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RESEARCH ARTICLE

Analysis of rubella disease model with non-local and non-singular fractional derivatives

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ABSTRACT

In this paper we investigate a possible applicability of the newly established fractional differentiation in the field of epidemiology. To do this we extend the model describing the Rubella spread by changing the derivative with the time fractional derivative for the inclusion of memory. Detailed analysis of existence and uniqueness of exact solution is presented using the Banach fixed point theorem. Finally some numerical simulations are showed to underpin the effectiveness of the used derivative.

1. Introduction

The aim of mathematical biology is to develop mathematical equations and to describe some physical problems encountered in biology. Noting that, the establishment of such mathematical formula is achieved using the concept of differentiation or more practically the notion of derivatives. There exist two classes of differentiation in the literatures. The first one is based on the concept of rate of change [8-11,21]. The second one is based on the convolution of some functions including exponential decay law and the generalized Mittag-Leffler law. The derivatives based on exponential appear naturally in many problems in nature as being able to describe the effect of fading memory. This class of derivative has been applied in several research papers for instance [5,7,13,15,16,18-20,22]. However, it was noted by several experts in the field that, this new derivative does not have a non-local kernel as its corresponding integral is not fractional, thus a new kernel was suggested by Atangana and Baleanu [6] where after some manipulations, the exponential decay kernel was replaced by the generalized Mittag-Leffler kernel. This last derivative, therefore appears to be a very powerful mathematical tools form modeling real world problems as the generalized Mittag-Leffler function is combination of the power law and exponential decay law.

Several research papers have been published using this new concept of fractional differentiation with Mittag-Leffler. More importantly the results obtained in [1-4,6] revealed that, the new concept of more adequate for modeling real world problems to take into account the non-locality and also to have a memory effect. We shall note that the choice of a kernel is very important when modeling real world problems. When looking at experimental data obtained from real world observations, we can see that, many biological problems may not always follow the power law based on the function $x^{-\alpha}$ which is the kernel mostly used in the literature nowadays. For instance the case of Rubella, which is also known as the German measles or more precisely the three-day measles is enveloped and has a single-stranded RNA genome. The virus spreads via breathing

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route and photocopies in the nasopharynx and lymph nodes. This virus can only be detected in the stream blood after a period of between 5 to 7 days when the infection has taken place, later spreads throughout the body. With its properties of teratogenic and the ability of overpass the placenta and infecting the fetus where it stops cells developing or destroys them. Such a complex dynamic will be suitable to portray a more advance concept of power with of course a non-local concept which is the property inherited by the newly established derivative with fractional order called Atangana-Baleanu derivatives [6]. This paper is therefore devoted to the analysis of the dynamic of the spread of Rubella virus exploring the Atangana-Baleanu fractional derivative. The aim of the research in this field, requires the use of the new fractional derivative for Rubella disease virus. The exactness and uniqueness of the solution of the fractional model is proved by applying the fixed-point theorem.

The remainder part of this paper is broken into sections. In Section 2, we give the definitions of the new fractional derivative with non-singular and non-local kernel. Section 3 deals with the existence of solutions for the spread of rubella disease model via Picard-Lindelof method. In Section 4, we provide a special solution of the model which is considered using Atangala-Balenau derivative in Caputo sense. Finally in Section 5, some numerical results obtained at different instances of fractional order are presented to justify the suitability of the adopted derivative.

2. New fractional derivative with non-singular and non-local kernel

Let us remind the definitions of the new fractional derivative with non-singular and non-local kernel [6].

Definition 1. *Let* $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$ *then, the definition of the new fractional derivative (Atangana-Baleanu derivative in Caputo sense) is given as:*

$$ {}_{a}^{ABC}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\int\limits_a^t f^{'}(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx, \tag{1}$$

where ${}_{a}^{ABC}D_t^{\alpha}$ *is fractional operator with Mittag-Leffler kernel in the Caputo sense with order* α *with respect to* t *and* $B(\alpha) = B(0) = B(1) = 1$ *is a normalization function [12].*

Definition 2. *Let* $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$ *and not differentiable then, the definition of the new fractional derivative (Atangana-Baleanu fractional derivative in Riemann-Liouville sense) is given as:*

$$ {}_{a}^{ABR}D_t^{\alpha}\left(f\left(t\right)\right) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int\limits_a^t f(x)E_{\alpha}\left[-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\right]dx. \tag{2}$$

Definition 3. *The fractional integral of order* α *of a new fractional derivative is defined as:*

$$ {}_{a}^{AB}I_t^{\alpha}\left\{f(t)\right\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int\limits_a^t f(y)(t-y)^{\alpha-1}dy. \tag{3}$$

When α *is zero, initial function is obtained and when* α *is 1, the ordinary integral is obtained.*

