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RESEARCH ARTICLE

On Hermite-Hadamard type inequalities for $S_\varphi-$preinvex functions by using Riemann-Liouville fractional integrals

Seda Kılınç *, Abdullah Akkurt and Hüseyin Yıldırım

Kahramanmaraş Sütçü İmam University, Department of Mathematics, Kahramanmaraş, Turkey
sedaaa_kilinc@hotmail.com, abdullahmat@gmail.com, hyildir@ksu.edu.tr

This paper is dedicated to the memory of our colleague, Dr. Hatice Yaldız, who recently passed away.

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ABSTRACT

In this study, we have obtained some Hermite-Hadamard type integral inequalities for $s_\varphi-$preinvex functions. These inequalities are a generalization of some of the results in the literature.

1. Introduction

Fractional calculus (see [1–3]) arise in the mathematical modeling of various problems in sciences and engineering such as mathematics, physics, chemistry and biology.

Many authors have been working to fractional integral operators (see [4–7]) due to many applications in differents areas of Mathematics, Engineering and Physics, etc (see [8, 9]). Also, these operators have allow to extended results about integral inequalities of many types (see [4, 10, 11]), for instance, Hermite-Hadamard integral inequalities (see [12–14]), Ostrowski type inequalities (see [7]).

In particular, in recent years, several extensions and generalizations have been considered for classical convexity (see [13, 15, 16]). A significant generalizations of convex functions is that of invex functions introduced by Hanson (see [17]).

In this work we derive several new inequalities of Hermite-Hadamard type for s_φ -preinvex function of first and second sense by using fractional integrals.

In this article, we define and recall some basic concepts and results. Let $\mathbb{R}^n$ be the finite dimensional Euclidian space, also $0 \leq \varphi \leq \frac{\pi}{2}$ be a continuous function.

Definition 1. ([7, 8]) *. Let $f \in L_1[a, b]$. Then Riemann-Liouville fractional integrals $J^\alpha_{a+}f$ and $J^\alpha_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad (1)$$

and

$$J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau)\, d\tau, \quad (2)$$

where Γ is the classical Gamma function.

Definition 2. *If $K_{\varphi\eta}$ in $\mathbb{R}^n$ set, is said to be $\varphi-$invex at u according to φ, if there exists a bifunction $\eta(.,.) : K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}^n$, so that,*

$$u + \tau e^{i\varphi}\eta(u, v) \in K_{\varphi\eta},\ \forall u, v \in K_{\varphi\eta}, \tau \in [0, 1].$$

*Corresponding Author

The $\varphi-$invex set $K_{\varphi\eta}$ is also called $\varphi\eta-$connected set. Note that the convex set with $\varphi = 0$ and $\eta(u,v) = v-u$ is a $\varphi-$invex set, but the converse is not true.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function defined on the interval $I = [a,b]$ of real numbers where $a < b$. Then, the following double inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2},$$

the above double inequality is known as Hermite-Hadamard type of inequality in the literature.

Let $\mathbb{R}$ be the set of real numbers. During the article $I = [a,b] \subset \mathbb{R}$ be the interval unless otherwise specified, also let $0 \leq \varphi \leq \frac{\pi}{2}$ be a continuous function.

Lemma 1. *Suppose a function $K_{\varphi\eta} \subseteq \mathbb{R}^n$ and $\varphi : K_{\varphi\eta} \to \mathbb{R}$, the $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$, $\eta(.,.) : K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}^n$, The $\varphi-$invex set $K_{\varphi\eta}$ and $0 \leq \varphi \leq \pi/2$ be a continuous function. Let $f'' \in L[a,b]$, afterward, we get the following equality for fractional integrals:*

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)\right.\\&\left.+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&=\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left(f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right.\\&\left.\times+f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right)d\tau.\end{aligned}$$

Proof. Let,

$$\begin{aligned}&\int_0^1(1-\tau)^{\alpha+1}\left[f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right.\\&\left.+f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\\&=\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\\&+\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\\&=I_1+I_2.\end{aligned}\tag{3}$$

