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RESEARCH ARTICLE

Analytical solutions of Phi-four equation

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ABSTRACT

This study bases attention on new analytical solutions of Phi-four equation. The modified $\exp(-\Omega(\xi))$-expansion function method (MEFM) has been used to obtain analytical solutions of the Phi-four equation. By using this method, dark soliton solutions and trigonometric function solution of the Phi-four equation have been found.

1. Introduction

Nonlinear evolution equations (NLEEs) are considerably used to identify a variety of physical circumstances in the areas such as quantum field theory, hydrodynamics, chemical kinematics, geochemistry, electricity, elastic media and plasma physics.
Recently, many researchers have introduced a lot of methods to acquire exact solutions of NLEEs such as G'/G-expansion method [1], modified extended tanh-function method [2], sine-cosine method [3], exp-function method [4], modified simple equation method [4], extended trial equation method [5], generalized Kudryashov method [6] . In this study, MEFM [7] will be implemented to find new analytical solutions of Phi-four equation.
We consider Phi-four equation [8-11],

$$u_{tt} - au_{xx} - u + u^3 = 0, a > 0, \quad (1)$$

where a is real constant. This equation can be investigated as a special form of the Klein-Gordon equation that patterns the phenomenon in particle physics where kink and anti-kink solitary waves interact [12].
Many scientists have used exact and numerical solutions of Phi-four equation to research some methods such as the sine-cosine method [8], the auxiliary equation method [9], the modified simple equation method [10], homotopy perturbation method [11], homotopy analysis method [11] and Adomian decomposition method [11].
In this article, the basic interest is to construct new exact solutions of Phi-four equation via MEFM. In Sec. 2, we clarify basic facts of MEFM. In Sec. 3, we find new exact solutions of the Phi-four equation via MEFM.

2. Basic facts of method

The fundamental properties of MEFM are introduced in this section. MEFM is predicated on the exp $(-\Omega(\xi))$-expansion function method [13-16]. In order to implement this method to the nonlinear partial differential equations, we handle it as follows:

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \cdots) = 0, \quad (2)$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its derivatives, in which the highest order derivatives and nonlinear terms are included and the subscripts demonstrate the partial derivatives. The fundamental stages of the method are defined as follows:
Step 1: Let us investigate the following traveling transformation identified by

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$$u(x,t)=U(\xi),\quad \xi=k(x-ct). \tag{3}$$

Using Eq. (3), we can turn Eq. (2) into a nonlinear ordinary differential equation (NODE) described by:

$$NODE(U,U',U'',U''',\cdots)=0, \tag{4}$$

where $NODE$ is a polynomial of U and its derivatives and the superscripts demonstrate the ordinary derivatives according to ξ.

Step 2: Assume the traveling wave solution of Eq. (4) can be shown as follows:

$$U(\xi)=\frac{\sum_{i=0}^{N} A_i\left[\exp(-\Omega(\xi))\right]^i}{\sum_{j=0}^{M} B_j\left[\exp(-\Omega(\xi))\right]^j} =\frac{A_0+A_1\exp(-\Omega)+\cdots+A_N\exp(\mathrm{N}(-\Omega))}{B_0+B_1\exp(-\Omega)+\cdots+B_M\exp(\mathrm{M}(-\Omega))}, \tag{5}$$

where $A_i, B_j, (0\le i\le N, 0\le j\le M)$ are constants to be described later, such that $A_N\neq 0, B_M\neq 0,$ and $\Omega=\Omega(\xi)$ is solution of the following ordinary differential equation:

$$\Omega'(\xi)=\exp(-\Omega(\xi))+\mu\exp(\Omega(\xi))+\lambda. \tag{6}$$

There are the following solution families of Eq. (6):

Family1: When $\mu\neq 0,\ \lambda^2-4\mu>0,$

$$\Omega(\xi)=\ln\left(\frac{-\sqrt{\lambda^2-4\mu}}{2\mu}\tanh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}(\xi+E)\right)-\frac{\lambda}{2\mu}\right). \tag{7}$$