3. Existence of solutions for the spread of rubella disease model

Let us consider the following model employing the Atangana-Baleanu fractional derivative in Caputo sense :

$$\begin{aligned}
{}_{0}^{ABC}D_t^{\alpha}S\left(t\right) &= B(a) - \left[\lambda(a,t) + P(a) + \mu(a)\right]S\left(t\right), \\
{}_{0}^{ABC}D_t^{\alpha}E\left(t\right) &= \lambda(a,t)S\left(t\right) - \left(\sigma + \mu(a)\right)E\left(t\right), \\
{}_{0}^{ABC}D_t^{\alpha}I\left(t\right) &= \sigma E\left(t\right) - \left(\beta + \mu(a)\right)I\left(t\right), \\
{}_{0}^{ABC}D_t^{\alpha}R\left(t\right) &= \beta I\left(t\right) - \mu(a)R\left(t\right), \\
{}_{0}^{ABC}D_t^{\alpha}V\left(t\right) &= D(a)S\left(t\right) - \mu(a)V\left(t\right),
\end{aligned} \tag{4}$$

where $S\left(t\right)$, $E\left(t\right)$, $I\left(t\right)$, $R\left(t\right)$, $V\left(t\right)$ are susceptible, latent, infectious, recovered and vaccinated parameters respectively. $P(a)$ is a parameter for which immunized by vaccination and $\lambda(a,t)$ is the force of infection of age a at time t. Finally, σ is the latent rate and β is the infection rate [14]. The aim of this section is to find existence of solutions for rubella disease model with Atangana-Balenau fractional derivative. The system state is made up with S, E, I, R, V. The above system (4) can be converted to Volterra type integral equation with the Atangana-Baleanu fractional integral.

Theorem 1. *The following time fractional ordinary differential equation*

$$ {}_{0}^{ABC}D_t^{\alpha}(f\left(t\right)) = u(t), \tag{5}$$

has a unique solution with taking the inverse Laplace transform and using the convolution theorem below [4]:

$$f(t) = \frac{1-\alpha}{B(\alpha)}u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t u(y)(t-y)^{\alpha-1}dy. \quad (6)$$

By the theorem above, the model can be written as (7):

$$\begin{cases} S(t) - g_1(t) = \frac{1-\alpha}{B(\alpha)}\left\{B(a) - \left[\lambda(a,t) + P(a) + \mu(a)\right]S(t)\right\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \\ \times \left\{B(a) - \left[\lambda(a,y) + P(a) + \mu(a)\right]S(y)\right\}dy, \\ E(t) - g_2(t) = \frac{1-\alpha}{B(\alpha)}\left\{\lambda(a,t)S(t) - (\sigma + \mu(a))E(t)\right\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \\ \times \left\{\lambda(a,y)S(y) - (\sigma+\mu(a))E(y)\right\}dy, \\ I(t) - g_3(t) = \frac{1-\alpha}{B(\alpha)}\left\{\sigma E(t) - (\beta+\mu(a))I(t)\right\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\left\{\sigma E(y) - (\beta+\mu(a))I(y)\right\}dy, \\ R(t) - g_4(t) = \frac{1-\alpha}{B(\alpha)}\left\{\beta I(t) - \mu(a)R(t)\right\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\left\{\beta I(y) - \mu(a)R(y)\right\}dy, \\ V(t) - g_5(t) = \frac{1-\alpha}{B(\alpha)}\left\{D(a)S(t) - \mu(a)V(t)\right\} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\left\{D(a)S(y) - \mu(a)V(y)\right\}dy, \end{cases} \quad (7)$$

The above system (7) of equations can be iteratively represented as:

$$\begin{cases} S_0(t) = g_1(t), \\ E_0(t) = g_2(t), \\ I_0(t) = g_3(t), \\ R_0(t) = g_4(t), \\ V_0(t) = g_5(t). \end{cases} \quad (8)$$

$$S_{n+1}(t) = \frac{1-\alpha}{B(\alpha)} \times \left\{B(a) - \left[\lambda(a,t) + P(a) + \mu(a)\right]S_n(t)\right\} \quad (9)$$
$$+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \times \left\{B(a) - \left[\lambda(a,y) + P(a) + \mu(a)\right]S_n(y)\right\}dy,$$

$$E_{n+1}(t) = \frac{1-\alpha}{B(\alpha)}\left\{\lambda(a,t)S_n(t) - (\sigma+\mu(a))E_n(t)\right\} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \times \left\{\lambda(a,y)S_n(y) - (\sigma+\mu(a))E_n(y)\right\}dy,$$

$$I_{n+1}(t) = \frac{1-\alpha}{B(\alpha)}\left\{\sigma E_n(t) - (\beta+\mu(a))I_n(t)\right\} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \times \left\{\sigma E_n(y) - (\beta+\mu(a))I_n(y)\right\}dy,$$

$$R_{n+1}(t) = \frac{1-\alpha}{B(\alpha)}\left\{\beta I_n(t) - \mu(a)R_n(t)\right\} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \times \left\{\beta I_n(y) - \mu(a)R_n(y)\right\}dy,$$

$$V_{n+1}(t) = \frac{1-\alpha}{B(\alpha)}\left\{D(a)S_n(t) - \mu(a)V_n(t)\right\} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1} \times \left\{D(a)S_n(y) - \mu(a)V_n(y)\right\}dy.$$

