RESEARCH ARTICLE

On Hermite-Hadamard type inequalities for $S_{\varphi}$–preinvex functions by using Riemann-Liouville fractional integrals

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This paper is dedicated to the memory of our colleague, Dr. Hatice Yaldız, who recently passed away.

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1. Introduction

Fractional calculus (see [1–3]) arise in the mathematical modeling of various problems in sciences and engineering such as mathematics, physics, chemistry and biology.

Many authors have been working to fractional integral operators (see [4–7]) due to many applications in different areas of Mathematics, Engineering and Physics, etc (see [8, 9]). Also, these operators have allow to extended results about integral inequalities of many types (see [4, 10, 11]), for instance, Hermite-Hadamard integral inequalities (see [12–14]), Ostrowski type inequalities (see [7]).

In particular, in recent years, several extensions and generalizations have been considered for classical convexity (see [13,15,16]). A significant generalizations of convex functions is that of invex functions introduced by Hanson (see [17]).

In this work we derive several new inequalities of Hermite-Hadamard type for $S_{\varphi}$–preinvex function of first and second sense by using fractional integrals.

In this article, we define and recall some basic concepts and results. Let $\mathbb{R}^n$ be the finite dimensional Euclidian space, also $0 \leq \varphi \leq \frac{\pi}{2}$ be a continuous function.

In the following, we give some basic concepts and results.

Definition 1. ([7,8]). Let $f \in L_1[a,b]$. Then Riemann-Liouville fractional integrals $J^\alpha_a f$ and $J^\alpha_b f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^\alpha_a f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\tau)^{\alpha-1} f(\tau) \, d\tau, \quad (1)$$

and

$$J^\alpha_b f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\tau-x)^{\alpha-1} f(\tau) \, d\tau, \quad (2)$$

where $\Gamma$ is the classical Gamma function.

Definition 2. If $K_{\varphi\eta}$ in $\mathbb{R}^n$ set, is said to be $\varphi-$invex at $u$ according to $\varphi$, if there exists a bi-function $\eta(.,.): K_{\varphi\eta} \times K_{\varphi\eta} \rightarrow \mathbb{R}^n$, so that,

$$u + \tau e^{\varphi \eta}(u,v) \in K_{\varphi\eta}, \forall u,v \in K_{\varphi\eta}, \tau \in [0,1].$$
The \( \varphi \)--\ invex set \( K_{\varphi \eta} \) is also called \( \varphi \eta \)--\ connected set. Note that the convex set with \( \varphi = 0 \) and \( \eta(u, v) = v - u \) is a \( \varphi \)--\ invex set, but the converse is not true.

### Theorem 1

Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function defined on the interval \( I = [a, b] \) of real numbers where \( a < b \). Then, the following double inequality

\[
f\left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2},
\]

the above double inequality is known as Hermite-Hadamard type of inequality in the literature.

Let \( \mathbb{R} \) be the set of real numbers. During the article \( I = [a, b] \subset \mathbb{R} \) be the interval unless otherwise specified, also let \( 0 \leq \varphi \leq \frac{\pi}{2} \) be a continuous function.

### Lemma 1

Suppose a function \( K_{\varphi \eta} \subseteq \mathbb{R}^n \) and \( \varphi : K_{\varphi \eta} \rightarrow \mathbb{R} \), the \( f : [a, b] \rightarrow \mathbb{R} \) be twice differentiable function on \( (a, b) \) with \( a < b \). \( \eta(\ldots) : K_{\varphi \eta} \times K_{\varphi \eta} \rightarrow \mathbb{R}^n \), The \( \varphi \)--\ invex set \( K_{\varphi \eta} \) and \( 0 \leq \varphi \leq \frac{\pi}{2} \) be a continuous function. Let \( f'' \in L[a, b] \), afterward, we get the following equality for fractional integrals:

\[
\frac{2^{n-1} \Gamma(\alpha+1)}{(e^{\varphi \eta(b,a)})^\alpha} \left[ J_{a^+}^{\alpha, \varphi} \left( a + \frac{e^{\varphi \eta(b,a)}}{2} \right) \right] f(a) \\
+ J_{a^+}^{\alpha, \varphi} \left( a + \frac{e^{\varphi \eta(b,a)}}{2} \right) f(b) - f \left( a + \frac{e^{\varphi \eta(b,a)}}{2} \right) \\
= \frac{|e^{\varphi \eta(b,a)}|^2}{8(\alpha+1)} \int_0^1 (1-\tau)^{\alpha+1} \left[ f''(a + \frac{1-\tau}{2} e^{\varphi \eta(b,a)}) \right] d\tau \\
+ f'' \left( a + \frac{1+\tau}{2} e^{\varphi \eta(b,a)} \right) \right) \}
\]

