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RESEARCH ARTICLE

New soliton properties to the ill-posed Boussinesq equation arising in nonlinear physical science

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ABSTRACT

In manuscript, with the help of the Wolfram Mathematica 9, we employ the modified exponential function method in obtaining some new soliton solutions to the ill-posed Boussinesq equation arising in nonlinear media. Results obtained with use of technique, and also, surfaces for soliton solutions are given. We also plot the 3D and 2D of each solution obtained in this study by using the same program in the Wolfram Mathematica 9.

1. Introduction

For some past decades explorations for the search of the new solutions to non-linear evolution equations (NLEEs) have attracted the attention of many scholars. Nonlinear evolution equation are often used to describe complex aspects of various models arising in the field of nonlinear sciences such as mathematical physics, chemical physics, chemistry, biological sciences etc. Attention from different researchers has been paid to this area in searching for new solutions to the different class of NLEEs where various powerful method are formulated such as the generalized and improved $\left(G'/G\right)$-expansion method [1], the Jacobi elliptic-function method [2], the modified simple equation method [3,4], the sine-Gordon expansion method [5-7], the extended tanh method [8], the improved Bernoulli sub-equation function method [9], the rational sine-cosine method [10], the Ricatti-Bernoulli sub-ODE method [11], the Homotopy perturbation method [12] and so on.

However, in this work we aim at investigating solution of the ill-posed Boussinesq equation [13] by using the modified exp $\left(-\psi\left(\xi\right)\right)$-expansion function method (MEFM) [14, 15]. The ill-posed Boussinesq equation also known as bad Boussinesq equation was derived by J. Boussinesq [16] to describe the propagation of long waves on the surface of water with a small amplitude in none-dimensional nonlinear lattices and in nonlinear strings [17].

Recently, some analytical methods for obtaining the solutions of ill-posed Boussinesq equation have been designed by different scientists, this include the solitary wave ansatz method and the Bernoulli sub-Ode [18], the exp function method [19], the Adomian decomposition method [20] etc.

2. Analysis of the method

In this section, we give the insight of the MEFM and how it can be applied to find solutions to some nonlinear partial differential equations. The MEFM is developed by improving the well-known exp $\left(-\psi\left(\xi\right)\right)$-expansion function method. To explore the search for the new solutions of any given nonlinear partial differential equation (Eq. (2.1)), we follow the following steps:

$$F\left(v, v_x, v_x v^2, v_{xx}, v_{xxt}, \ldots\right) = 0, \qquad (2.1)$$

where $v = v\left(x,t\right)$ is unknown function, F is a

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polynomial in $v(x,t)$ and its derivatives in which the highest order derivatives and the nonlinear terms are involved and the subscript stand for the partial derivatives.

Step 1: Consider the wave transformation given by

$$v(x,t)=V(\xi),\quad \xi=k(x-ct) \tag{2.2}$$

applying Eq. (2.2) on Eq. (2.1), gives the following nonlinear ordinary differential equation (NODE):

$$Q\left(V,V^2,V',V'',...\right)=0, \tag{2.3}$$

where Q is a polynomial of V and its derivatives and the superscripts stand for the ordinary derivatives of V with respect to ξ.

Step 2: Assuming that the wave solutions of Eq. (2.3) can be written in the following form:

$$V(\xi)=\frac{\sum_{i=0}^{N}A_i\left[e^{-\psi(\xi)}\right]^i}{\sum_{j=0}^{M}B_j\left[e^{-\psi(\xi)}\right]^j} =\frac{A_0+A_1e^{-\psi}+\ldots+A_Ne^{-N\psi}}{B_0+B_1e^{-\psi}+\ldots+B_Me^{-M\psi}}, \tag{2.4}$$

where $A_i, B_j, (0\le i\le N, 0\le j\le M)$ are constants to be found later, such that $A_N\neq 0,\ B_M\neq 0$ and $\psi=\psi(\xi)$ simplifies the following ODE:

$$\psi'(\xi)=e^{-\psi(\xi)}+\mu e^{\psi(\xi)}+\lambda, \tag{2.5}$$

Eq.(2.5) has the following families of solution [21-23]:

Family 1: When $\mu\neq 0,\ \lambda^2-4\mu>0$,

$$\psi(\xi)=In\left(\frac{-\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+E)\right)-\frac{\lambda}{2\mu}\right). \tag{2.6}$$