As the exact solution of the iterative formula of a Picard series used here converges toward the exact solution as the number of series terms tends to infinity. If we take the limit with greater than n, we expect to obtain the exact solution of equation as below:

$$\begin{cases} \lim_{n\to\infty} S_n(t) = S(t), \\ \lim_{n\to\infty} E_n(t) = E(t), \\ \lim_{n\to\infty} I_n(t) = I(t), \\ \lim_{n\to\infty} R_n(t) = R(t), \\ \lim_{n\to\infty} V_n(t) = V(t). \end{cases}$$

3.1. Existence of solution via Picard-Lindelof method

Let us define the following operator for showing the existence of solution:

$$\begin{aligned} f_1(a,t) &= B(a) - \left[\lambda(a,t) + P(a) + \mu(a)\right]S(t), \\ f_2(a,t) &= \lambda(a,t)S(t) - (\sigma+\mu(a))E(t), \\ f_3(a,t) &= \sigma E(t) - (\beta+\mu(a))I(t), \\ f_4(a,t) &= \beta I(t) - \mu(a)R(t), \\ f_5(a,t) &= D(a)S(t) - \mu(a)V(t). \end{aligned} \quad (10)$$

Let

$$\begin{aligned} N_1 &= \sup_{C[b,c_1]} \left\|f_1(a,t)\right\|, \; N_2 = \sup_{C[b,c_2]} \left\|f_2(a,y)\right\|, \\ N_3 &= \sup_{C[b,c_3]} \left\|f_3(a,z)\right\|, \; N_4 = \sup_{C[b,c_4]} \left\|f_4(a,p)\right\|, \\ N_5 &= \sup_{C[b,c_5]} \left\|f_5(a,r)\right\|, \end{aligned} \quad (11)$$

where

$$\begin{aligned} C[b,c_1] &= [t-b,t+b]\times[x-c_1,x+c_1] = B_1\times C_1,\\ C[b,c_2] &= [t-b,t+b]\times[x-c_2,x+c_2] = B_1\times C_2,\\ C[b,c_3] &= [t-b,t+b]\times[x-c_3,x+c_3] = B_1\times C_3,\\ C[b,c_4] &= [t-b,t+b]\times[x-c_4,x+c_4] = B_1\times C_4,\\ C[b,c_5] &= [t-b,t+b]\times[x-c_5,x+c_5] = B_1\times C_5. \end{aligned} \tag{12}$$

We will make use of Banach fixed-point theorem using the metric on $C[b,c_i]$, $(i=1,2,...,5)$ made by the uniform norm

$$\|X(t)\|_\infty = \sup_{t\in[t-b,t+b]} |f(t)|. \tag{13}$$

The next operator is defined between the two functional spaces of continuous functions, Picard's operator as follows:

$$\begin{aligned} O &: C(B_1,C_1,C_2,C_3,C_4,C_5)\\ &\to C(B_1,C_1,C_2,C_3,C_4,C_5). \end{aligned} \tag{14}$$

For simplicity, let us define $f_i(a,t) = X(t)$, $f_i(a,0) = X_0(t)$, $(i=1,2,...,5)$. Then the system is reduced the following:

$$\begin{aligned} OX(t) &= X_0(t) + F(t,X(t))\frac{1-\alpha}{B(\alpha)}\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}F(y,X(y))dy, \end{aligned} \tag{15}$$

where X is the matrice of given as

$$X(t) = \begin{cases} S(t)\\ E(t)\\ I(t)\\ R(t)\\ V(t)\end{cases}, X_0(t) = \begin{cases} S(0)\\ E(0)\\ I(0)\\ R(0)\\ V(0)\end{cases}, \tag{16}$$

$$F(a,X(t)) = \begin{cases} f_1(a,t)\\ f_2(a,t)\\ f_3(a,t)\\ f_4(a,t)\\ f_5(a,t)\end{cases}.$$

