

RESEARCH ARTICLE

## A hybrid approach for the regularized long wave-Burgers equation

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### ABSTRACT

In this paper, a new hybrid approach based on sixth-order finite difference and seventh-order weighted essentially non-oscillatory finite difference scheme is proposed to capture numerical simulation of the regularized long wave-Burgers equation which represents a balance relation among dissipation, dispersion and nonlinearity. The corresponding approach is implemented to the spatial derivatives and then MacCormack method is used for the resulting system. Some test problems discussed by different researchers are considered to apply the suggested method. The produced results are compared with some earlier studies, and to validate the accuracy and efficiency of the method, some error norms are computed. The obtained solutions are in good agreement with the literature. Furthermore, the accuracy of the method is higher than some previous works when some error norms are taken into consideration.



## 1. Introduction

In describing many models in a great deal of fields of science, nonlinear partial differential equations (PDEs) play a significant role. Hence, reaching exact or well approximate solutions of nonlinear PDEs is still important. These kinds of partial differential equations may not have an exact solution by reason of their nonlinearity. So, it is of interest to introduce a new method or develop an existing technique to obtain accurate numerical results. One of the popular nonlinear partial differential equations studied for its numerical solutions is the regularized long wave-Burgers (RLW-Burgers) equation also known as Benjamin-Bona-Mahony-Burgers (BBMB) equation. This equation describes the propagation of surface water waves in a channel [1]. The RLW-Burgers equation is considered as follows with physical boundary conditions  $u \rightarrow 0$  as  $x \rightarrow \pm\infty$ :

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + g(u)_x = 0, \quad -\infty < x < \infty, \quad t \geq 0 \quad (1)$$
$$u(x, 0) = \phi(x) \rightarrow 0, \quad x \rightarrow \mp\infty.$$

The subscripts  $t$  and  $x$  are time and space derivatives, and denote the horizontal coordinate along the channel and the elapsed time, respectively.  $\phi(x)$  is a known function as initial condition,  $\alpha$  is a positive constant,  $\beta \in \mathbb{R}$  and  $g(u)$  is a  $C^2$  smooth nonlinear

function. Eq. (1) represents a balance relation among dissipation, dispersion and nonlinearity [2]. Due to the fact that this equation is important for understanding the nonlinear wave phenomena, many researchers [2-9] have studied on it for many years.

Since the numerical methods are good means of understanding these types of equations, the effort of finding a more accurate numerical approach is still in progress. Investigating an effective and accurate numerical method encourages us to produce a new hybrid approach based on some high order finite difference (FD) schemes for analyzing the behavior of the RLW-Burgers equation. One of these FD schemes is a seventh-order weighted essentially non-oscillatory (WENO7) [10, 11, 16] method. It can be clear from the literature that the WENO method based on ENO schemes is one of the popular numerical methods for PDEs in conservative form  $u_t + f(u)_x = 0$ . High order accuracy can be achieved in the smooth regions and discontinuities can be computed without spurious oscillations [12]. Some studies in recent years have introduced several versions of the WENO scheme derived for improving ENO properties [10-16]. However, some researchers have combined the WENO schemes with a high order method to overcome some drawbacks [17-19]. Inspired by these drawbacks in the corresponding studies, we prefer to

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combine the corresponding WENO scheme with the sixth-order finite difference (FD6) scheme [20, 21] because the FD6 gives convergent approximations as well as being effective, reliable and easy to implement. To validate the accuracy and efficiency of the proposed method, some error norms are presented and the obtained results are compared with the previous works in the literature.

The arrangement of this paper is as follows: The suggested scheme in both space and time are introduced in Section 2. Five test problems including different  $\alpha$ ,  $\beta$  parameters and  $g(u)$ ,  $\phi(x)$  functions are solved to show the efficiency and accuracy of the proposed method, and the computed results are compared with others selected from the available literature in Section 3. Finally, the last section includes the summary of findings in the paper.

## 2. Construction of the method

One can rewrite problem (1) with the following form:

$$u_t = v, \quad (2)$$

$$v_{xx} - v = f(u)_x - \alpha u_{xx}, \quad f(u)_x = (\beta u + g(u))_x. \quad (3)$$

As  $\alpha$ ,  $\beta$  and  $g(u)$  change, Eq. (3) changes for each test problem. It can also be seen from the above system, there is no time derivative term in Eq. (3). The proposed approach is involved the FD6 and WENO7 finite difference formulations to the spatial derivatives, and the MacCormack discretization is taken into account for the time derivative. Details of the implementation of the present method are introduced in the following subsections.