Family2: When $\mu\neq 0,\ \lambda^2-4\mu<0,$

$$\Omega(\xi)=\ln\left(\frac{\sqrt{-\lambda^2+4\mu}}{2\mu}\tan\left(\frac{\sqrt{-\lambda^2+4\mu}}{2}(\xi+E)\right)-\frac{\lambda}{2\mu}\right). \tag{8}$$

Family3: When $\mu=0,\ \lambda\neq 0,$ and $\lambda^2-4\mu>0,$

$$\Omega(\xi)=-\ln\left(\frac{\lambda}{\exp(\lambda(\xi+E))-1}\right). \tag{9}$$

Family4: When $\mu\neq 0,\ \lambda\neq 0,$ and $\lambda^2-4\mu=0,$

$$\Omega(\xi)=\ln\left(-\frac{2\lambda(\xi+E)+4}{\lambda^2(\xi+E)}\right). \tag{10}$$

Family5: When $\mu=0,\ \lambda=0,$ and $\lambda^2-4\mu=0,$

$$\Omega(\xi)=\ln(\xi+E). \tag{11}$$

such that $A_0,A_1,A_2,\cdots A_N,B_0,B_1,B_2,\cdots B_M,E,\lambda,\mu$ are constants to be described later. The positive integers N and M can be identified by taking into consideration the homogeneous balance between the highest order derivatives and the nonlinear terms arising in Eq. (5).

Step 3: Embedding Eqs. (6) and (7–11) into Eq. (5), we attain a polynomial of $\exp(-\Omega(\xi))$. We compensate all the coefficients of same power of $\exp(-\Omega(\xi))$ to zero. This process provides a system of equations which can be solved to obtain $A_0,A_1,A_2,\cdots A_N,B_0,B_1,B_2,\cdots B_M,E,\lambda,\mu$ by using Wolfram Mathematica 9. Putting the values of $A_0,A_1,A_2,\cdots A_N,B_0,B_1,B_2,\cdots B_M,E,\lambda,\mu$ into Eq. (5), the general solutions of Eq. (5) fulfil the determination of the solution of Eq. (1).

3. MEFM for Phi-Four Equation

In this section, we look for the exact solutions of Eq. (1) by using MEFM.

We find the travelling wave solutions of Eq. (1) by using the wave variables

$$u(x,t)=u(\xi),\ \xi=k(x-ct), \tag{12}$$

where k and c are arbitrary constants.

Putting Eq. (13) into Eq. (1),

$$u_{tt}=k^2c^2u'',\ u_{xx}=k^2u'', \tag{13}$$

we obtain following equation

$$k^2(c^2-a)u''-u+u^3=0, \tag{14}$$

where the prime indicates the derivative with regard to ξ.

Using balance principle in Eq. (14), we obtain

$$N=M+1. \tag{15}$$

If we take $M=1$ so $N=2$, we can acquire

$$U=\frac{A_0+A_1\exp(\text{-}\Omega)+A_2\exp(2(\text{-}\Omega))}{B_0+B_1\exp(\text{-}\Omega)}=\frac{\Upsilon}{\Psi}, \tag{16}$$

and

$$U'=\frac{\Upsilon'\Psi-\Psi'\Upsilon}{\Psi^2}, \tag{17}$$

$$U''=\frac{\Upsilon''\Psi^3-\Psi^2\Upsilon'\Psi'-(\Psi''\Upsilon+\Psi'\Upsilon')\Psi^2+2(\Psi')^2\Upsilon\Psi}{\Psi^4}, \\ \vdots \tag{18}$$

where $A_2\neq 0$ and $B_1\neq 0$. When we use Eq.(16) and Eq.(18) in Eq.(14) we get a system of algebraic equations from the coefficients of polynomial of $\exp(-\Omega(\xi))$. By solving this system of algebraic equations by using Wolfram Mathematica 9, it yields us the following coefficients:

Case 1:

$$A_0 = \frac{\lambda B_0}{\sqrt{\lambda^2 - 4\mu}}, A_1 = \frac{2B_0 + \lambda B_1}{\sqrt{\lambda^2 - 4\mu}},$$
$$A_2 = \frac{2B_1}{\sqrt{\lambda^2 - 4\mu}}, c = -\frac{\sqrt{-2 + ak^2\left(\lambda^2 - 4\mu\right)}}{\sqrt{k^2\left(\lambda^2 - 4\mu\right)}}, \qquad (19)$$
$$B_0 = B_0, B_1 = B_1, \lambda = \lambda, \mu = \mu.$$