Let us assume that the physical problem under investigation satisfies followings:

$$\|X(t)\|_\infty \le \max\{c_1,c_2,c_3,c_4,c_5\}. \tag{17}$$

$$\begin{aligned} &\|OX(t) - X_0(t)\| \qquad (18)\\ &= \left\| F(t,X(t))\tfrac{1-\alpha}{B(\alpha)} + \tfrac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}F(y,X(y))dy \right\|\\ &\le \frac{1-\alpha}{B(\alpha)}\|F(t,X(t))\|\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\|F(y,X(y))\|dy\\ &\le \frac{1-\alpha}{B(\alpha)}N = \max\{N_1,N_2,N_3,N_4,N_5\}\\ &+ \frac{\alpha}{B(\alpha)}Nb^\alpha < bN \le c = \max\{c_1,c_2,c_3,c_4,c_5\}, \end{aligned}$$

where we demand that

$$b < \frac{c}{N}.$$

Also we evaluate the following equality

$$\|OX_1 - OX_2\|_\infty = \sup_{t\in B}|X_1 - X_2|. \tag{19}$$

Nonetheless using the definition of our defined operator, we have

$$\|OX_1 - OX_2\|$$

$$= \left\| \begin{array}{c} \{F(t,X_1(t)) - F(t,X_2(t))\}\frac{1-\alpha}{B(\alpha)}\\ +\frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-l)^{\alpha-1}\left\{\begin{array}{c} F(l,X_1(l))\\ -F(l,X_2(l))\end{array}\right\}dl \end{array}\right\| \tag{20}$$

$$\begin{aligned} &\le \frac{1-\alpha}{B(\alpha)}\|F(t,X_1(t)) - F(t,X_2(t))\| \qquad (21)\\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\\ &\times \|F(l,X_1(y)) - F(l,X_2(y))\|dy\\ &\le \frac{1-\alpha}{B(\alpha)}q\|X_1(t) - X_2(t)\|\\ &+ \frac{\alpha q}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\|X_1(y) - X_2(y)\|dy\\ &\le \left\{\frac{1-\alpha}{B(\alpha)}q + \frac{\alpha q b^\alpha}{B(\alpha)\Gamma(\alpha)}\right\}\|X_1(t) - X_2(t)\|\\ &\le bq\|X_1(t) - X_2(t)\|. \end{aligned}$$

with $q < 1$ since F is a contraction we have that $bq < 1$, thus the defined operator O is a contraction. So system has a unique set of solution.

4. Special solutions via iteration approach

The aim of this section is to provide a special solution of the model which is considered using

Atangala-Balenau derivative in Caputo sense. Let us apply the Sumudu transform on both sides of equation (4) together with an iterative method. We shall give the Sumudu transform of Atangana-Balenau fractional derivative in Caputo sense below:

Theorem 2. *Let $f \in H^1(a,b)$, $b > a$, $\alpha \in [0,1]$ then, the Sumudu transform of Atangana-Balenau fractional derivative in Caputo sense is given as:*

$$ST\left\{{}_0^{ABC}D_t^{\alpha}(f(t))\right\} = \frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha(-\frac{1}{1-\alpha}p^\alpha)\right) \times \left(ST(f(t)) - f(0)\right). \tag{22}$$

Proof. Proof of the theorem can be found in [4]. □

To solve Equation (4), we apply the Sumudu transform of the Atangana-Balenau fractional derivative of $f(t)$ on system with both sides. Then we obtain below:

$$\begin{aligned}
&\tfrac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha(-\tfrac{1}{1-\alpha}p^\alpha)\right)\left(ST(S(t))-S(0)\right)\\
&= ST\left\{B(a) - [\lambda(a,t)+P(a)+\mu(a)]\,S(t)\right\},\\
&\tfrac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha(-\tfrac{1}{1-\alpha}p^\alpha)\right)\left(ST(E(t))-E(0)\right)\\
&= ST\left\{\lambda(a,t)S(t) - (\sigma+\mu(a))\,E(t)\right\},\\
&\tfrac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha(-\tfrac{1}{1-\alpha}p^\alpha)\right)\left(ST(I(t))-I(0)\right)\\
&= ST\left\{\sigma E(t) - (\beta+\mu(a))\,I(t)\right\},\\
&\tfrac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha(-\tfrac{1}{1-\alpha}p^\alpha)\right)\left(ST(R(t))-R(0)\right)\\
&= ST\left\{\beta I(t) - \mu(a)R(t)\right\},\\
&\tfrac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha(-\tfrac{1}{1-\alpha}p^\alpha)\right)\left(ST(V(t))-V(0)\right)\\
&= ST\left\{D(a)S(t) - \mu(a)V(t)\right\}.
\end{aligned} \tag{23}$$