Integration by part respectively:

$$\begin{aligned}I_1&=\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\\&=-\left.\frac{2(1-\tau)^{\alpha+1}f'\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)}\right|_0^1-\frac{2(\alpha+1)}{e^{i\varphi}\eta(b,a)}\\&\times\int_0^1(1-\tau)^{\alpha}f'\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\\&=\frac{2}{e^{i\varphi}\eta(b,a)}f'\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&-\frac{2(\alpha+1)}{e^{i\varphi}\eta(b,a)}\left[\frac{2}{e^{i\varphi}\eta(b,a)}f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right.\\&\left.-\frac{2^{\alpha+1}\Gamma(\alpha+1)}{e^{i\varphi}\eta(b,a)}\times J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)\right]\\&=\frac{2}{e^{i\varphi}\eta(b,a)}f'\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&-\frac{4(\alpha+1)}{\left|e^{i\varphi}\eta(b,a)\right|^2}f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&+\frac{2^{\alpha+2}\Gamma(\alpha+2)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+2}}J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a),\end{aligned}$$

and,

$$\begin{aligned}I_2&=\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\\&=\left.\frac{2(1-\tau)^{\alpha+1}f'\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)}{e^{i\varphi}\eta(b,a)}\right|_0^1\\&+\frac{2(\alpha+1)}{e^{i\varphi}\eta(b,a)}\int_0^1(1-\tau)^{\alpha}f'\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\\&=-\frac{2}{e^{i\varphi}\eta(b,a)}f'\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&+\frac{2(\alpha+1)}{e^{i\varphi}\eta(b,a)}\left[-\frac{2}{e^{i\varphi}\eta(b,a)}f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right.\\&\left.+\frac{2^{\alpha+1}\Gamma(\alpha+1)}{e^{i\varphi}\eta(b,a)}J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]\\&=-\frac{2}{e^{i\varphi}\eta(b,a)}f'\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&-\frac{4(\alpha+1)}{\left|e^{i\varphi}\eta(b,a)\right|^2}f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&+\frac{2^{\alpha+2}\Gamma(\alpha+2)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha+2}}J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b).\end{aligned}$$

Using I_1 and I_2 in (3), and afterwards multiplying both sides by $\dfrac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}$ the proof is done. □

If we take $\alpha = 1$ in Lemma 1, we obtain to following result.

Lemma 2. *Let $K_{\varphi\eta} \subseteq \mathbb{R}^n$ and $\varphi : K_{\varphi\eta} \to \mathbb{R}$, the $f : [a,b] \to \mathbb{R}$ be twice differentiable function on (a,b) with $a < b$. Let $f'' \in L[a,b]$, afterward, $\eta(.,.) : K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}^n$, the $\varphi-$invex set $K_{\varphi\eta}$ and $0 \leq \varphi \leq \frac{\pi}{2}$ be a continuous function. We get the following equality for fractional integrals:*

$$\begin{aligned}&\frac{1}{e^{i\varphi}\eta(b,a)}\int_a^{a+\frac{e^{i\varphi}\eta(b,a)}{2}}f(x)dx-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\\&=\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{16}\int_0^1(1-\tau)^2\\&\left[f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)+f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau.\end{aligned}$$

If we take $\eta(b,a) = b-a$, $\varphi = 0$, then we have,

$$\left[a, a+e^{i\varphi}\eta(b,a)\right] = \left[a, a+\eta(b,a)\right] = [a,b].$$

2. Inequalities for S_φ-preinvex of second sense

In order to obtain main results introduced by [18] the $s_\varphi-$preinvex function of second sense.

Definition 3. *[18] A function f on the set $K_{\varphi\eta}$ is said to be $s_\varphi-$preinvex function of second sense according to φ and η, we get*

$$f\left(u+\tau e^{i\varphi}\eta\left(v,u\right)\right)\leq\left(1-\tau\right)^{s}f(u)+\tau^{s}f(v),\quad (4)$$

where $\forall u,v\in K_{\varphi\eta},\tau\in[0,1]$.

