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RESEARCH ARTICLE

New function method to the (n+1)-dimensional nonlinear problems

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ABSTRACT

In this study, a new approach that assumes $u' = f\left(\cos\left(u\right)\right)$ and $u' = f\left(\sinh\left(u\right)\right)$ is applied to construct the traveling wave solutions of the (N + 1)-dimensional double sine-Gordon and (N + 1)-dimensional double sinh-cosh-Gordon equations. Some new elliptic integral function solutions are respectively obtained by this method, and then these solutions are converted into the Jacobi elliptic function solutions. According these results, one can easily see that this method is very effective mathematical tool for the (N+1)-dimensional nonlinear physical problems.

1. Introduction

The nonlinear evolution equations have become of ever greater interest in the modelling of real life problems. Thus, many methods are constructed and applied for these problems. Some of them can be respectively given as the trial equation method [1], the extended trial equation method [2,3], the Weierstrass transform method [4], the tanh function method [5], the Kudryashovs method [6], and so on. In this paper, the investigation of various traveling wave solutions to (N+1)-dimensional double sine-Gordon and (N+1)-dimensional double sinh-cosh-Gordon equations have been widely studied by many authors [7-9]

$$\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} - \alpha\sin\left(u\right) - \beta\sin\left(2u\right) = 0, \quad (1)$$

$$\sum_{j=1}^{N} u_{x_j x_j} - u_{tt} - \alpha\cosh\left(u\right) - \beta\sinh\left(2u\right) = 0. \quad (2)$$

The sine-cosine-Gordon and the sinh-cosh-Gordon equations have importance in the fields of integrable quantum field theory, kink dynamics, and fluid dynamics. On the other hand, a variety of effective methods have been defined to construct the traveling wave solutions of nonlinear partial differential equations. It is given the new function method as one of most important methods and its applications [10-14]. In this paper, we apply the new function method, based on sine, sinh functions, to (N+1)-dimensional double sine-Gordon and (N+1)-dimensional double sinh-cosh-Gordon equations. Thus, some new Jacobi elliptic function solutions are obtained by the using of this method. The obtained results reveal that the new function method is powerful mathematical tool for solving the (N+1)-dimensional sine-Gordon and sinh-cosh-Gordon equations.

2. New function method

The new function methods have been proposed by using the exponential function, trigonometric function [10,11]. In this paper, we apply the new function method by depending on the hyperbolic and trigonometric functions. Firstly, we take the general form of the generalized (N+1)-dimensional sine-cosine-Gordon or sinh-cosh-Gordon equations,

$$P\begin{pmatrix} u, \sin\left(u\right), \cosh\left(u\right), \sin\left(2u\right), \\ , \sinh\left(2u\right) u_{tt}, u_{x_1x_1}, u_{x_2x_2}, \ldots u_{x_Nx_N} \end{pmatrix} = 0. \quad (3)$$

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Then use the wave transformation $u=u(x,t)=u(\mu)=u\left(k\left(\sum_{j=1}^{N}x_j-ct\right)\right)$,

$\mu=k\left(\sum_{j=1}^{N}x_j-ct\right)$

where $c\neq 0$. Thus, we have a nonlinear ordinary differential equation

$$N(u,u',u'',\ldots)=0. \qquad (4)$$

The new function method assumes that the function u provides

$$F(u'')=G(g(u)), \qquad (5)$$

where F, G and g are any functions. Here, we use the equations

$$u'=f(g(u)),$$
$$u''=f(g(u))g'(u)f'(g(u)). \qquad (6)$$

Substituting Eq. (6) into Eq. (5), we have

$$F\left(f(g(u))g'(u)f'(g(u))\right)=G(g(u)). \qquad (7)$$

If we take $\psi=g(u)$, then we can write

$$F\left(\psi' f(\psi)f'(\psi)\right)=G(\psi). \qquad (8)$$

Solving Eq. (8), is sometimes a variable separated ordinary differential equation, yields the function f. By integration, we can obtain the solutions as follows:

$$\frac{du}{f(g(u))}=d\mu\Rightarrow\int\frac{du}{f(g(u))}=\int d\mu=\mu+P, \qquad (9)$$

where P is an integration constant. The explicit solutions can be derived by the inverse function. Otherwise, the implicit solutions can be retrieved if the above integration is much complex.