Rearranging, we obtain following inequalities where,

$$\begin{aligned}
&ST(S(t)) = S(0)\\
&+\theta * ST\left\{B(a) - [\lambda(a,t)+P(a)+\mu(a)]\,S(t)\right\},\\
&ST(E(t)) = E(0)\\
&+\theta * ST\left\{\lambda(a,t)S(t) - (\sigma+\mu(a))\,E(t)\right\},\\
&ST(I(t)) = I(0)\\
&+\theta * ST\left\{\sigma E(t) - (\beta+\mu(a))\,I(t)\right\},\\
&ST(R(t)) = R(0)\\
&+\theta * ST\left\{\beta I(t) - \mu(a)R(t)\right\},\\
&ST(V(t)) = V(0)\\
&+\theta * ST\left\{D(a)S(t) - \mu(a)V(t)\right\}.
\end{aligned}$$

For simplicity, here

$$\theta = \frac{1-\alpha}{B(\alpha)\left(\alpha\Gamma(\alpha+1)E_\alpha(-\frac{1}{1-\alpha}p^\alpha)\right)},$$

is considered and "$*$" means multiplication sign . We next obtain the following recursive formula;

$$\begin{aligned}
S_{n+1}(t) &= S_n(0)\\
&+ST^{-1}\left\{\theta * ST\left\{B(a) - [\lambda(a,t)+P(a)+\mu(a)]\,S_n(t)\right\}\right\},\\
E_{n+1}(t) &= E_n(0)\\
&+ST^{-1}\left\{\theta * ST\left\{\lambda(a,t)S_n(t) - (\sigma+\mu(a))\,E_n(t)\right\}\right\},\\
I_{n+1}(t) &= I_n(0)\\
&+ST^{-1}\left\{\theta * ST\left\{\sigma E_n(t) - (\beta+\mu(a))\,I_n(t)\right\}\right\},\\
R_{n+1}(t) &= R_n(0)\\
&+ST^{-1}\left\{\theta * ST\left\{\beta I_n(t) - \mu(a)R_n(t)\right\}\right\},\\
V_{n+1}(t) &= V_n(0)\\
&+ST^{-1}\left\{\theta * ST\left\{D(a)S_n(t) - \mu(a)V_n(t)\right\}\right\}.
\end{aligned} \tag{24}$$

Therefore, the solution of equation (24) approximate to following

$$\begin{aligned}
S(t) &= \lim_{n\to\infty} S_n(t),\\
E(t) &= \lim_{n\to\infty} E_n(t),\\
I(t) &= \lim_{n\to\infty} I_n(t),\\
R(t) &= \lim_{n\to\infty} R_n(t),\\
V(t) &= \lim_{n\to\infty} V_n(t).
\end{aligned} \tag{25}$$

4.1. Application of fixed-point theorem for stability analysis of iteration method

Let $(X, \|.\|)$ be a Banach space and H a self-map of X. Let $y_{n+1} = g(H, y_n)$ be recurcive procedure. Suppose that, $F(H)$ the fixed-point set of

H has at least one element and that $y_n$converges to a point $p \in F(H)$. Let $\{x_n\} \subseteq X$ and define $e_n = \|x_{n+1} - g(H, x_n)\|$. If $\lim_{n\to\infty} e_n = 0$ implies that $\lim_{n\to\infty} x_n = p$, then the iteration method $y_{n+1} = g(H, y_n)$ is H-Stable. Then let we assume that, our sequence $\{x_n\}$ has an upper boundary . If all these conditions are satisfied for $y_{n+1} = Hy_n$ which is known as Picard's iteration, consequently the iteration is H-Stable. We shall then state the following theorem.

Theorem 3. *Let $(X, \|.\|)$ be a Banach space and H a self-map of X satisfying*

$$\|H_x - H_y\| \le K\|x - H_x\| + k\|x - y\|,$$

for all x, y in X where $0 \le K$, $0 \le k < 1$. Suppose that H is Picard H-Stable [17].

Let us consider the following recursive formula equation (27) with (4) where

$$\theta = \frac{1-\alpha}{B(\alpha)\left(\alpha\Gamma(\alpha+1)E_\alpha(-\frac{1}{1-\alpha}p^\alpha)\right)}, \tag{26}$$

is the fractional Lagrange multiplier.