Theorem 2. *Suppose a function $K_{\varphi\eta}\subseteq\mathbb{R}^n$ be a open invex set according to bifunction $\eta(.,.):K_{\varphi\eta}\times K_{\varphi\eta}\to\mathbb{R}^n$, The $\varphi-$invex set $K_{\varphi\eta}$ where $\eta(b,a)>0$. Also $\varphi:K_{\varphi\eta}\to\mathbb{R}$. Let $f''\in L_1\left[a,a+e^{i\varphi}\eta(b,a)\right]$ and $|f''|$ is s_φ-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^{-}}f(a)\right.\right.\\&\left.\left.+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^{+}}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\\&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{s+3}(\alpha+1)}\left\{\left(\frac{1}{s+1}\left[2^{s+1}-1\right]\right)+\frac{1}{s+\alpha+2}\right\}\\&\times\left[\left|f''(a)\right|+\left|f''(b)\right|\right].\end{aligned}\quad (5)$$

Proof. Via Lemma 1 and the fact that $|f''|$ is s_φ-preinvex function of second sense, we get

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^{-}}f(a)\right.\right.\\&\left.\left.+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^{+}}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\\&=\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\right.\\&\times\left[f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right.\\&\left.\left.+f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\right|\\&\leq\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\right.\\&\left.\times f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|\\&+\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^{2}}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\right.\\&\left.\times f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|\\&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\\&\times\left|f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|d\tau\\&+\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\\&\times\left|f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|d\tau\end{aligned}$$

where

$$\int_0^1(1-\tau)^{\alpha+1}(1+\tau)^{s}d\tau\leq\int_0^1(1+\tau)^{s}d\tau$$

the above selection will be accepted, namely,

$$\begin{aligned}&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{8(\alpha+1)}\left[\int_0^1(1-\tau)^{\alpha+1}\right.\\&\left.\times\left(\left(\tfrac{1+\tau}{2}\right)^{s}\left|f''(a)\right|+\left(\tfrac{1-\tau}{2}\right)^{s}\left|f''(b)\right|\right)d\tau\right]\\&+\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{8(\alpha+1)}\left[\int_0^1(1-\tau)^{\alpha+1}\right.\\&\left.\times\left(\left(\tfrac{1-\tau}{2}\right)^{s}\left|f''(a)\right|+\left(\tfrac{1+\tau}{2}\right)^{s}\left|f''(b)\right|\right)d\tau\right]\\&=\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{s+3}(\alpha+1)}\left[\left|f''(a)\right|\int_0^1(1-\tau)^{\alpha+1}(1+\tau)^{s}d\tau\right.\\&\left.+\left|f''(b)\right|\int_0^1(1-\tau)^{\alpha+1}(1-\tau)^{s}d\tau\right]\\&+\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{s+3}(\alpha+1)}\left[\left|f''(a)\right|\int_0^1(1-\tau)^{\alpha+1}(1-\tau)^{s}d\tau\right.\\&\left.+\left|f''(b)\right|\int_0^1(1-\tau)^{\alpha+1}(1+\tau)^{s}d\tau\right]\\&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{s+3}(\alpha+1)}\left[\left|f''(a)\right|\left(\tfrac{1}{s+1}\left(2^{s+1}-1\right)\right)\right.\\&+\left|f''(b)\right|\tfrac{1}{s+\alpha+2}+\left|f''(a)\right|\tfrac{1}{s+\alpha+2}\\&\left.+\left|f''(b)\right|\left(\tfrac{1}{s+1}\left(2^{s+1}-1\right)\right)\right]\\&=\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{s+3}(\alpha+1)}\left\{\left(\tfrac{1}{s+1}\left[2^{s+1}-1\right]\right)+\tfrac{1}{s+\alpha+2}\right\}\\&\times\left[\left|f''(a)\right|+\left|f''(b)\right|\right].\end{aligned}$$