3. Applications

3.1.Solutions for (N+1)-dimensional double sine-gordon equation

By the travelling wave transformation to Eq. (1), we find

$$k^2\left(N-c^2\right)u''-\alpha\sin(u)-\beta\sin(2u)=0. \qquad (10)$$

We assume that the equation

$$u'=f(\cos(u)), \qquad (11)$$

defined by $u'(\mu)$ and $\cos(u)$ satisfies Eq. (10). From Eq. (11), we can write

$$u''=-\sin(u)f(\cos(u))f'(\cos(u)). \qquad (12)$$

By substituting Eq. (12) into Eq. (10), we derive

$$\begin{aligned}&k^2\left(c^2-N\right)f(\cos(u))f'(\cos(u))=\\&=\alpha+2\beta\cos(u).\end{aligned} \qquad (13)$$

Let u' is a function of $\cos(u)$ and $\cos(u)=\psi$, then f is a function of ψ. Therefore we can easily write

$$u=\arccos((\psi))$$
$$u'=-\frac{\psi'}{\sqrt{1-\psi^2}}=f(\cos(u))=f(\psi) \qquad (14)$$

Now, we can try to have the form of the function f:

$$k^2\left(c^2-N\right)f(\psi)f'(\psi)=\alpha+2\beta\psi. \qquad (15)$$

Eq. (15) is an ordinary differential equation of variable separated:

$$\frac{k^2\left(c^2-N\right)}{2}f^2(\psi)=\alpha\psi+\beta\psi^2+P, \qquad (16)$$

where P is a constant of integration. From Eq. (16), we can easily compute

$$f(\psi)=-\sqrt{\frac{2}{k^2(c^2-N)}\left(\alpha\psi+\beta\psi^2+P\right)}. \qquad (17)$$

Using the equation $-\frac{\psi'}{\sqrt{1-\psi^2}}=f(\psi)$, we have

$$\begin{aligned}&\psi'=\frac{d\psi}{d\mu}=\\&=-\sqrt{\frac{2}{k^2(c^2-N)}\left(1-\psi^2\right)\left(\alpha\psi+\beta\psi^2+P\right)}\end{aligned} \qquad (18)$$

By using of the symbolic computation software program Mathematica, Eq. (18) that is a variable separated ordinary differential equation is solved.

So, the following elliptic integral function F solution to Eq. (1) is obtained as

$$\begin{aligned}&\mu+Q=-\sqrt{\frac{2k^2\left(c^2-N\right)}{\beta(\psi_2-\psi_3)(\psi_1-\psi_4)}}\\&EllipticF\left[\begin{array}{l}\arcsin\left(\sqrt{\frac{(\psi-\psi_2)(\psi_1-\psi_4)}{(\psi-\psi_1)(\psi_2-\psi_4)}}\right),\\ \frac{(\psi_1-\psi_3)(\psi_2-\psi_4)}{(\psi_2-\psi_3)(\psi_1-\psi_4)}\end{array}\right],\end{aligned} \qquad (19)$$

where Q is a constant of integration. $\psi_i\ (i=1,\cdots,4)$ are roots of equation

$$\left(\alpha\psi+\beta\psi^2+P\right)\left(1-\psi^2\right)=0$$

$$\begin{aligned}&\psi_1=-1,\quad \psi_2=1,\\ &\psi_3=\frac{-\alpha-\sqrt{\alpha^2-4P\beta}}{2\beta}.\\ &\psi_4=\frac{-\alpha+\sqrt{\alpha^2-4P\beta}}{2\beta}\end{aligned} \tag{20}$$

Then, we find

$$\psi=\frac{\psi_2\left(\psi_1-\psi_4\right)-\psi_1\left(\psi_2-\psi_4\right)\operatorname{sn}^2\left[\varphi,\ell^2\right]}{\psi_1-\psi_4-\left(\psi_2-\psi_4\right)\operatorname{sn}^2\left[\varphi,\ell^2\right]}. \tag{21}$$

Replace ψ with $\cos u$, μ with $\mu=k\left(\sum_{j=1}^{N}x_j-ct\right)$ in (21), and then the explicit solutions for Eq. (1) can be obtained as follows:

$$u=\arccos\left[\frac{\psi_2\left(\psi_1-\psi_4\right)-\psi_1\left(\psi_2-\psi_4\right)\operatorname{sn}^2\left(\varphi,\ell^2\right)}{\psi_1-\psi_4-\left(\psi_2-\psi_4\right)\operatorname{sn}^2\left(\varphi,\ell^2\right)}\right], \tag{22}$$

where,

$$\varphi=-\sqrt{\frac{\beta\left(\psi_1-\psi_4\right)\left(\psi_2-\psi_3\right)}{2k^2\left(c^2-N\right)}}\left(\tau\left(\sum_{j=1}^{N}x_j-ct\right)+Q\right)$$

and $\ell^2=\dfrac{\left(\psi_1-\psi_3\right)\left(\psi_2-\psi_4\right)}{\left(\psi_2-\psi_3\right)\left(\psi_1-\psi_4\right)}$.