Theorem 4. *Let H be a self-map defined as (27) as below.*

$$\begin{aligned} H(S_n(t)) &= S_{n+1}(t) = S_n(t) \qquad (27)\\ &+ST^{-1}\{\theta * ST\{B(a) - [\lambda(a,t) + P(a) + \mu(a)]S_n(t)\}\},\\ H(E_n(t)) &= E_{n+1}(t) = E_n(t)\\ &+ST^{-1}\{\theta * ST\{\lambda(a,t)S_n(t) - (\sigma + \mu(a))E_n(t)\}\},\\ H(I_n(t)) &= I_{n+1}(t) = I_n(t)\\ &+ST^{-1}\{\theta * ST\{\sigma E_n(t) - (\beta + \mu(a))I_n(t)\}\},\\ H(R_n(t)) &= R_{n+1}(t) = R_n(t)\\ &+ST^{-1}\{\theta * ST\{\beta I_n(t) - \mu(a)R_n(t)\}\},\\ H(V_n(t)) &= V_{n+1}(t) = V_n(t)\\ &+ST^{-1}\{\theta * ST\{D(a)S_n(t) - \mu(a)V_n(t)\}\}. \end{aligned}$$

Then (27) is H-stable in $L^1(a, b)$ if following statement can be obtained.

$$\begin{aligned} (1 - [\lambda(a,t) + P(a) + \mu(a)]A(\gamma)) &< 1, \qquad (28)\\ (1 + \lambda(a,t)B(\gamma) - (\sigma + \mu(a))C(\gamma)) &< 1,\\ (1 + \sigma D(\gamma) - (\beta + \mu(a))E(\gamma)) &< 1,\\ (1 + \beta F(\gamma) - \mu(a)G(\gamma)) &< 1,\\ (1 + D(a)H(\gamma) - \mu(a)J(\gamma)) &< 1. \end{aligned}$$

Proof. Let we start with showing that H has a fixed point. To achieve this, we evaluate the followings for all $(n, m) \in \mathbb{N} \times \mathbb{N}$.

$$\begin{aligned} H(S_n(t)) - H(S_m(t)) &= S_n(t) - S_m(t) \qquad (29)\\ &+ST^{-1}\{\theta * ST\{B(a) - [\lambda(a,t) + P(a) + \mu(a)]S_n(t)\}\}\\ &-ST^{-1}\{\theta * ST\{B(a) - [\lambda(a,t) + P(a) + \mu(a)]S_m(t)\}\}. \end{aligned}$$

Let us consider (29) and apply norm on both sides and without loss of generality

$$\begin{aligned} &\|H(S_n(t)) - H(S_m(t))\| \qquad (30)\\ &= \left\| \begin{array}{c} S_n(t) - S_m(t) \\ +ST^{-1}\left\{\theta * ST\left\{ \begin{array}{c} B(a) - [\lambda(a,t) + P(a) + \mu(a)]S_n(t) \\ -(B(a) - [\lambda(a,t) + P(a) + \mu(a)]S_m(t)) \end{array} \right\}\right\} \end{array} \right\| \end{aligned}$$

$$\begin{aligned} &\le \|S_n(t) - S_m(t)\| \qquad (31)\\ &+\|ST^{-1}\{\theta * ST\{-[\lambda(a,t) + P(a) + \mu(a)](S_n(t) - S_m(t))\}\}\|. \end{aligned}$$

Now we obtain :

$$\begin{aligned} &\|H(S_n(t)) - H(S_m(t))\| \le \|S_n(t) - S_m(t)\|\\ &\times(1 - [\lambda(a,t) + P(a) + \mu(a)]A(\gamma)), \qquad (32) \end{aligned}$$

where $A(\gamma)$ is the $ST^{-1}\{\theta * ST\}$. Since all solutions have same role also we have following:

$$\begin{aligned} &\|H(E_n(t)) - H(E_m(t))\| \le \|E_n(t) - E_m(t)\|\\ &\times(1 + \lambda(a,t)B(\gamma) - (\sigma + \mu(a))C(\gamma)),\\ &\|H(I_n(t)) - H(I_m(t))\| \le \|I_n(t) - I_m(t)\|\\ &\times(1 + \sigma D(\gamma) - (\beta + \mu(a))E(\gamma)),\\ &\|H(R_n(t)) - H(R_m(t))\| \le \|R_n(t) - R_m(t)\|\\ &\times(1 + \beta F(\gamma) - \mu(a)G(\gamma)),\\ &\|H(V_n(t)) - H(V_m(t))\| \le \|V_n(t) - V_m(t)\|\\ &\times(1 + D(a)H(\gamma) - \mu(a)J(\gamma)). \qquad (33) \end{aligned}$$