which completes the proof of Theorem. □

Theorem 3. *Suppose a function $K_{\varphi\eta}\subseteq\mathbb{R}^n$ and $\varphi:K_{\varphi\eta}\to\mathbb{R}$, $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$, $\eta(.,.):K_{\varphi\eta}\times K_{\varphi\eta}\to\mathbb{R}^n$.Also $\mathbb{R}^n$ be the finite dimensional Euclidian space. The $\varphi-$invex set $K_{\varphi\eta}$. Let $f''\in L[a,b]$ and $|f''|^q$ is s_φ-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^{-}}f(a)\right.\right.\\&\left.\left.+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^{+}}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\\&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\&\times\left[\left\{\left|f''(a)\right|^{q}\left(2^{s+1}-1\right)+\left|f''(b)\right|^{q}\right\}^{\frac{1}{q}}\right]\\&+\frac{\left|e^{i\varphi}\eta(b,a)\right|^{2}}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\\&\times\left[\left\{\left|f''(a)\right|^{q}+\left|f''(b)\right|^{q}\left(2^{s+1}-1\right)\right\}^{\frac{1}{q}}\right].\end{aligned}\quad (6)$$

Proof. From Lemma 1, Holder's inequality and the fact that $|f''|^q$ is s_φ-preinvex function of second sense, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(e^{i\varphi}\eta(b,a))^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a) + J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right] - f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|$$
$$=\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1} \times\left[f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right) + +f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\right|$$
$$\leq\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1} \times f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right| + \left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1} \times f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-\tau)^{(\alpha+1)p}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left|f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}\right\}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-\tau)^{(\alpha+1)p}d\tau\right)^{\frac{1}{p}} \times\left(\int_0^1\left|f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \times\left\{\left(|f''(a)|^q\int_0^1\left(\frac{1+\tau}{2}\right)^s d\tau^{\frac{1}{q}} + |f''(b)|^q\int_0^1\left(\frac{1-\tau}{2}\right)^s d\tau\right)^{\frac{1}{q}}\right\}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \times\left\{\left(|f''(a)|^q\int_0^1\left(\frac{1-\tau}{2}\right)^s d\tau^{\frac{1}{q}} + |f''(b)|^q\int_0^1\left(\frac{1+\tau}{2}\right)^s d\tau\right)^{\frac{1}{q}}\right\}$$
$$=\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times\left[\left\{|f''(a)|^q\left(2^{s+1}-1\right)+|f''(b)|^q\right\}^{\frac{1}{q}}\right]$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}} \times\left[\left\{|f''(a)|^q+|f''(b)|^q\left(2^{s+1}-1\right)\right\}^{\frac{1}{q}}\right],$$

which completes the proof of Theorem. □

Theorem 4. *Suppose a function$K_{\varphi\eta}\subseteq\mathbb{R}^n$ and $\varphi : K_{\varphi\eta}\to\mathbb{R}$, $f : [a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$, $\eta(.,.) : K_{\varphi\eta}\times K_{\varphi\eta}\to\mathbb{R}^n$. Also $\mathbb{R}^n$ be the finite dimensional Euclidian space .The $\varphi-$invex set $K_{\varphi\eta}$. Let $f''\in L[a,b]$ and $|f''|^q$ is $s_\varphi-$preinvex function of second sense, afterward, we get the following inequality for fractional integrals:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(e^{i\varphi}\eta(b,a))^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a) + J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right] - f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times\left[\left\{|f''(a)|^q\left(\frac{1}{s+1}\left[2^{s+1}-1\right]\right)^{\frac{1}{q}} + |f''(b)|^q\frac{1}{\alpha+s+2}\right\}^{\frac{1}{q}}\right] \tag{7}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times\left[|f''(a)|^q\frac{1}{\alpha+s+2}\right\} + |f''(b)|^q\left(\frac{1}{s+1}\left[2^{s+1}-1\right]\right)\Big\}^{\frac{1}{q}}\Big].$$