3.2. Solutions for (N+1)-dimensional sinh-cosh-gordon equation

By the travelling wave transformation to Eq. (2), we get

$$k^2\left(N-c^2\right)u''-\alpha\cosh\left(u\right)-\beta\sinh\left(2u\right)=0. \tag{23}$$

We assume that the equation

$$u'=f\left(\sinh\left(u\right)\right), \tag{24}$$

defined by $u'\left(\mu\right)$ and $\sinh\left(u\right)$ satisfies Eq. (23). From Eq. (24), we can compute

$$u''=\left(\cosh\left(u\right)\right)f\left(\sinh\left(u\right)\right)f'\left(\sinh\left(u\right)\right) \tag{25}$$

By substituting Eq. (25) into Eq. (23), we derive

$$\begin{aligned}&k^2\left(N-c^2\right)f\left(\sinh\left(u\right)\right)f'\left(\sinh\left(u\right)\right)=\\ &=\alpha+2\beta\sinh\left(u\right).\end{aligned} \tag{26}$$

Let u' is a function of $\sinh\left(u\right)$ and $\sinh\left(u\right)=\psi$, then f is a function of ψ. Therefore we can easily write

$$\begin{aligned}&u=\operatorname{arcsinh}\left(\left(\psi\right)\right)\\ &u'=\frac{\psi'}{\sqrt{1+\psi^2}}=f\left(\sinh\left(u\right)\right)=f\left(\psi\right)\end{aligned} \tag{27}$$

Now, we can try to have the form of the function f :

$$k^2\left(N-c^2\right)f\left(\psi\right)f'\left(\psi\right)=\alpha+2\beta\psi. \tag{28}$$

Eq. (27) is an ordinary differential equation of variable separated:

$$\frac{k^2\left(N-c^2\right)}{2}f^2\left(\psi\right)=\alpha\psi+\beta\psi^2+P, \tag{29}$$

where P is a constant of integration. From Eq. (28), we can easily compute

$$f(\psi)=\pm\sqrt{\frac{2}{k^2(N-c^2)}\left(\alpha\psi+\beta\psi^2+P\right)}. \tag{30}$$

Using the equation $\dfrac{\psi'}{\sqrt{1+\psi^2}}=f\left(\psi\right)$, we have

$$\begin{aligned}&\psi'=\frac{d\psi}{d\mu}=\\ &=\pm\sqrt{\frac{2}{k^2(N-c^2)}\left(1+\psi^2\right)\left(\alpha\psi+\beta\psi^2+P\right)}\end{aligned} \tag{31}$$

By using of the symbolic computation software program Mathematica, Eq. (30) that is a variable separated ordinary differential equation is solved. So, the following elliptic integral function F solution to Eq. (2) is obtained as

$$\begin{aligned}&\mu+Q=\\ &=\frac{EllipticF\left[\arcsin\left(\sqrt{\frac{\left(\psi-\psi_2\right)\left(\psi_1-\psi_4\right)}{\left(\psi-\psi_1\right)\left(\psi_2-\psi_4\right)}}\right),\ell^2\right]}{\varphi},\end{aligned} \tag{32}$$

where Q is a constant of integration. $\psi_i\ \left(i=1,\cdots,4\right)$ are roots of equation $\left(\alpha\psi+\beta\psi^2+P\right)\left(1+\psi^2\right)=0$

$$\begin{aligned}&\psi_1=-i,\quad \psi_2=i,\\ &\psi_3=\frac{-\alpha-\sqrt{\alpha^2-4P\beta}}{2\beta},\\ &\psi_4=\frac{-\alpha+\sqrt{\alpha^2-4P\beta}}{2\beta}\end{aligned} \tag{33}$$

Then, we find

$$\psi=\frac{\psi_2(\psi_1-\psi_4)-\psi_1(\psi_2-\psi_4)\operatorname{sn}^2\left[\varphi,\ell^2\right]}{\psi_1-\psi_4-(\psi_2-\psi_4)\operatorname{sn}^2\left[\varphi,\ell^2\right]} \quad (34)$$

Replace ψ with $\sinh u$, μ with $\mu=k\left(\sum_{j=1}^{N}x_j-ct\right)$ in (33), and then the explicit solutions for Eq. (2) can be obtained as follows:

$$u=\operatorname{arcsinh}\left[\frac{\psi_2(\psi_1-\psi_4)-\psi_1(\psi_2-\psi_4)\operatorname{sn}^2(\varphi,\ell^2)}{\psi_1-\psi_4-(\psi_2-\psi_4)\operatorname{sn}^2(\varphi,\ell^2)}\right], \quad (35)$$

where,

$$\varphi=-\sqrt{\frac{\beta(\psi_1-\psi_4)(\psi_2-\psi_3)}{2k^2(c^2-N)}}\left(\tau\left(\sum_{j=1}^{N}x_j-ct\right)+Q\right)$$

and $\ell^2=\dfrac{(\psi_1-\psi_3)(\psi_2-\psi_4)}{(\psi_2-\psi_3)(\psi_1-\psi_4)}$.