### Proof

Let,

\[
I_1 = \int_0^1 (1-\tau)^{\alpha+1} \left[ f''(a + \frac{1-\tau}{2} e^{\varphi \eta(b,a)}) \right] d\tau \\
I_2 = \int_0^1 (1-\tau)^{\alpha+1} \left[ f''(a + \frac{1+\tau}{2} e^{\varphi \eta(b,a)}) \right] d\tau
\]

Integration by part respectively:

\[
I_1 = \int_0^1 (1-\tau)^{\alpha+1} f''(a + \frac{1-\tau}{2} e^{\varphi \eta(b,a)}) \, d\tau \\
= \frac{1}{2} \int_0^1 (1-\tau)^{\alpha+1} f'(a + \frac{\tau}{2} e^{\varphi \eta(b,a)}) \, d\tau \\
= \frac{1}{2} \int_0^1 (1-\tau)^{\alpha+1} f'(a + \frac{\tau}{2} e^{\varphi \eta(b,a)}) \, d\tau \\
- \frac{1}{2} \int_0^1 (1-\tau)^{\alpha+1} f'(a) \, d\tau \\
= \frac{2}{\alpha+1} \int_0^1 (1-\tau)^{\alpha+1} f'(a) \, d\tau
\]

Using \( I_1 \) and \( I_2 \) in (3), and afterwards multiplying both sides by \( \frac{(e^{\varphi \eta(b,a)})^\alpha}{8(\alpha+1)} \) the proof is done. \( \square \)

If we take \( \alpha = 1 \) in Lemma 1, we obtain to following result.

### Lemma 2

Let \( K_{\varphi \eta} \subseteq \mathbb{R}^n \) and \( \varphi : K_{\varphi \eta} \rightarrow \mathbb{R} \), the \( f : [a, b] \rightarrow \mathbb{R} \) be twice differentiable function on \( (a, b) \) with \( a < b \). Let \( f'' \in L[a, b] \), afterward, \( \eta(\ldots) : K_{\varphi \eta} \times K_{\varphi \eta} \rightarrow \mathbb{R}^n \), the \( \varphi \)--\ invex set \( K_{\varphi \eta} \) and \( 0 \leq \varphi \leq \frac{\pi}{2} \) be a continuous function. We get the following equality for fractional integrals:

\[
\int_a^{e^{\varphi \eta(b,a)}} e^{\varphi \eta(b,a)} f(x) \, dx - f \left( a + \frac{e^{\varphi \eta(b,a)}}{2} \right) \\
= \frac{|e^{\varphi \eta(b,a)}|^2}{16} \int_0^1 (1-\tau)^2 \left[ f''(a + \frac{1-\tau}{2} e^{\varphi \eta(b,a)}) + f''(a + \frac{1+\tau}{2} e^{\varphi \eta(b,a)}) \right] \, d\tau.
\]

If we take \( \eta(b,a) = b - a \), \( \varphi = 0 \), then we have,

\[ [a, a + e^{\varphi \eta(b,a)}] = [a, a + \eta(b,a)] = [a, b]. \]
2. Inequalities for $S_{\varphi}$-preinvex of second sense

In order to obtain main results introduced by [18] the $s_{\varphi}$-preinvex function of second sense.

**Definition 3.** [18] A function $f$ on the set $K_{\varphi\eta}$ is said to be $s_{\varphi}$-preinvex function of second sense according to $\varphi$ and $\eta$, we get

$$f(u + \tau e^{i\varphi} \eta(v, u)) \leq (1 - \tau)^s f(u) + \tau^s f(v), \quad (4)$$

where $\forall u, v \in K_{\varphi\eta}, \tau \in (0, 1]$.

**Theorem 2.** Suppose a function $K_{\varphi\eta} \subseteq \mathbb{R}^n$ be an open set according to bistunction $\eta(., .): K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}^n$, The $\varphi$-invex set $K_{\varphi\eta}$ where $\eta(b, a) > 0$. Also $\varphi: K_{\varphi\eta} \to \mathbb{R}$. Let $f''(\alpha \in \mathbb{R}, f'' \in L_1[a, a + e^{i\varphi}\eta(b, a)]$ and $|f''|$ is $s_{\varphi}$-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:

$$\left| \int_{\alpha}^{a + e^{i\varphi}\eta(b, a)} f(a) \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f'(a) \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f''(a) \right|$$

**Proof.** Via Lemma [1] and the fact that $|f''|$ is $s_{\varphi}$-preinvex function of second sense, we get

$$\left| \int_{\alpha}^{a + e^{i\varphi}\eta(b, a)} f(a) \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f'(a) \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f''(a) \right|$$

where $\int_0^1 (1 - \tau)^{\alpha+1} (1 + \tau)^s d\tau \leq \int_0^1 (1 + \tau)^s d\tau$

the above selection will be accepted, namely,

$$\left| \frac{|e^{i\varphi}\eta(b, a)|^2}{2^{\alpha+1}\Gamma(\alpha+1)} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f'(a) \right| \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f''(a) \right|$$