Case 2:

$$A_0 = \frac{A_1 B_0}{2B_1}, A_1 = A_1, A_2 = \frac{A_1 B_1}{2B_0},$$
$$\lambda = \frac{2B_0}{B_1}, \mu = B_0^2\left(-\frac{4}{A_1^2} + \frac{1}{B_1^2}\right), \qquad (20)$$
$$c = -\frac{\sqrt{-\frac{A_1^2}{2} + 4ak^2 B_0^2}}{2kB_0}, B_0 = B_0, B_1 = B_1.$$

Case 3:

$$A_0 = \frac{B_0\left(A_1^2 + B_1^2\right)}{2A_1 B_1}, A_2 = \frac{A_1 B_1}{2B_0},$$
$$\lambda = \frac{2B_0}{B_1}, \mu = B_0^2\left(-\frac{1}{A_1^2} + \frac{1}{B_1^2}\right), \qquad (21)$$
$$c = -\frac{\sqrt{-\frac{A_1^2}{2} + 4ak^2 B_0^2}}{2kB_0}, A_1 = A_1,$$
$$B_0 = B_0, B_1 = B_1.$$

Case 4:

$$A_0 = \frac{A_1 B_0}{2B_1} + \frac{B_0 B_1}{A_1}, A_2 = \frac{A_1 B_1}{2B_0},$$
$$\lambda = \frac{2B_0}{B_1}, \mu = B_0^2\left(\frac{2}{A_1^2} + \frac{1}{B_1^2}\right), \qquad (22)$$
$$c = -\frac{\sqrt{-\frac{A_1^2}{2} + 4ak^2 B_0^2}}{2kB_0}, A_1 = A_1,$$
$$B_0 = B_0, B_1 = B_1.$$

Embedding Eq. (19) together with Eqs. (3) and (7) in Eq. (16), we obtain dark soliton solution for Eq. (1) as follows:

$$u_1(x,t) = \frac{\lambda^2 - 4\mu + \lambda\sqrt{\lambda^2 - 4\mu}\tanh\left[f(x,t)\right]}{\sqrt{\lambda^2 - 4\mu}\left(\lambda + \sqrt{\lambda^2 - 4\mu}\tanh\left[f(x,t)\right]\right)}, \qquad (23)$$

where

$$f(x,t) = \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\left[E + k\left(x + \frac{t\sqrt{-2 + ak^2\left(\lambda^2 - 4\mu\right)}}{\sqrt{k^2\left(\lambda^2 - 4\mu\right)}}\right)\right],$$

and $\mu \neq 0, \lambda^2 - 4\mu > 0.$

Substituting Eq. (20) together with Eqs. (3) and (7) in Eq. (16), we find dark soliton solution for Eq. (1) as follows:

$$u_2(x,t) = \frac{A_1 \tanh\left[g(x,t)\right] + 2B_1}{A_1 + 2B_1 \tanh\left[g(x,t)\right]}, \qquad (24)$$

where

$$g(x,t) = \frac{4(E + kx)B_0 + \sqrt{2}t\sqrt{-A_1^2 + 8ak^2 B_0^2}}{2A_1}, \text{ and}$$
$$-4B_0^2\left(-\frac{4}{A_1^2} + \frac{1}{B_1^2}\right) + \frac{4B_0^2}{B_1^2} > 0.$$

Putting Eq. (21) together with Eqs. (3) and (7) in Eq. (16), we find dark soliton solution for Eq. (1) as follows:

$$u_3(x,t) = \frac{1}{2}\left[\frac{A_1}{B_1} + \frac{B_1}{A_1} + \frac{A_1^2 - B_1^2}{A_1\left(B_1 + A_1 \tanh\left[h(x,t)\right]\right)}\right] + \frac{1}{2}\left[\frac{-A_1^2 + B_1^2}{B_1\left(A_1 + B_1 \tanh\left[h(x,t)\right]\right)}\right], \qquad (25)$$

where

$$h(x,t) = \frac{B_0}{A_1}\left(E + kx + t\frac{\sqrt{-\frac{A_1^2}{2} + 4ak^2 B_0^2}}{2B_0}\right),$$

and $-4B_0^2\left(-\frac{1}{A_1^2} + \frac{1}{B_1^2}\right) + \frac{4B_0^2}{B_1^2} > 0.$