### 2.1. Space discretization with the hybrid scheme

First of all, we divide the domain of problem  $[a, b]$  into  $N$  subintervals such as  $a = x_1 < x_2 < \dots < x_N < x_{N+1} = b$  with the spatial step size  $h = \Delta x = x_{i+1} - x_i$  for  $i = 1, 2, \dots, N$ . Also,  $(n+1)$ -th time level is defined by  $t^{n+1} = t^n + \Delta t$  where  $t^n$  is the initial time for  $n = 0$ . Thus, the numerical solution of  $u$  is represented by  $u_i^n$  at grid point  $(x_i, t^n)$ . We use the FD6 scheme derived for the second order derivatives to discretize the terms  $v_{xx}$  and  $u_{xx}$  in Eq. (3). The FD6 scheme can briefly be introduced as follows:

$v'$  and  $v''$  in space, can be approximated by the following FD6 formulae used 7-point stencil

$$v'_i = \frac{1}{h} \sum_{j=-L}^R a_{j+L} v_{i+j}, \quad v''_i = \frac{1}{h^2} \sum_{j=-L}^R \tilde{a}_{j+L} v_{i+j} \quad (4)$$

for  $1 \leq i \leq N+1$ .

In Equations (4),  $(N+1)$  denotes the number of grid points,  $a_k$  and  $\tilde{a}_k$  ( $k = 0, \dots, R+L$ ) are unknown constants,  $R$  and  $L$  denote the number of grid points in the right and left hand side for the taken stencil, respectively. At internal points,  $R$  and  $L$  is equal while they are different for the boundary nodes. The coefficients  $a_k$  and  $\tilde{a}_k$  can be determined with Taylor series expansions about the related point and they are given in Table 1.

**Table 1.** The coefficients  $a_k$  and  $\tilde{a}_k$

$i$	Coefficients*	$k$						
		$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
1	$a_k$	-147	360	-450	400	-225	72	-10
	$\tilde{a}_k$	812	-3132	5265	-5080	2970	-972	137
2	$a_k$	-10	-77	150	-100	50	-15	2
	$\tilde{a}_k$	137	-147	-255	470	-285	93	-13
3	$a_k$	2	-24	-35	80	-30	8	-1
	$\tilde{a}_k$	-13	228	-420	200	15	-12	2
Internal Nodes	$a_k$	-1	9	-45	0	45	-9	1
	$\tilde{a}_k$	2	-27	270	-490	270	-27	2
$N-1$	$a_k$	1	-8	30	-80	35	24	-2
	$\tilde{a}_k$	2	-12	15	200	-420	228	-13
$N$	$a_k$	-2	15	-50	100	-150	77	10
	$\tilde{a}_k$	-13	93	-285	470	-255	-147	137
$N+1$	$a_k$	10	-72	225	-400	450	-360	147
	$\tilde{a}_k$	137	-972	2970	-5080	5265	-3132	812

\*Each given values of  $a_k$  and  $\tilde{a}_k$  in the table must be divided by 60 and 180, respectively

For the term  $f(u)_x$  in Eq. (3), the WENO7 scheme is implemented together with the FD6 scheme. The WENO schemes are based on ENO schemes and it was first suggested by Liu et al [22]. They provide high order accurate solutions in smooth regions and have a good convergence since they use a convex combination of all candidate stencils against the ENO schemes. In the literature, many researchers have focused on the WENO schemes in order to improve them. Taking inspiration from those studies, the present work discusses a combination of the WENO7 finite difference scheme with the FD6 scheme in computing highly accurate results. The mentioned WENO scheme is applied to internal nodes and the FD6 formulae given in above are implemented for near the boundaries. We can then introduce the WENO7 procedure with its main points herein below [10, 11, 16]:

The WENO schemes for discretization of the spatial derivatives in the following hyperbolic conservation law

$$u_t + f(u)_x = 0 \quad (5)$$

are successful in terms of the numerical approximation. A reconstruction procedure based on the local smoothness of numerical solution is used as the main point of the WENO finite difference scheme in order to produce high order accurate solutions. The term  $f(u)_x$  is approximated by

$$f(u)_x|_{x=x_j} \approx \frac{1}{\Delta x} \left( \hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}} \right), \quad (6)$$

where  $\hat{f}_{j+\frac{1}{2}}$  represents the numerical flux. The

WENO7 scheme uses 7 candidate stencils written as a set  $S = \{x_{j-3}, \dots, x_{j+3}\}$  for these numerical fluxes. It is

$$\begin{aligned} \beta_0 &= f(u_{j-3})[547f(u_{j-3}) - 3882f(u_{j-2}) + 4642f(u_{j-1}) - 1854f(u_j)] + f(u_{j-2})[7043f(u_{j-2}) - 17246f(u_{j-1}) + 7042f(u_j)] \\ &\quad + f(u_{j-1})[11003f(u_{j-1}) - 9402f(u_j)] + f(u_j)[2107f(u_j)], \\ \beta_1 &= f(u_{j-2})[267f(u_{j-2}) - 1642f(u_{j-1}) + 1602f(u_j) - 494f(u_{j+1})] + f(u_{j-1})[2843f(u_{j-1}) - 5966f(u_j) + 1922f(u_{j+1})] \\ &\quad + f(u_j)[3443f(u_j) - 2522f(u_{j+1})] + f(u_{j+1})[547f(u_{j+1})], \\ \beta_2 &= f(u_{j-1})[547f(u_{j-1}) - 2522f(u_j) + 1922f(u_{j+1}) - 494f(u_{j+2})] + f(u_j)[3443f(u_j) - 5966f(u_{j+1}) + 1602f(u_{j+2})] \\ &\quad + f(u_{j+1})[2843f(u_{j+1}) - 1642f(u_{j+2})] + f(u_{j+2})[267f(u_{j+2})], \\ \beta_3 &= f(u_j)[2107f(u_j) - 9402f(u_{j+1}) + 7042f(u_{j+2}) - 1854f(u_{j+3})] + f(u_{j+1})[11003f(u_{j+1}) - 17246f(u_{j+2}) + 4642f(u_{j+3})] \\ &\quad + f(u_{j+2})[7043f(u_{j+2}) - 3882f(u_{j+3})] + f(u_{j+3})[547f(u_{j+3})]. \end{aligned}$$

**Table 2.** The coefficients  $b_m$  for the WENO7 scheme

$b_m$	$i = 0$	$i = 1$	$i = 2$	$i = 3$
$m = 0$	25/12	-23/12	13/12	-1/4
$m = 1$	1/4	13/12	-5/12	1/12
$m = 2$	-1/12	7/12	7/12	-1/12
$m = 3$	1/12	-5/12	13/12	1/4

divided into four subset as  $S^m = \{x_{j-3+m}, \dots, x_{j+m}\}$ ,  $m = 0, 1, 2, 3$ . The numerical flux  $\hat{f}_{j+\frac{1}{2}}$  is written using

stencil sets  $S^m$  as

$$\hat{f}_{j+\frac{1}{2}} = \sum_{m=0}^3 \omega_m \hat{f}_{j+\frac{1}{2}}^{(m)}, \quad \hat{f}_{j+\frac{1}{2}}^{(m)} = \sum_{i=0}^3 b_{mi} f_{j+m-i} \quad (7)$$

for  $m = 0, 1, 2, 3$ . In Eq. (7),  $\omega_m$  are called non-linear weights defined by

$$\omega_m = \frac{\alpha_m}{\sum_{k=0}^3 \alpha_k}, \quad \alpha_m = d_m \left( 1 + \left( \frac{\tau_7}{(\varepsilon + \beta_m)} \right)^q \right)$$

with  $\tau_7 = |\beta_0 - \beta_3|$  and the linear weights  $d_0 = 1/35$ ,  $d_1 = 12/35$ ,  $d_2 = 18/35$ ,  $d_3 = 4/35$ . The coefficients  $b_{mi}$  can be calculated with the approach inspired by Xie [23] using a fourth order polynomial

$$h(x) = A + B(x - x_{j+1/2}) + C(x - x_{j+1/2})^2 + D(x - x_{j+1/2})^3 + E(x - x_{j+1/2})^4$$

with the four candidate stencils above and are presented in Table 2. The coefficients required for  $\hat{f}_{j-\frac{1}{2}}$  can be found using the same stencils in a similar way. In calculations,  $q = 2$ , and  $\varepsilon$  is used to avoid the division by zero and it is selected to be quite small,  $\varepsilon = 10^{-10}$ . The smoothness indicators,  $\beta_m$ , are given by

For more details of the WENO finite difference scheme, interested readers are referred to as the literature such as [10-13, 16].