For

$$\begin{aligned} (1 - [\lambda(a,t) + P(a) + \mu(a)]A(\gamma)) &< 1,\\ (1 + \lambda(a,t)B(\gamma) - (\sigma + \mu(a))C(\gamma)) &< 1,\\ (1 + \sigma D(\gamma) - (\beta + \mu(a))E(\gamma)) &< 1,\\ (1 + \beta F(\gamma) - \mu(a)G(\gamma)) &< 1,\\ (1 + D(a)H(\gamma) - \mu(a)J(\gamma)) &< 1, \qquad (34) \end{aligned}$$

then H-self mapping has a fixed point. Also nonlinear mapping H has to satisfy the conditions. So let we assume

$$K = (0, 0, 0, 0, 0)$$

$$k = \left\{ \begin{array}{c} (1 - [\lambda(a,t) + P(a) + \mu(a)]A(\gamma)) \\ (1 + \lambda(a,t)B(\gamma) - (\sigma + \mu(a))C(\gamma)) \\ (1 + \sigma D(\gamma) - (\beta + \mu(a))E(\gamma)) \\ (1 + \beta F(\gamma) - \mu(a)G(\gamma)) \\ (1 + D(a)H(\gamma) - \mu(a)J(\gamma)) \end{array} \right., \tag{35}$$

then all conditions of Theorem 3 hold. This completes the proof. □

5. Numerical Simulation

In this part, we present the numerical replication of the model for different values of fractional order using the proposed numerical scheme. The numerical simulations are shown in figure 1, 2, 3, and 4. Figures 1 is considered alpha to be 0.95, figure 2 is considered alpha to be 0.65, figure 3 is considered alpha to be 0.45 and finally in figure 4 is considered alpha to be 0.05. The paremeters used in this simulations are given below:

$$\begin{array}{ccc} B = 100, & P = 0.3, & \lambda = 0.4, \\ \mu = 0.4, & \sigma = 0.3, & \beta = 0.4 \end{array}. \tag{36}$$