Proof. From Lemma 1, power-mean inequality and the fact that $|f''|^q$ is $s_\varphi-$preinvex function of second sense, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(e^{i\varphi}\eta(b,a))^{\alpha}}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a) + J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right] - f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|$$
$$=\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left[f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right) + f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\right|$$
$$\leq\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right| + \left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-\tau)^{\alpha+1}d\tau\right)^{1-\frac{1}{q}} + \left(\int_0^1(1-\tau)^{\alpha+1}\left|f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}\right\}$$
$$\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left\{\left(\int_0^1(1-\tau)^{\alpha+1}d\tau\right)^{1-\frac{1}{q}} + \left(\int_0^1(1-\tau)^{\alpha+1}\left|f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}\right\}$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times\left[\left\{|f''(a)|^q\left(\frac{1}{s+1}\left[2^{s+1}-1\right]\right)+|f''(b)|^q\frac{1}{\alpha+s+2}\right\}^{\frac{1}{q}}\right]$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}} \times\left[\left\{|f''(a)|^q\frac{1}{\alpha+s+2}+|f''(b)|^q\left(\frac{1}{s+1}\left[2^{s+1}-1\right]\right)\right\}^{\frac{1}{q}}\right]$$

which completes the proof of Theorem. □

Remark 1. *If we take $\varphi=0$ in Theorem 4, we obtain the results in [7].*

3. Inequalities for S_φ-convex funstions of first sense

In order to obtain main results introduced by [18] the $s_\varphi-$preinvex function of first sense.

Definition 4. *[18] Suppose a function f on the set $K_{\varphi\eta}$ is said to be $s_\varphi-$preinvex function of first sense according to φ and η, let*

$$f\left(u+\tau e^{i\varphi}\eta\left(v,u\right)\right)\leq\left(1-\tau^s\right)f(u)+\tau^s f(v), \tag{8}$$

$\forall u,v\in K_{\varphi\eta},\tau\in[0,1]$.

Theorem 5. *Suppose a function $K_{\varphi\eta}\subseteq\mathbb{R}^n$ and $\varphi:K_{\varphi\eta}\to\mathbb{R}$, the $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$, $\eta(.,.):K_{\varphi\eta}\times K_{\varphi\eta}\to\mathbb{R}^n$.Also $\mathbb{R}^n$ be the finite dimensional Euclidian space. The $\varphi-$invex set $K_{\varphi\eta}$. Let $f''\in L[a,b]$ and $|f''|$ is s_φ-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)(\alpha+2)}\left[\left|f''(a)\right|+\left|f''(b)\right|\right].$$

Proof. From Lemma 1 and the fact that $|f''|$ is s_φ-preinvex function of first sense, we get

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\\
&=\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left[f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)+f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\right|\\
&\leq\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|+\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|\\
&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left|f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|d\tau+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left|f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|d\tau\\
&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left\{1-\left(\tfrac{1-\tau}{2}\right)^s\right\}\left|f''(a)\right|\Big]d\tau+\left(\tfrac{1-\tau}{2}\right)^s\left|f''(b)\right|\Big]d\tau\\
&+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left[\int_0^1(1-\tau)^{\alpha+1}\left(\tfrac{1-\tau}{2}\right)^s\left|f''(a)\right|+\left(1-\left(\tfrac{1-\tau}{2}\right)^s\right)\left|f''(b)\right|\right]d\tau\\
&=\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\frac{1}{\alpha+2}\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\end{aligned}$$