### 2.2. Time discretization with MacCormack method

After the implementation of the aforementioned main schemes to Eq. (3), the values of variable  $v$  are found and then the MacCormack method is used to find new values of  $u$  at the next time level from Eq. (2). This method is widely used for solving nonlinear PDEs representing fluid flows and provides accurate results [24]. Let us consider the following general form of governing equation

$$\frac{du_i}{dt} = Pu_i.$$

In this form,  $P$  represents a spatial differential operator, and each values on the right hand side of the above equation are already known through the method described in the previous subsection. In order to solve this semi-discrete equation, the MacCormack approach is then implemented via the following process:

Pre. Step:  $u_i^{\bar{n+1}} = u_i^n + \Delta t Pu_i^n,$

Cor. Step:  $u_i^{n+1} = u_i^{n+1/2} + \frac{\Delta t}{2} Pu_i^{\bar{n+1}}, u_i^{n+1/2} = \frac{u_i^n + u_i^{\bar{n+1}}}{2}.$

### 3. Numerical Illustrations

In this section, we implement the previous procedure to five test problems for producing numerical solutions of the RLW-Burgers equation. The accuracy of the numerical solutions is observed by using absolute error and the following error norms

$$L_2 = \sqrt{h \sum_{j=1}^N |u_j^{analytical} - u_j^{numerical}|^2},$$

$$L_\infty = \max |u_j^{analytical} - u_j^{numerical}|.$$

which measure the mean and maximum differences between the numerical and analytical solutions. To show the behaviors of corresponding problems, some figures are also plotted.

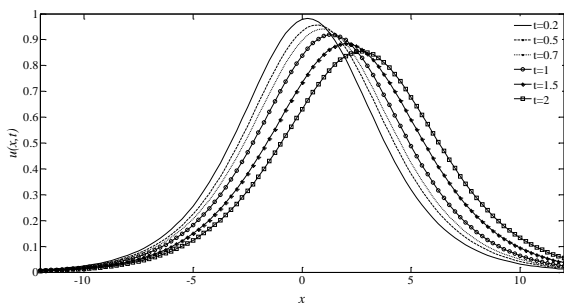
**Example 1.** As the first test problem, Eq. (1) with  $\alpha = 1, \beta = 1$  and  $g(u) = u^2/2$  is considered by the following initial condition

$$u(x, 0) = \text{sech}^2(x/4).$$

Table 3 gives the obtained results using  $h = 0.25$  and  $\Delta t = 0.01$  in the interval  $-12 \leq x \leq 12$ . It can be clearly seen that the produced results are compatible with the results of Zarebnia and Parvaz [8]. Also, the solutions at various times are qualitatively presented in Figure 1. As is the case in the study of Zarebnia and Parvaz [8] and as naturally expected, the amplitude of wave slightly decreases as the time goes on (see Figure 1).

**Table 3.** Numerical results with the parameters  $h = 0.25$  and  $\Delta t = 0.01$  for Example 1

		Present method					
		$t$					
$x$		0.2	0.5	0.7	1	1.5	2
-10		0.024700	0.022144	0.020606	0.018518	0.015529	0.013038
-5		0.256292	0.224794	0.206475	0.182379	0.149529	0.123687
0		0.978142	0.933383	0.897619	0.838349	0.733532	0.631524
5		0.319370	0.380986	0.423415	0.487534	0.589834	0.676862
10		0.032031	0.041918	0.049799	0.063792	0.093622	0.132550
		[8]					
-10		0.022951	0.019822	0.017950	0.015435	0.011934	0.009165
-5		0.256278	0.224742	0.206391	0.182231	0.149239	0.123215
0		0.978102	0.933352	0.897596	0.83834	0.733537	0.631526
5		0.319376	0.380993	0.42342	0.487532	0.589817	0.676809
10		0.030420	0.039796	0.047263	0.060502	0.088659	0.12528