Family 2: When $\mu\neq 0,\ \lambda^2-4\mu<0$,

$$\psi(\xi)=In\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tanh\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\xi+E)\right)-\frac{\lambda}{2\mu}\right). \tag{2.7}$$

Family 3: When $\mu=0,\ \lambda\neq 0$ and $\lambda^2-4\mu<0$,

$$\psi(\xi)=-In\left(\frac{\lambda}{e^{\lambda(\xi+E)}-1}\right). \tag{2.8}$$

Family 4: When $\mu\neq 0,\ \lambda\neq 0$ and $\lambda^2-4\mu=0$,

$$\psi(\xi)=In\left(-\frac{2\lambda(\xi+E)+4}{\lambda^2(\xi+E)}\right). \tag{2.9}$$

Family 5: When $\mu=0,\ \lambda=0$ and $\lambda^2-4\mu=0$,

$$\psi(\xi)=In(\xi+E). \tag{2.10}$$

Where $A_i, B_j, (0\le i\le N, 0\le j\le M),\ E,\lambda,\mu$ are coefficients to be found later and M, N are positive integers that can be obtained by using the homogeneous balance principle.

Step 3: Inserting Eq. (2.4) and its derivatives along with the Eq. (2.5) and simplifying, we obtain an equation involving polynomial of $e^{-\psi(\xi)}$. We extract system of equations from that polynomial of $e^{-\psi(\xi)}$ by summing all the terms of the same power and equating each summation to zero. To determine the new solutions of (2.1), we solve the system of equations by using the Wolfram Mathematica 9 to obtain the values of the various coefficients $A_i, B_j, (0\le i\le N, 0\le j\le M),\ E,\lambda,\mu$.

Substituting the obtained values of the coefficients along with one of Eqs. (2.6-2.10) into Eq. (2.4), yields new solution to (2.1).

3. Application

Let us consider the ill-posed Boussinesq equation [13] given by:

$$v_{tt}-v_{xx}-\left(v^2\right)_{xx}-v_{xxxx}=0, \tag{3.1}$$

$$v(x,t)=V(\xi),\quad \xi=x-ct, \tag{3.2}$$

where c is the wave velocity. Using Eq. (3.2) on Eq. (3.1), we get the following NODE;

$$\left(c^2-1\right)V-V^2-V''=0. \tag{3.3}$$

Balancing the highest power nonlinear term V^2 and the highest power derivative V'' by using the balancing principle, yields the following relationship between N and M;

$$N=M+2\qquad N,M\in\mathbb{Z}^+. \tag{3.4}$$

Choosing $M=1$, yields $N=3$. Considering $M=1$, $N=3$ along with Eq. (2.4), yields;

$$V(\xi)=\frac{A_0+A_1e^{-\psi(\xi)}+A_2e^{-2\psi(\xi)}+A_3e^{-3\psi(\xi)}}{B_0+B_1e^{-\psi(\xi)}}. \tag{3.5}$$

We insert Eq. (3.5) and its second derivative into Eq. (3.3), doing this we get a polynomial of $e^{-\psi(\xi)}$. We extract a system of equations from the obtained polynomial by summing all the term that have the same power and equating each summation to zero. We solve the extracted system of equations withthe help of Wolfram Mathematica 9 and obtain the values of the coefficients that are involved in the system of equations. We classify the results of the coefficients into different cases. To obtain some new solutions for Eq. (3.1), we consider each case thereby putting the results of the coefficients along with one of Eqs. (2.6-2.10) (depending on the condition) into Eq. (3.5).

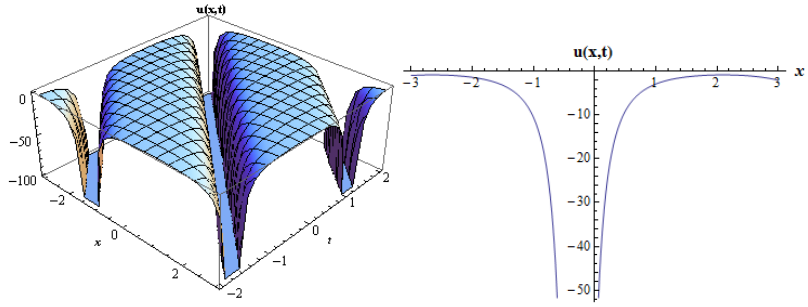
Case 1.1.