4. 2D and 3D graphics of solution

4.1. 2D graphic of solution

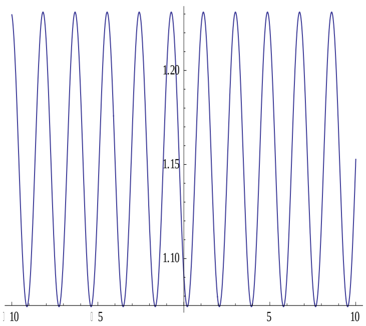

Figure 1. 2D graphic represents the solution (22) at $t=1$.

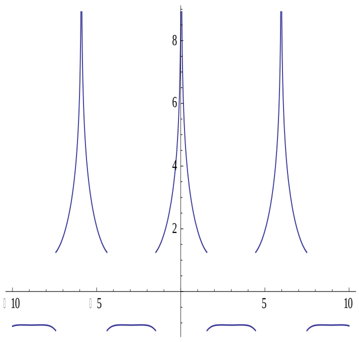

Figure 2. The solution (35) is shown real part at $t=1$.

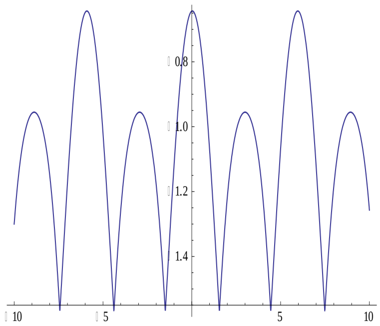

Figure 3. The solution (35) is shown imaginary part at $t=1$.

4.2. 3D graphic of solution

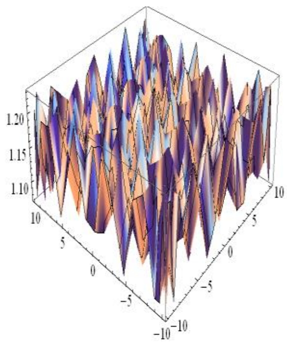

Figure 4. The solution (22) is shown at $\alpha=4$, $P=0$, $\beta=3$, $N=1$, $Q=0$, $c=3$, and $k=1$

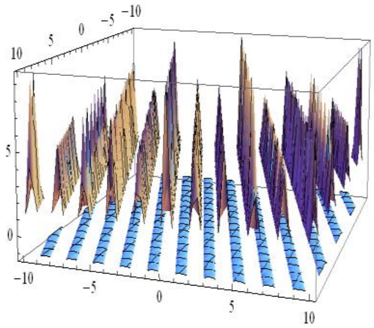

Figure 5. The solution (35) is shown real part at $\alpha=6$, $P=1$, $\beta=8$, $n=1$, $Q=0$, $c=1$, and $k=3$

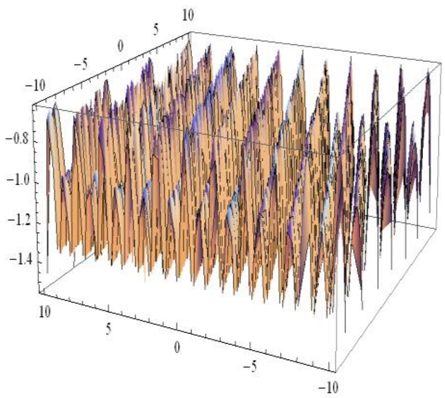

Figure 6. The solution (35) is shown imaginary part at $\alpha=6$, $P=1$, $\beta=8$, $n=1$, $Q=0$, $c=1$, and $k=3$

5. Conclusion

We consider the (N+1)-dimensional double sine-Gordon and sinh-cosh-Gordon equations to construct new traveling wave solutions by using of the new function method. By these applications, we get some new elliptic integral function solutions. Using simple mathematical transformations, we obtain some new exact solutions based on the Jacobi elliptic function *sn*. The obtained results show that the new function method is very effective mathematical tool for solving the (N+1)-dimensional nonlinear evolution equations.

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