**Figure 1.** The behavior of the wave in Example 1.

**Example 2.** Consider Eq. (1) with the initial condition

$$u(x, 0) = -\frac{9}{5} - \frac{6}{5} \tanh\left(\frac{x}{2}\right) + \frac{3}{5} \tanh^2\left(\frac{x}{2}\right)$$

using  $\alpha = 1, \beta = 1$  and  $g(u) = u^2/2$ . The analytical solution is given by

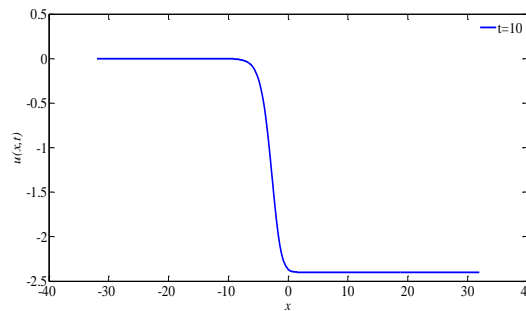
$$u(x, t) = -\frac{9}{5} - \frac{6}{5} \tanh\left(\frac{5x+t}{10}\right) + \frac{3}{5} \tanh^2\left(\frac{5x+t}{10}\right).$$

For this problem the parameters are chosen as  $h = 0.2$  and  $\Delta t = 0.01$  in the interval  $-32 \leq x \leq 32$ . In Table 4,

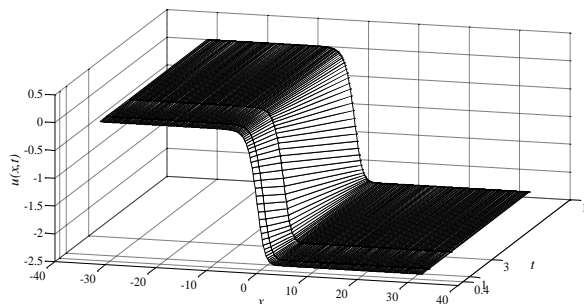
the produced  $L_2$ ,  $L_\infty$  errors are given. The obtained error values are quite good even for the larger time  $t=10$ . Furthermore, absolute errors at various points in the corresponding domain are presented and compared with the study of Alquran and Al-Khaled [4] for some time values in Table 5. It can be said that our results are at least three decimal digits better than the results of Alquran and Al-Khaled [4]. The qualitative behavior of solutions at  $t=10$  and at various times are exhibited in Figure 2(a)-2(b), respectively.

**Table 4.**  $L_2$  and  $L_\infty$  error norms for Example 2

$t$	$L_2$	$L_\infty$
0.2	1.826363E-07	1.678531E-07
0.4	3.371454E-07	3.068812E-07
1	6.734666E-07	6.326276E-07
3	1.096361E-06	9.812815E-07
10	1.754436E-05	1.546722E-05



(a)



(b)

**Figure 2.** Numerical solutions for Example 2 using  $h = 0.2$  and  $\Delta t = 0.01$  for (a)  $t = 10$  and (b) various times.

**Table 5.** Absolute errors at various times for Example 2

$x \setminus t$	Present Method	[4]	Present Method	[4]	Present Method	Present Method	Present Method
	0.2	0.2	0.4	0.4	1	3	10
0.2	1.28E-07	6.76E-04	2.22E-07	5.02E-04	3.39E-07	7.71E-08	1.92E-07
0.4	6.42E-08	5.05E-04	1.00E-07	4.52E-04	9.16E-08	2.84E-07	2.11E-07
0.6	1.63E-09	4.82E-04	1.16E-08	6.02E-04	1.02E-07	3.27E-07	1.97E-07
0.8	4.01E-08	4.14E-04	8.19E-08	6.02E-04	1.97E-07	2.47E-07	1.65E-07
1	5.50E-08	3.21E-04	1.03E-07	2.02E-04	1.98E-07	1.12E-07	1.26E-07
1.2	4.85E-08	6.05E-05	8.56E-08	7.51E-05	1.39E-07	1.90E-08	8.94E-08
1.4	3.07E-08	5.85E-05	5.05E-08	5.98E-05	6.02E-08	1.14E-07	5.81E-08
1.8	4.73E-09	1.36E-05	1.39E-08	2.37E-05	5.65E-08	1.76E-07	1.66E-08
2.4	1.96E-08	1.25E-05	3.75E-08	1.01E-05	7.90E-08	1.08E-07	5.72E-09
3	1.20E-08	6.23E-06	2.19E-08	7.78E-06	4.08E-08	3.76E-08	7.24E-09
5	3.92E-10	4.82E-06	6.01E-10	6.42E-06	4.69E-10	2.01E-09	6.30E-10