Embedding Eq. (22) together with Eqs. (3) and (7) in Eq. (16), we obtain trigonometric function solution for Eq. (1) as follows:

$$u_4(x,t) = \frac{\left(A_1^2 + 2B_1^2\right)\sec^2\left[k(x,t)\right]}{\left(2B_1 + \sqrt{2}A_1 \tan\left[k(x,t)\right]\right).\left(-A_1 + \sqrt{2}\,B_1 \tan\left[k(x,t)\right]\right)}, \qquad (26)$$

where

$$k(x,t) = \frac{\sqrt{2}B_0}{A_1}\left(E + kx + t\frac{\sqrt{-\frac{A_1^2}{2} + 4ak^2 B_0^2}}{2B_0}\right),$$

and $-4B_0^2\left(\frac{2}{A_1^2} + \frac{1}{B_1^2}\right) + \frac{4B_0^2}{B_1^2} > 0.$

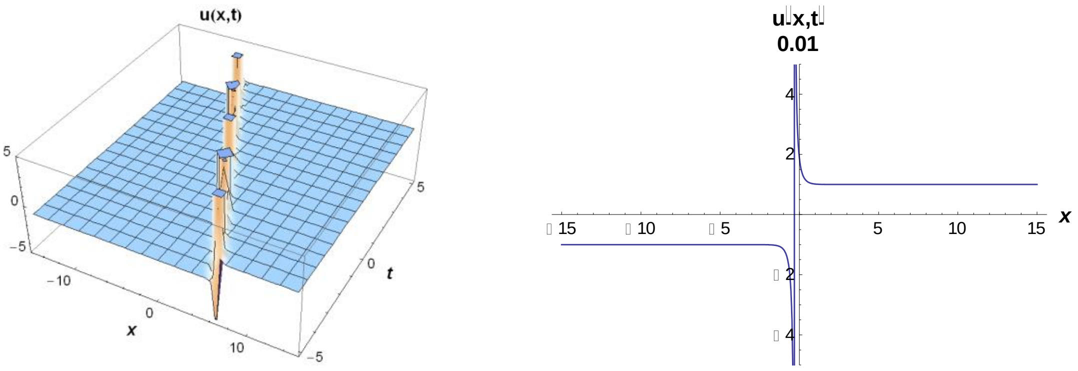

Figure 1. The 3D and 2D surfaces of Eq. (23) for $\lambda = 0.3, \mu = -0.2, k = 4, a = 2, E = 0.5, -15 < x < 15, -5 < t < 5$ and $t = 0.01$ for 2D surface.

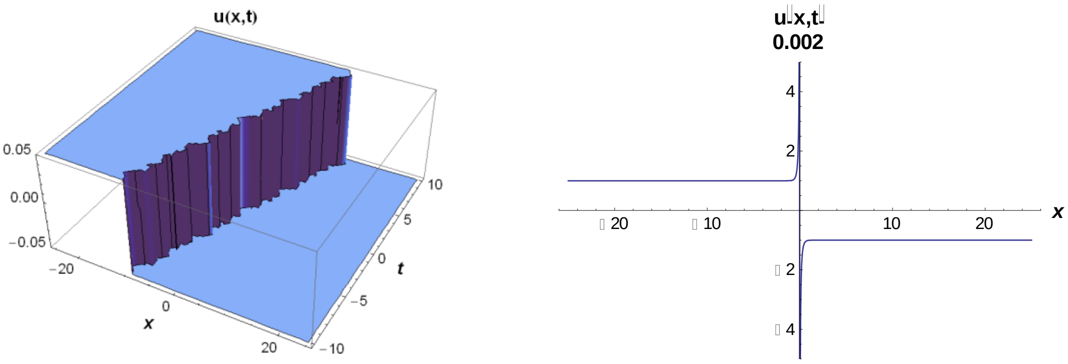

Figure 2. The 3D and 2D surfaces of Eq. (24) for $k = 3, a = 1, A_1 = 2, B_0 = -1, B_1 = 4, E = 0.3, -25 < x < 25, -10 < t < 10$ and $t = 0.002$ for 2D surface.