which completes the proof of Theorem.. □

Theorem 6. *Suppose a function $K_{\varphi\eta}\subseteq\mathbb{R}^n$ and $\varphi:K_{\varphi\eta}\to\mathbb{R}$, the $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$, $\eta(.,.):K_{\varphi\eta}\times K_{\varphi\eta}\to\mathbb{R}^n$, Also $\mathbb{R}^n$ be the finite dimensional Euclidian space. The $\varphi-$invex set $K_{\varphi\eta}$. If $f''\in L[a,b]$ and $|f''|^q$ is s_φ-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\\
&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac1p}\left(\frac{1}{s+1}\right)^{\frac1q}\times\left\{\left(\left|f''(a)\right|^q\left(2^s(s+1)-1\right)+\left|f''(b)\right|^q\right)^{\frac1q}\right\}\\
&+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac1p}\left(\frac{1}{s+1}\right)^{\frac1q}\times\left\{\left(\left|f''(a)\right|^q+\left|f''(b)\right|^q\left(2^s(s+1)-1\right)\right)^{\frac1q}\right\}.\end{aligned}$$

Proof. From Lemma 1, Hölder inequality and the fact that $|f''|^q$ is s_φ-preinvex function of second sense, we get

$$\begin{aligned}&\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|\\
&=\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left[f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)+f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\right|\\
&\leq\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|+\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\tfrac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|\\
&\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\int_0^1(1-\tau)^{(\alpha+1)p}d\tau\right)^{\frac1p}\times\left(\int_0^1\left|f''\left(a+\tfrac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac1q}\end{aligned}$$

$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\int_0^1(1-\tau)^{(\alpha+1)p}d\tau\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\left|f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q\right.$$
$$\left.\times\int_0^1\left(1-\left(\frac{1-\tau}{2}\right)^s\right)d\tau+|f''(b)|^q\int_0^1\left(\frac{1-\tau}{2}\right)^s d\tau\right)^{\frac{1}{q}}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(|f''(a)|^q\int_0^1\left(\frac{1-\tau}{2}\right)^s d\tau\right.$$
$$\left.+|f''(b)|^q\int_0^1\left\{1-\left(\frac{1-\tau}{2}\right)^s\right\}d\tau\right)^{\frac{1}{q}}$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\left\{\left(|f''(a)|^q\left(2^s(s+1)-1\right)+|f''(b)|^q\right)^{\frac{1}{q}}\right\}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{2^{\frac{3q+s}{q}}(\alpha+1)}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}$$
$$\times\left\{\left(|f''(a)|^q+|f''(b)|^q\left(2^s(s+1)-1\right)\right)^{\frac{1}{q}}\right\},$$

which completes the proof of Theorem. □

Theorem 7. *Suppose a function $K_{\varphi\eta}\subseteq\mathbb{R}^n$ and $\varphi:K_{\varphi\eta}\to\mathbb{R}$, the $f:[a,b]\to\mathbb{R}$ be twice differentiable function on (a,b) with $a<b$, $\eta(.,.):K_{\varphi\eta}\times K_{\varphi\eta}\to\mathbb{R}^n$. Also $\mathbb{R}^n$ be the finite dimensional Euclidian space. The $\varphi-$invex set $K_{\varphi\eta}$. Let $f''\in L[a,b]$ and $|f''|^q$ is s_φ-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)\right.\right.$$
$$\left.+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]$$
$$\left.-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(|f''(a)|^q\left(\frac{1}{\alpha+2}-\frac{1}{2^s}\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)\right.$$
$$\left.+|f''(b)|^q\frac{1}{2^s}\frac{\Gamma(s+\alpha+2)}{\Gamma(s+\alpha+3)}\right)^{\frac{1}{q}}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left[|f''(a)|^q\frac{1}{2^s}\frac{1}{\alpha+s+2}\right.$$
$$\left.+|f''(b)|^q\left(\frac{1}{\alpha+2}-\frac{1}{2^s}\frac{1}{\alpha+s+2}\right)\right]^{\frac{1}{q}}.$$

Proof. From Lemma 1, power-mean inequality and the fact that $|f''|^q$ is s_φ-preinvex function of second sense, we get