**Example 3.** We now consider Eq. (1) with the parameters  $\alpha=1$ ,  $\beta=1$  and  $g(u)=u^2/2$  with the initial condition

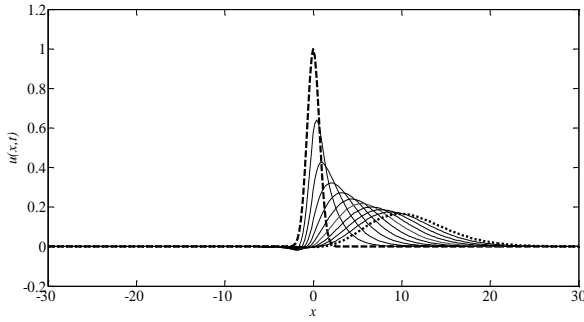
$$u(x,0) = \exp(-x^2).$$

In this example, the domain is taken to be  $-30 \leq x \leq 30$  and the behavior of the problem is examined up to time  $t=10$ . We use  $h=0.2$  and  $\Delta t=0.1$  in the proposed scheme and the recorded values are presented in Table 6. Furthermore, the profile of the wave is plotted in Figure 3 from  $t=0$  to  $t=10$ . It can be seen from the figure that the amplitude of wave and the position of that amplitude changes in time. The amplitude of wave is equal to 1 located at  $x=0$  for initial time, while that value

decreases as the times goes on and it becomes about 0.2 located close by  $x=10$ .

**Table 6.** Numerical results for Example 3 with  $h = 0.2$  and  $\Delta t = 0.1$

$x \setminus t$	1	2	5	10
-30	-5.100E-13	-3.027E-12	-6.602E-11	-1.174E-09
-15	-4.952E-08	-4.754E-08	-1.567E-08	-1.462E-09
0	5.726E-01	2.871E-01	2.032E-02	-2.175E-03
15	2.199E-05	2.396E-04	8.813E-03	9.039E-02
30	1.653E-10	5.346E-09	2.186E-06	4.348E-04



**Figure 3.** The behavior of the wave in Example 3 from  $t = 0$  to  $t = 10$  using  $h = 0.2$  and  $\Delta t = 0.1$ .

**Example 4.** As the fourth test problem, Eq. (1) under the consideration of parameters  $\alpha = 1$ ,  $\beta = 1$  and  $g(u) = 6u^2$  is studied using the following initial condition

$$u(x, 0) = -\frac{23}{120} - \frac{1}{5} \tanh x + \frac{1}{10} \tanh^2 x$$

extracted from the exact solution

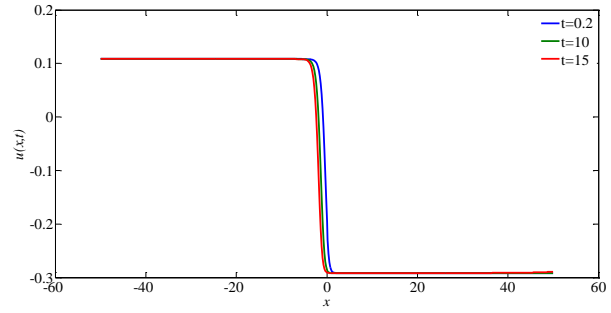
$$u(x, t) = -\frac{23}{120} - \frac{1}{5} \tanh\left(x + \frac{t}{10}\right) + \frac{1}{10} \tanh^2\left(x + \frac{t}{10}\right).$$

For comparison with an early work by Zhao et al. [2],  $L_\infty$  errors are calculated for various times over the domain  $-50 \leq x \leq 50$ . The parameter  $h$  is taken to be 0.2 with both  $\Delta t = 0.01$  and  $\Delta t = 0.1$ , and the results are presented up to time  $t = 10$  in Table 7. It is seen that the obtained  $L_\infty$  errors are less than the compared results. Furthermore, the presented errors still decrease when  $\Delta t = 0.01$ . The behavior of the problem for three different times are given in Figure 4.