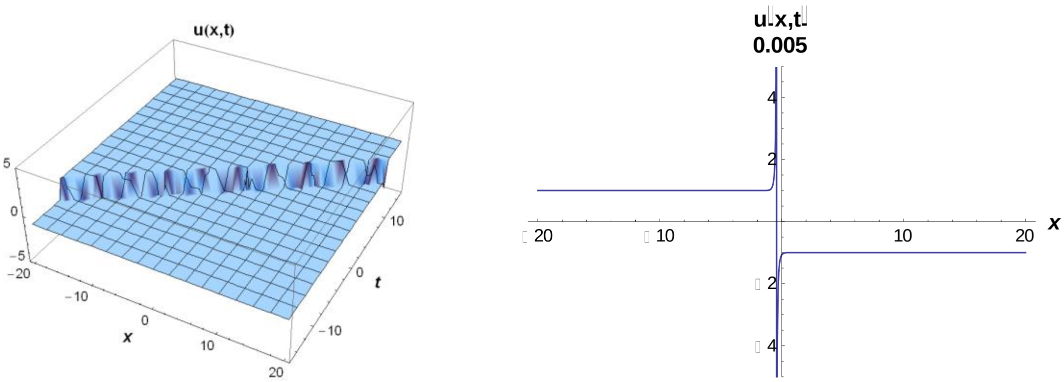

Figure 3. The 3D and 2D surfaces of Eq. (25) for $k = 1, a = 4, A_1 = 1, B_0 = -2, B_1 = 3, E = 0.6, -20 < x < 20, -15 < t < 15$ and $t = 0.005$ for 2D surface.

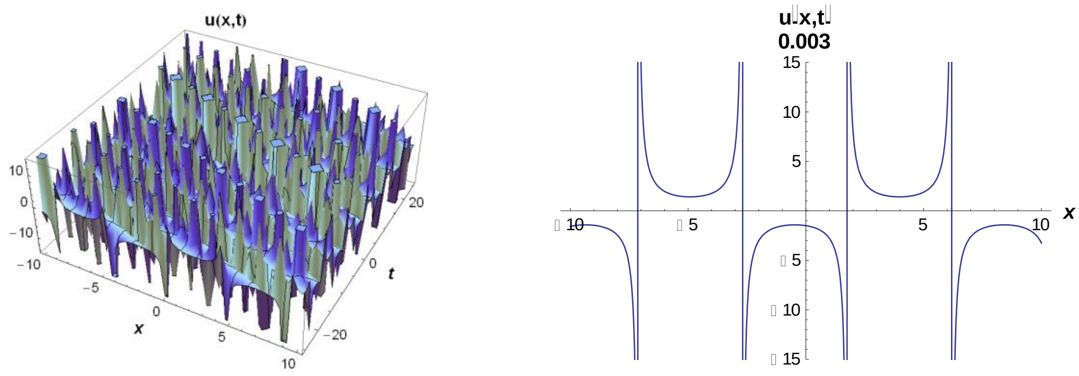

Figure 4. The 3D and 2D surfaces of Eq. (26) for $k=-1, a=3, A_1=4, B_0=-1, B_1=5, E=0.3, -10<x<10, -25<t<25$ and $t=0.003$ for 2D surface.

Remark The exact solutions of Eq. (1) were obtained via MEFM and were controlled by use of Mathematica Release 9. As far as we know, the solutions of Eq. (1) that we found in this study are new and are not indicated before.

4. Conclusion

In this paper, we use MEFM to find exact solutions of Phi-four equation. Then, in Figures 1-4, we plot 2D and 3D surfaces of dark soliton solutions and trigonometric function solution of Phi-four equation by using Mathematica Release 9.

According to these data and observation, it has been deduced that this method has been influential for the exact solutions of these NLEEs and this method is highly effective and dependable in the sense that reaching analytical solutions. Thus, we can say that this method has a substantial position to attain exact solutions of NLEEs.

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