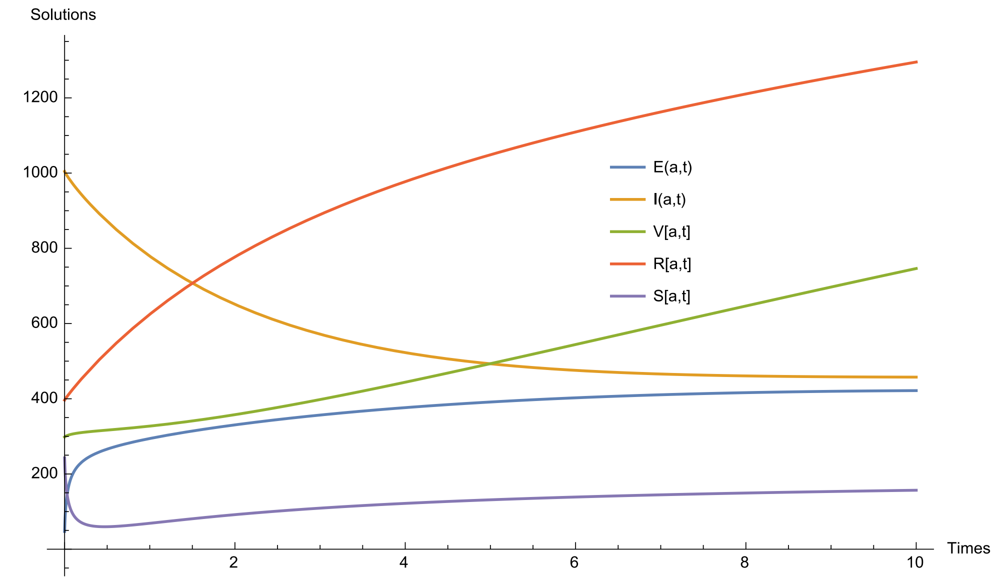

Figure 1 : Numerical simulation of solution for $\alpha = 0.95$

n

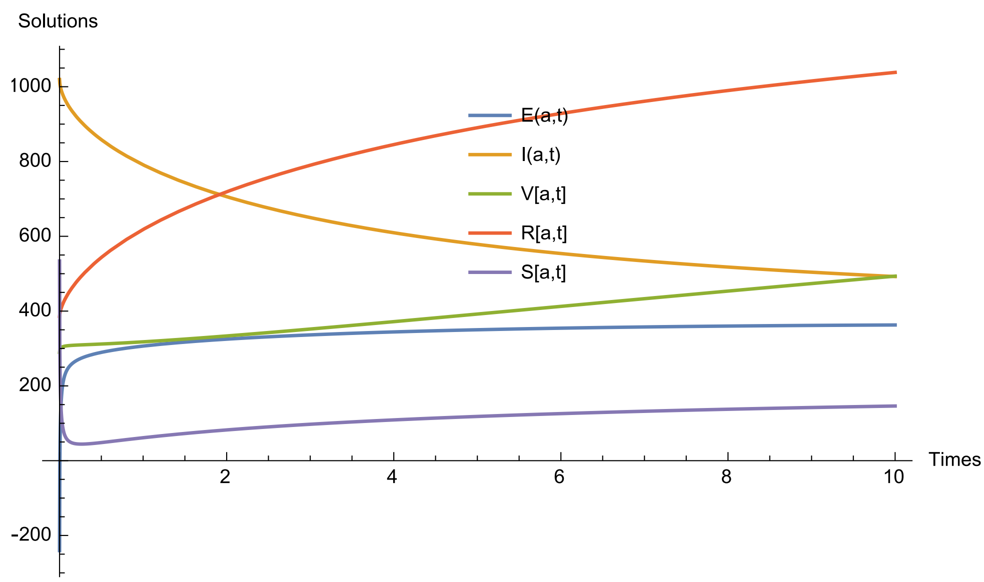

Figure 2 : Numerical simulation of solution for $\alpha = 0.65$

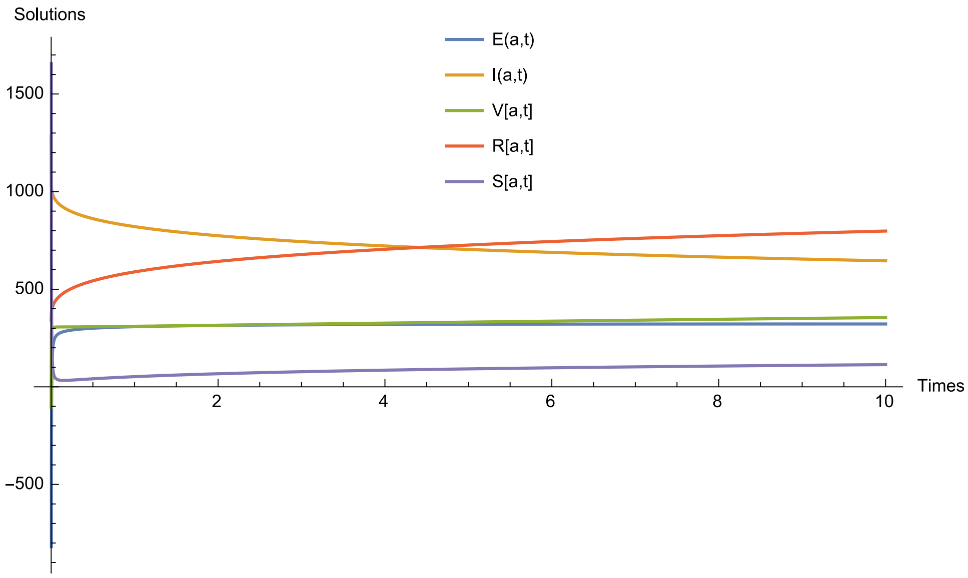

Figure 3 : Numerical simulation of solution for $\alpha = 0.45$

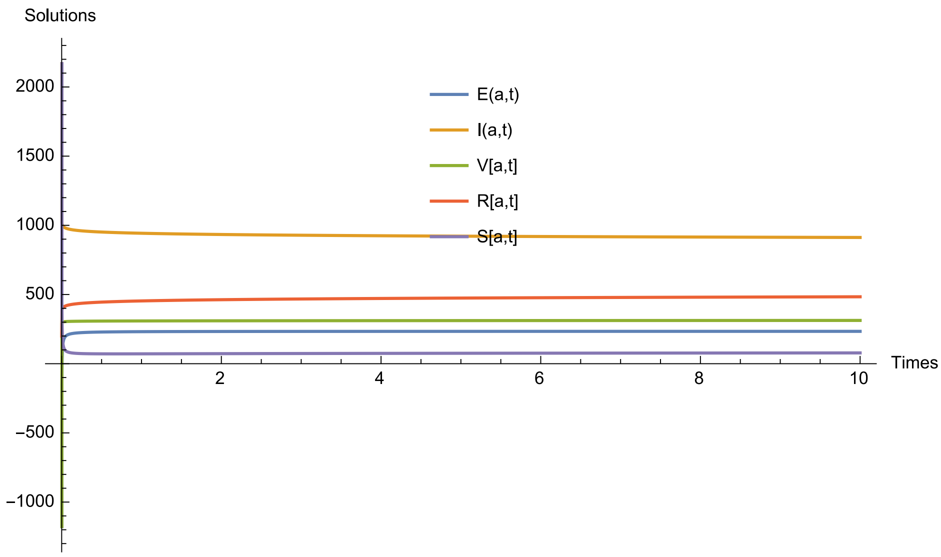

Figure 4 : Numerical simulation of solution for $\alpha = 0.05$

6. Conclusion

In this work, we have extended the model of rubella disease to the concept of fractional differential based on the Mittag-Leffler. We studied the existence of the generalized model using the fixed-point theorem. We presented the derivation of the solution using the Sumudu transform of Atanagana-Balenau derivative in Caputo sense. The stability analysis of the method is validated with the H-stable approach. Finally. Numerical simulations presented for different values of α.

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