)\right|^q}{8\alpha}+\frac{b^{q\alpha(\alpha-1)}\left|D_\alpha(f)(b^\alpha)\right|}{8\alpha}\right)^{1/q}.$$

Proof. Using Lemma 3 and from Hölder's inequality, it follows that

$$\left|\frac{\alpha}{b^\alpha-a^\alpha}\int_a^b f\left(x^\alpha\right)d_\alpha x-f\left(\frac{a^\alpha+b^\alpha}{2}\right)\right|$$

$$\leq\left\{\int_0^{\frac{1}{2^{1/\alpha}}} t^\alpha\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|d_\alpha t\right.$$

$$\left.+\int_{\frac{1}{2^{1/\alpha}}}^1(1-t^\alpha)\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|d_\alpha t\right\}$$

$$\leq\left\{\left(\int_0^{\frac{1}{2^{1/\alpha}}} t^{p\alpha}d_\alpha t\right)^{1/p}\right.$$

$$\times\left(\int_0^{\frac{1}{2^{1/\alpha}}}\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|^q d_\alpha t\right)^{1/q}$$

$$+\left(\int_{\frac{1}{2^{1/\alpha}}}^1(1-t^\alpha)^p d_\alpha t\right)^{1/p}$$

$$\left.\left(\int_{\frac{1}{2^{1/\alpha}}}^1\left|D_\alpha(f)\left(t^\alpha a^\alpha+(1-t^\alpha)b^\alpha\right)\right|^q d_\alpha t\right)^{1/q}\right\}.$$

By using (35), it follows that

$$\left|\frac{\alpha}{b^{\alpha}-a^{\alpha}}\int_{a}^{b}f\left(x^{\alpha}\right)d_{\alpha}x-f\left(\frac{a^{\alpha}+b^{\alpha}}{2}\right)\right|$$

$$\leq\alpha\left(b^{\alpha}-a^{\alpha}\right)\left(\frac{1}{\alpha\left(p+1\right)2^{p+1}}\right)^{1/p}$$

$$\times\left\{\left(\int\limits_{0}^{\frac{1}{2^{1/\alpha}}}\left[t^{\alpha}a^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(a^{\alpha}\right)\right|^{q}\right.\right.\right.$$

$$\left.\left.+\left(1-t^{\alpha}\right)b^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(b^{\alpha}\right)\right|^{q}\right]d_{\alpha}t\right)^{1/q}$$

$$+\left(\int\limits_{\frac{1}{2^{1/\alpha}}}^{1}\left[t^{\alpha}a^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(a^{\alpha}\right)\right|^{q}\right.\right.$$

$$\left.\left.\left.+\left(1-t^{\alpha}\right)b^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(b^{\alpha}\right)\right|^{q}\right]d_{\alpha}t\right)^{1/q}\right\}$$

$$=\alpha\left(b^{\alpha}-a^{\alpha}\right)\left(\frac{1}{\alpha\left(p+1\right)2^{p+1}}\right)^{1/p}$$

$$\times\left\{\left(\frac{a^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(a^{\alpha}\right)\right|^{q}}{8\alpha}\right.\right.$$

$$\left.+\frac{3b^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(b^{\alpha}\right)\right|}{8\alpha}\right)^{1/q}$$

$$+\left(\frac{3a^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(a^{\alpha}\right)\right|^{q}}{8\alpha}\right.$$

$$\left.\left.+\frac{b^{q\alpha(\alpha-1)}\left|D_{\alpha}\left(f\right)\left(b^{\alpha}\right)\right|}{8\alpha}\right)^{1/q}\right\}.$$

Thus, the proof of completed. □

Remark 8. *If we choose $\alpha=1$ in (38), then inequality (38) become the inequality (2.1) in Theorem 2.3. in [6].*

6. Conclusion

In this work, we have obtained some new Hermite-Hadamard type integral inequalities for conformable integrals and we will investigate some integral inequalities connected with the left and right hand side of the Hermite-Hadamard type inequalities for conformable fractional integral. The results presented here would provide generalizations of those given in earlier works and we show that some our results are better than the other results with respect to midpoint inequalities.

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