**Theorem 3.** Suppose a function $K_{\varphi\eta} \subseteq \mathbb{R}^n$ and $\varphi: K_{\varphi\eta} \to \mathbb{R}, f: [a, b] \to \mathbb{R}$ be twice differentiable function on $(a, b)$ with $a < b, \eta(., .): K_{\varphi\eta} \times K_{\varphi\eta} \to \mathbb{R}^n$. Also $\mathbb{R}^n$ be the finite dimensional Euclidian space. The $\varphi$-invex set $K_{\varphi\eta}$. Let $f'' \in L[a, b]$ and $|f''|$ is $s_{\varphi}$-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:

$$\left| \int_{\alpha}^{a + e^{i\varphi}\eta(b, a)} f(a) \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f'(a) \right| \leq \frac{2^{\alpha+1}\Gamma(\alpha+1)}{|e^{i\varphi}\eta(b, a)|^\alpha} \left| \int_0^{a + e^{i\varphi}\eta(b, a)} f''(a) \right|$$

where $\int_0^1 (1 - \tau)^{\alpha+1} (1 + \tau)^s d\tau \leq \int_0^1 (1 + \tau)^s d\tau$
Proof. From Lemma [I] Holder’s inequality and the fact that $|f''|^q$ is $s_{\varphi}$-preinvex function of second sense, we get

$$
\left|\frac{2^{n-1}\Gamma(a+1)}{(e^{s_{\varphi}\eta(b,a)})^n}\int_{a}^{b} f(x)^q \, dx - J \left(\frac{a^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) \right| \\
\leq \frac{|f''|^{q+1}}{2^{1+(q-1)(a-1)}} \left[ f(a) + f(b) \right] \\
+ \frac{1}{2^{1+(q-1)(a-1)}} \left[ f'' \left(\frac{a^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) + f'' \left(\frac{b^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) \right].
$$

Second sense, afterward, we get the following inequality for fractional integrals:

$$
\left|\frac{2^{n-1}\Gamma(a+1)}{(e^{s_{\varphi}\eta(b,a)})^n}\int_{a}^{b} f(x)^q \, dx - J \left(\frac{a^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) \right| \\
\leq \frac{|f''|^{q+1}}{2^{1+(q-1)(a-1)}} \left[ f(a) + f(b) \right] \\
+ \frac{1}{2^{1+(q-1)(a-1)}} \left[ f'' \left(\frac{a^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) + f'' \left(\frac{b^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) \right].
$$

Proof. From Lemma [I] power-mean inequality and the fact that $|f''|^q$ is $s_{\varphi}$-preinvex function of second sense, we get

$$
\left|\frac{2^{n-1}\Gamma(a+1)}{(e^{s_{\varphi}\eta(b,a)})^n}\int_{a}^{b} f(x)^q \, dx - J \left(\frac{a^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) \right| \\
\leq \frac{|f''|^{q+1}}{2^{1+(q-1)(a-1)}} \left[ f(a) + f(b) \right] \\
+ \frac{1}{2^{1+(q-1)(a-1)}} \left[ f'' \left(\frac{a^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) + f'' \left(\frac{b^{s_{\varphi}\eta(b,a)}}{e^{s_{\varphi}\eta(b,a)}}\right) \right].
$$

Theorem 4. Suppose a function $K_{\varphi} \subseteq \mathbb{R}^n$ and $\varphi : K_{\varphi} \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ be twice differentiable function on $(a, b)$ with $a < b$, $\eta(\cdot, \cdot) : K_{\varphi} \times K_{\varphi} \to \mathbb{R}$. Also $\mathbb{R}^n$ be the finite dimensional Euclidean space. The $\varphi$-invex set $K_{\varphi}$. Let $f'' \in L[a, b]$ and $|f''|^q$ is $s_{\varphi}$-preinvex function of first sense,
which completes the proof of Theorem.

\[ \Box \]

**Remark 1.** If we take \( \varphi = 0 \) in Theorem 4, we obtain the results in [7].

### 3. Inequalities for \( s_\varphi \)-convex functions of first sense

In order to obtain main results introduced by [13], the \( s_\varphi \)-preinvex function of first sense.