$$A_0=-B_0\left(\lambda^2+2\mu\right),\ A_1=-6B_0\lambda-B_1\left(\lambda^2+2\mu\right),$$

$A_2=-6\left(B_0+B_1\lambda\right), A_3=-6B_1, c=-\sqrt{1-\lambda+4\mu},$ substituting these coefficients along with the suitable family solution of Eq. (2.5) into Eq. (3.5), produces the following solutions:

Solution 1.1. When $\mu\neq 0,\ \lambda^2-4\mu>0$, we get

$$v_{1.1}(x,t)=-\frac{1}{\left(\lambda+\sqrt{\lambda^2-4\mu}\tanh\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(E+x+\sqrt{1-\lambda^2+4\mu}\,t\right)\right]\right)^2}$$
$$\left(\left(\lambda^2-4\mu\right)\left(\lambda^2-6\mu+2\lambda\sqrt{\lambda^2-4\mu}\tanh\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(E+x+\sqrt{1-\lambda^2+4\mu}\,t\right)\right]\right.\right. \tag{3.6}$$
$$\left.\left.+\left(\lambda^2+2\mu\right)\tanh^2\left[\frac{1}{2}\sqrt{\lambda^2-4\mu}\left(E+x+\sqrt{1-\lambda^2+4\mu}\,t\right)\right]\right)\right).$$

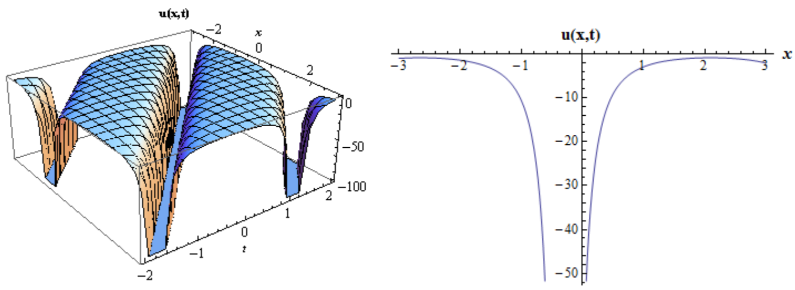


(a) Sub-figure 1. **(b)** Sub-figure 1.

Figure 1. The singular periodic wave solution shape of Eq. (3.6)by substituting the values $\mu=2,\ \lambda=2.5,\ E=1.5,\ -3<x<3,\ -2<t<2$ (a) and $t=0.25$ for the 2D graphic (b).

Solution 1.2. When $\mu\neq 0,\ \lambda^2-4\mu<0$, we get

$$v_{1.2}(x,t)=\frac{1}{\left(\lambda-\sqrt{-\lambda^2+4\mu}\tan\left[\frac{1}{2}\sqrt{-\lambda^2+4\mu}\left(E+x+\sqrt{1-\lambda^2+4\mu}\,t\right)\right]\right)^2}$$
$$\left(\left(\lambda^2-4\mu\right)\left(-\lambda^2+6\mu+2\lambda\sqrt{-\lambda^2+4\mu}\tan\left[\frac{1}{2}\sqrt{-\lambda^2+4\mu}\left(E+x+\sqrt{1-\lambda^2+4\mu}\,t\right)\right]\right.\right. \tag{3.7}$$
$$\left.\left.+\left(\lambda^2+2\mu\right)\tan^2\left[\frac{1}{2}\sqrt{-\lambda^2+4\mu}\left(E+x+\sqrt{1-\lambda^2+4\mu}\,t\right)\right]\right)\right).$$

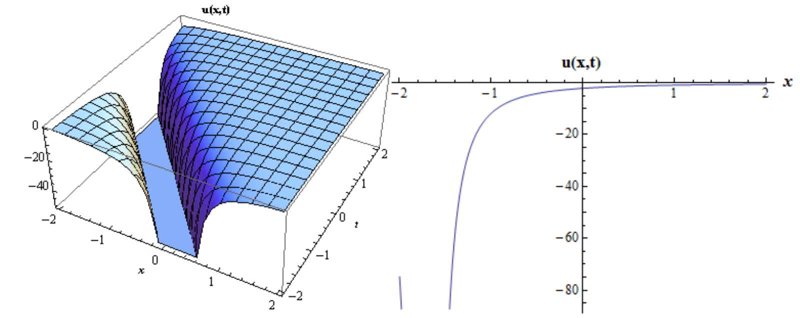


(a) Sub-figure 2. **(b)** Sub-figure 2.