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(e^{i\varphi}\eta(b,a)\right)^\alpha}\left[J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^-}f(a)\right.\right.$$
$$\left.+J^{\alpha,\varphi}_{\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)^+}f(b)\right]$$
$$\left.-f\left(a+\frac{e^{i\varphi}\eta(b,a)}{2}\right)\right|$$
$$=\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}\left[f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right.\right.$$
$$\left.\left.+f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right]d\tau\right|$$
$$\leq\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|$$
$$+\left|\frac{\left(e^{i\varphi}\eta(b,a)\right)^2}{8(\alpha+1)}\int_0^1(1-\tau)^{\alpha+1}f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)d\tau\right|$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\int_0^1(1-\tau)^{\alpha+1}d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1(1-\tau)^{\alpha+1}\left|f''\left(a+\frac{1-\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\int_0^1(1-\tau)^{\alpha+1}d\tau\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1(1-\tau)^{\alpha+1}\right.$$
$$\left.\times\left|f''\left(a+\frac{1+\tau}{2}e^{i\varphi}\eta(b,a)\right)\right|^q d\tau\right)^{\frac{1}{q}}$$
$$\leq\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}\left\{\int_0^1(1-\tau)^{\alpha+1}\right.$$
$$\times\left[\left(1-\left(\frac{1-\tau}{2}\right)^s\right)\right.$$
$$\left.\left.\times|f''(a)|^q+\left(\frac{1-\tau}{2}\right)|f''(b)|^q\right]d\tau\right\}^{\frac{1}{q}}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left\{\int_0^1(1-\tau)^{\alpha+1}\left[\left(1-\left(\frac{1-\tau}{2}\right)^s\right)|f''(b)|^q\right.\right.$$
$$\left.\left.+\left(\frac{1-\tau}{2}\right)|f''(a)|^q\right]d\tau\right\}^{\frac{1}{q}}$$
$$=\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(|f''(a)|^q\left(\frac{1}{\alpha+2}-\frac{1}{2^s}\frac{1}{\alpha+s+2}\right)\right.$$
$$\left.+|f''(b)|^q\frac{1}{2^s}\frac{1}{\alpha+s+2}\right)^{\frac{1}{q}}$$
$$+\frac{\left|e^{i\varphi}\eta(b,a)\right|^2}{8(\alpha+1)}\left(\frac{1}{\alpha+2}\right)^{1-\frac{1}{q}}$$
$$\times\left(|f''(a)|^q\frac{1}{2^s}\frac{1}{\alpha+s+2}\right.$$
$$\left.+|f''(b)|^q\left(\frac{1}{\alpha+2}-\frac{1}{2^s}\frac{1}{\alpha+s+2}\right)\right)^{\frac{1}{q}}$$

which completes the proof of Theorem. □

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Seda Kılınç *graduated from Kahramanmaraş Sütçü İmam University in 2014. In 2016, she received a master's degree from Kahramanmaraş Sütçü İmam University. In 2017, she started her Doctor of Philosophy (Ph.D) degree programme at Kahramanmaraş Sütçü İmam University.*

Abdullah Akkurt *holds Bachelor of Mathematics and Master of Science degrees from the University of Kahramanmaraş Sütçü İmam, Turkey. He is an Research Assistant in the Department of Mathematics in the University of Kahramanmaraş Sütçü İmam. His research interests are in special functions and integral inequalities. Presently, he is undertaking his Doctor of Philosophy (Ph.D) degree programme at University of Kahramanmaraş Sütçü İmam.*

Hüseyin Yıldırım *received his BSc (Maths) degree from Atatürk University, Erzurum, Turkey in 1986. He received his M.Sc. degree from Van Yüzüncü Yıl University in 1990. In 1995, he received a PhD (Maths) degrees from Ankara University. At present, he is working as a professor in the Department of Mathematics at Kahramanmaraş Sütçü İmam University (Turkey) and is the head of the department. He is the author or coauthor of more than 100 papers in the field of theory of inequalities, potential theory, integral equations and transforms, special functions, timescales.*