**Definition 4.** [18] Suppose a function \( f \) on the set \( K_{\varphi n} \) is said to be \( s_\varphi \)-preinvex function of first sense according to \( \varphi \) and \( \eta \), let

\[
 f(u + \tau e^{i\varphi} \eta(v, u)) \leq (1 - \tau^q) f(u) + \tau^q f(v),
\]

\( \forall u, v \in K_{\varphi n}, \tau \in [0, 1]. \)

**Theorem 5.** Suppose a function \( K_{\varphi n} \subseteq \mathbb{R}^n \) and \( \varphi : K_{\varphi n} \to \mathbb{R} \), the \( f : [a, b] \to \mathbb{R} \) be twice differentiable function on \( a, b \) with \( a < b \), \( \eta(\cdot, \cdot) : K_{\varphi n} \times K_{\varphi n} \to \mathbb{R} \). Also \( \mathbb{R}^n \) be the finite dimensional Euclidean space. The \( \varphi \)-invex set \( K_{\varphi n} \). Let \( f'' \in L[a, b] \) and \( |f''| \) is \( s_\varphi \)-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:

\[
\left| \frac{2^{n-1} \Gamma((n+1)/2)}{\Gamma((n+1)/2)} \right| \left[ J^{\varphi, \alpha} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) - f(a) \right] + J^{\varphi, \alpha} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) \left| f(b) \right| - f \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) \left| \right| \leq \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right|.
\]

**Proof.** From Lemma 1 and the fact that \( |f''| \) is \( s_\varphi \)-preinvex function of first sense, we get

\[
\left| \frac{2^{n-1} \Gamma((n+1)/2)}{\Gamma((n+1)/2)} \right| \left[ J^{\varphi, \alpha} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) - f(a) \right] + J^{\varphi, \alpha} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) \left| f(b) \right| - f \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) \left| \right| \leq \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right| \leq \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right|.
\]

\[ \Box \]

**Theorem 6.** Suppose a function \( K_{\varphi n} \subseteq \mathbb{R}^n \) and \( \varphi : K_{\varphi n} \to \mathbb{R} \), the \( f : [a, b] \to \mathbb{R} \) be twice differentiable function on \( a, b \) with \( a < b \), \( \eta(\cdot, \cdot) : K_{\varphi n} \times K_{\varphi n} \to \mathbb{R} \). Also \( \mathbb{R}^n \) be the finite dimensional Euclidean space. The \( \varphi \)-invex set \( K_{\varphi n} \). If \( f'' \in L[a, b] \) and \( |f''| \) is \( s_\varphi \)-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:

\[
\left| \frac{2^{n-1} \Gamma((n+1)/2)}{\Gamma((n+1)/2)} \right| \left[ J^{\alpha, \varphi} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) - f(a) \right] + \left[ J^{\alpha, \varphi} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) \right] \left| \right| + \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right| \leq \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right|.
\]

**Proof.** From Lemma 1 and Hölder inequality and the fact that \( |f''| \) is \( s_\varphi \)-preinvex function of second sense, we get

\[
\left| \frac{2^{n-1} \Gamma((n+1)/2)}{\Gamma((n+1)/2)} \right| \left[ J^{\alpha, \varphi} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) - f(a) \right] + \left[ J^{\alpha, \varphi} \left( \frac{e^{i\varphi} \eta(a, b)}{2} \right) \right] \left| \right| + \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right| \leq \left| \frac{e^{i\varphi} \eta(a, b)}{2} \right| \left| \left| f''(a) \right| + |f''(b)| \right|.
\]
Proof. From Lemma 1, power-mean inequality and the fact that $|f''|^{q}$ is $s_{\varphi}$-preinvex function of second sense, we get

$$
\left| J_{0}^{1} (1 - \tau)^{-\alpha+1} \right| \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) = \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) \leq \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) \leq \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) \leq \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a)
$$

which completes the proof of Theorem.

Theorem 7. Suppose a function $K_{\varphi} \subseteq \mathbb{R}^{n}$ and $\varphi : K_{\varphi} \rightarrow \mathbb{R}$, the $f : [a, b] \rightarrow \mathbb{R}$ be twice differentiable function on $(a, b)$ with $a < b$, $\eta(\cdot) : K_{\varphi} \times K_{\varphi} \rightarrow \mathbb{R}^{n}$. Also $\mathbb{R}^{n}$ be the finite dimensional Euclidean space. The $\varphi$-invers set $K_{\varphi}$. Let $f'' \in L[a, b]$ and $|f''|^{q}$ is $s_{\varphi}$-preinvex function of second sense, afterward, we get the following inequality for fractional integrals:

$$
\left| J_{0}^{1} (1 - \tau)^{-\alpha+1} \right| \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) = \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) \leq \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) \leq \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a) \leq \frac{2^\alpha \Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} \left( f(a)^{\alpha} + \frac{s_{\varphi}(b-a)}{2^{\alpha+1}} f(b) \right) \frac{1}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)} - f(a)
$$

which completes the proof of Theorem.

References


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