Figure 2. The singular periodic wave solution shape of Eq. (3.7)by substituting the values $\mu=2,\ \lambda=2.5,\ E=1.5,\ -3<x<3,\ -2<t<2$ (a) and $t=0.25$ for the 2D graphic (b).

Solution 1.3. When $\mu=0,\ \lambda\neq 0$ and $\lambda^2-4\mu>0$,
we get

$$v_{1.3}\left(x,t\right)=\lambda^2\left(-1-\frac{3}{-1+\cosh\left[\lambda\left(E+x+\sqrt{1-\lambda^2}\,t\right)\right]}\right). \tag{3.8}$$

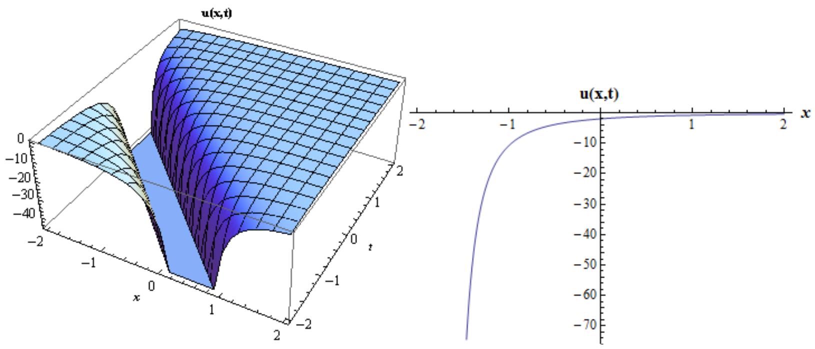


(a) Sub-figure 3. **(b)** Sub-figure 3.

Figure 3. The singular soliton solution shape of Eq. (3.8) by substituting the values $\lambda=0.5,\ E=1.5,\ -2<x<2,\ -2<t<2$ (a) and $t=0.25$ for the 2D graphic (b).

Solution 1.4. When $\mu=0,\ \lambda=0$ and $\lambda^2-4\mu=0$
we get

$$v_{1.4}\left(x,t\right)=-\frac{6}{\left(E+x+t\right)^2}. \tag{3.9}$$



(a) Sub-figure 4. **(b)** Sub-figure 4.

Figure 4. The singular soliton solution shape of Eq. (3.9)by substituting the values $E=1.5,\ -2<x<2,\ -2<t<2$ (a) and $t=0.25$ for the 2D graphic (b).

Case 2.1.

$$A_0=-\frac{1}{2}B_0\left(c^2+3\lambda^2-1\right),$$

$$A_1=\frac{1}{2}\left(-12B_0\lambda-B_1\left(c^2+3\lambda^2-1\right)\right),$$

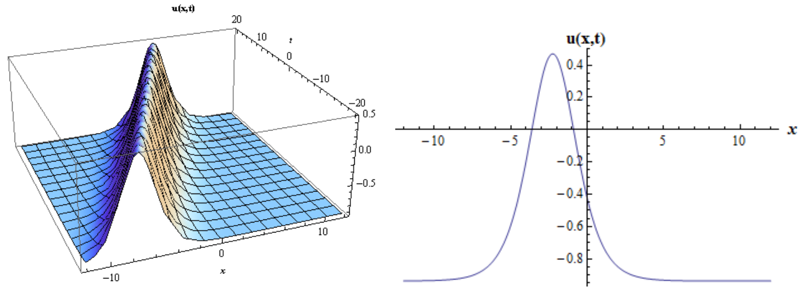
$$A_2=-6\left(B_0+B_1\lambda\right),\ A_3=-6B_1,$$

$$\mu=\frac{1}{4}\left(c^2+\lambda^2-1\right),$$

substituting these coefficients along with the suitable family solution of Eq. (2.5) into Eq. (3.5), produces the following solutions:

Solution 2.1. When $\mu \neq 0,\ \lambda^2 - 4\mu > 0$, we get

$$v_{2.1}(x,t) = \frac{1}{2}\left(1 - c^2 - \frac{3\left(1 - c^2 + \lambda\sqrt{1-c^2}\tanh\left[\frac{1}{2}\sqrt{1-c^2}\,(E + x - ct)\right]\right)^2}{\left(\lambda + \sqrt{1-c^2}\tanh\left[\frac{1}{2}\sqrt{1-c^2}\,(E + x - ct)\right]\right)^2}\right). \tag{3.10}$$

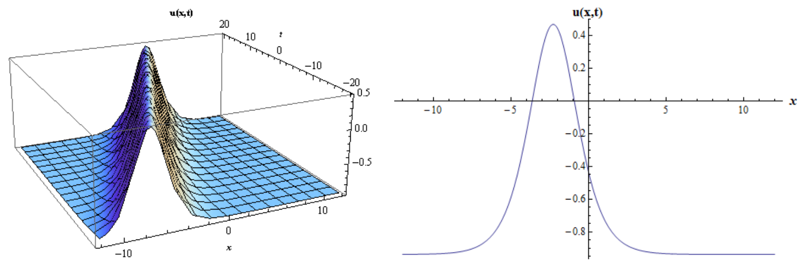


(a) Sub-figure 5. **(b)** Sub-figure 5.

Figure 5. The soliton solution shape of Eq. (3.10) by substituting the values $c = 0.25,\ \lambda = 2.5,\ E = 1.5,\ -12 < x < 12,\ -20 < t < 20$ (a) and $t = 0.35$ for the 2D graphic (b).

Solution 2.2. When $\mu \neq 0,\ \lambda^2 - 4\mu < 0$, we get

$$v_{2.2}(x,t) = \frac{1}{2}\left(1 - c^2 - \frac{3\left(c^2 - 1 + \lambda\sqrt{c^2 - 1}\tan\left[\frac{1}{2}\sqrt{c^2 - 1}\,(E + x - ct)\right]\right)^2}{\left(\lambda + \sqrt{c^2 - 1}\tan\left[\frac{1}{2}\sqrt{c^2 - 1}\,(E + x - ct)\right]\right)^2}\right). \tag{3.11}$$

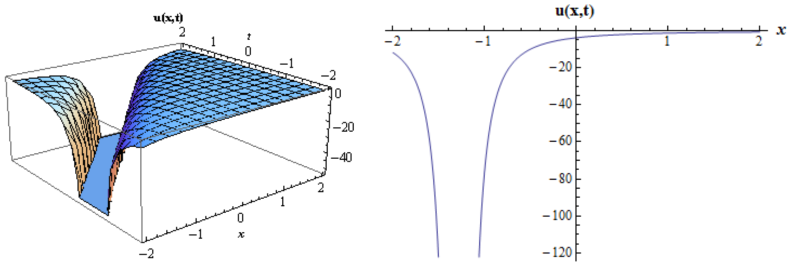


(a) Sub-figure 6. **(b)** Sub-figure 6.

Figure 6. The soliton solution shape of Eq. (3.11) by substituting the values $c = 0.25,\ \lambda = 2.5,\ E = 1.5,\ -12 < x < 12,\ -20 < t < 20$ (a) and $t = 0.25$ for the 2D graphic (b).

Solution 2.3. When $\mu = 0,\ \lambda \neq 0$ and $\lambda^2 - 4\mu > 0$, we get

$$v_{2.3}(x,t) = \lambda^2\left(-1 - \frac{3}{-1 + \cosh\left[\lambda\left(E + x - \sqrt{1-\lambda^2}\,t\right)\right]}\right) \tag{3.12}$$



(a) Sub-figure 7. **(b)** Sub-figure 7.

Figure 7. The singular soliton solution shape of Eq. (3.12) by substituting the values $\lambda=0.5,\ E=1.5,\ 2<x<2,\ -2<t<2$ (a) and $t=0.25$ for the 2D graphic (b).

Case 3.1.

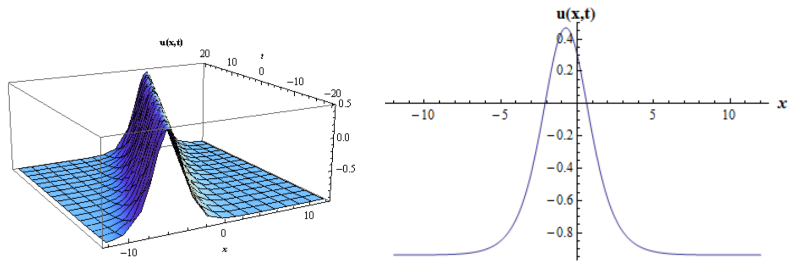
$$A_0=B_0\left(c^2-6\mu-1\right),$$
$$A_1=B_1\left(c^2-6\mu-1\right)+6B_0\sqrt{1-c^2+4\mu},$$
$$A_2=6B_1\sqrt{1-c^2+4\mu}-6B_0,$$
$$A_3=-6B_1,\ \lambda=-\sqrt{1-c^2+4\mu},$$

substituting these coefficients along with the suitable family solution of Eq. (2.5) into Eq. (3.5), produces the following solutions:

Solution 3.1. When $\mu\neq 0,\ \lambda^2-4\mu>0$, we get

$$v_{3.1}(x,t)=c^2-6\mu-1+\frac{12\mu\left(1-c^2+2\mu-\sqrt{1-c^2}\,\sqrt{1-c^2+4\mu}\tanh\left[f_1(x,t)\right]\right)}{\left(\sqrt{1-c^2+4\mu}-\sqrt{1-c^2}\tanh\left[f_1(x,t)\right]\right)^2} \tag{3.13}$$

where $f_1(x,t)=\frac{1}{2}\sqrt{1-c^2}\left(E+x-ct\right)$.



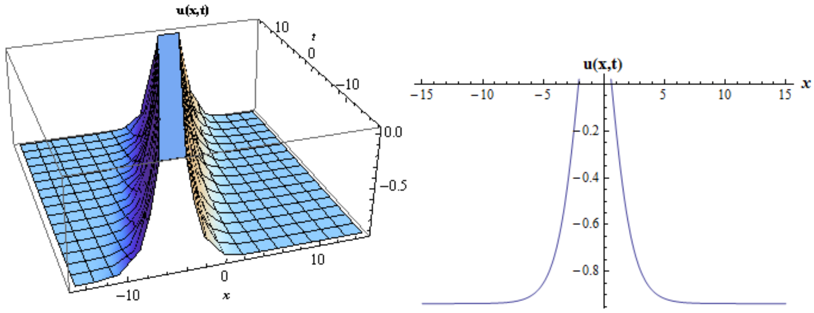
(a) Sub-figure 8. **(b)** Sub-figure 8.

Figure 8. The soliton solution shape of Eq. (3.13) by substituting the values $c=0.25,\ \mu=2,\ E=1.5,\ -12<x<12,\ -20<t<20$ (a) and $t=0.25$ for the 2D graphic (b).

Solution 3.2. When $\mu\neq 0,\ \lambda^2-4\mu>0$, we get

$$v_{3.2}(x,t)=c^2-6\mu-1+\frac{12\mu\left(1-c^2+2\mu-\sqrt{c^2-1}\,\sqrt{1-c^2+4\mu}\tan\left[f_2(x,t)\right]\right)}{\left(\sqrt{1-c^2+4\mu}-\sqrt{c^2-1}\tan\left[f_2(x,t)\right]\right)^2} \tag{3.14}$$

where $f_2(x,t)=\frac{1}{2}\sqrt{c^2-1}\left(E+x-ct\right)$.



(a) Sub-figure 9. **(b)** Sub-figure 9.

Figure 9. The soliton solution shape of Eq. (3.14) by substituting the values $c=0.25,\ \mu=2,\ E=1.5,\ -15<x<15,\ -18<t<18$ (a) and $t=0.25$ for the 2D graphic (b).

4. Conclusion

In this study, with the aid of the Wolfram Mathematica 9, the modified exp $\left(-\psi\left(\xi\right)\right)$-expansion function method (MEFM) [14, 15] is used in investigating new solutions to the well-known ill-posed Boussinesq equation [13] which arises in shallow water waves and nonlinear lattices. It is very important to look for some new solutions to this equation as it plays a vital roles in the field of applied mathematics. We precede in getting some new solutions with new structures such as trigonometric, hyperbolic and rational function structures. When we checked our solutions, we have seen that they all satisfied the ill-posed Boussinesq equation. We observed that our results are new when compared with the results obtained by some existing techniques in the literature. We plot the 2- and 3-dimensional of each solution obtained in the paper and we also give the physical interpretations of each figure. From the results we obtained, we observed that the modified exp $\left(-\psi\left(\xi\right)\right)$-expansion function method (MEFM) is a powerful and efficient mathematical tool that can be applied to the various nonlinear evolution equations that arise in the different field of nonlinear sciences.

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