**Table 7.**  $L_\infty$  errors at various times for Example 4 using

t	$L_\infty$		
	Present Method		[2]
	$h = 0.2,$ $\Delta t = 0.01$	$h = 0.2,$ $\Delta t = 0.1$	$h = 0.2,$ $\Delta t = 0.1$
0.2	6.048E-06	6.553E-06	7.650E-05
0.3	9.310E-06	1.003E-05	6.954E-05
0.4	1.274E-05	1.365E-05	1.490E-04
0.5	1.632E-05	1.739E-05	1.334E-04
0.6	2.001E-05	2.123E-05	2.160E-04
0.7	2.377E-05	2.510E-05	1.918E-04
0.8	2.752E-05	2.895E-05	2.774E-04
0.9	3.119E-05	3.270E-05	2.474E-04
1	3.472E-05	3.629E-05	3.385E-04
2	5.823E-05	6.143E-05	-
3	9.216E-05	9.493E-05	-
5	1.481E-04	1.511E-04	-
10	2.871E-04	2.889E-04	-

**Example 5.** For the last problem, the nonlinear function  $g(u)$  is chosen as  $u^3/3$ ,  $u^5/5$  and  $u^9/9$ , respectively, for Eq. (1) with  $\alpha = 1/2, \beta = 1$ .



**Figure 4.** Numerical solutions for Example 4 at different times.

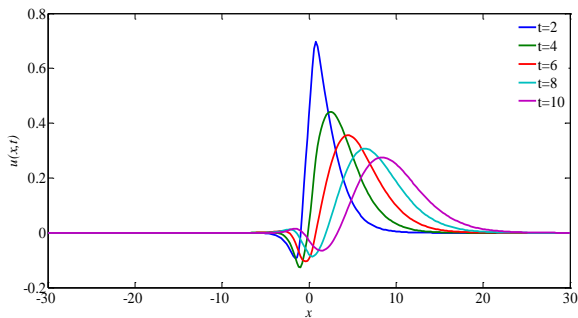
The initial condition is taken to be

$$u(x, 0) = 1/(1+x^4).$$

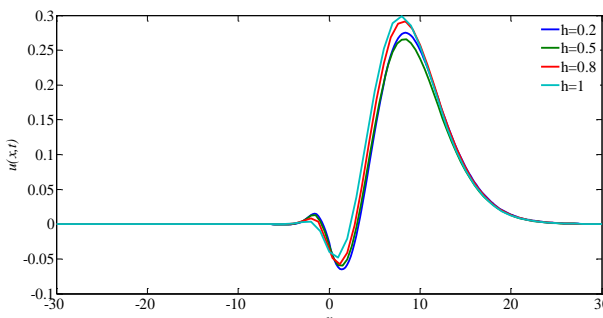
In this example, the solutions with considering various nonlinear function  $g(u)$  are investigated up to time  $t = 10$ . In Figure 5, the solutions are plotted at different times using  $h = 0.2$  and  $\Delta t = 0.1$ . To show the effect of parameter  $h$ , the solutions at  $t = 10$  are also displayed in Figure 6 using  $\Delta t = 0.1$  and various  $h$  values. It is observed from the corresponding figures that due to the value of  $\alpha$  parameter, a slight oscillation occurs at the beginning of the wave, and the amplitude of both wave and oscillation decreases as the time goes on. To see the effect of  $\alpha$  parameter on the behavior of the wave, Figure 7 is presented for various values of  $\alpha$ . In the calculations, the parameters are taken to be  $h = 0.2$ ,  $\Delta t = 0.1$ ,  $g(u) = u^3/3$ . Figure 7 shows that as the value of  $\alpha$  decreases, the amplitude of the wave slightly increases. However, any oscillation does not appear in the wave motion if  $\alpha$  value is taken larger. It can be also seen that as the  $\alpha$  value decreases, some oscillations occur. The results in the above examples revealed that the proposed method has been seen to be usually more convergent and easier than its rival methods from the literature.

#### 4. Conclusion

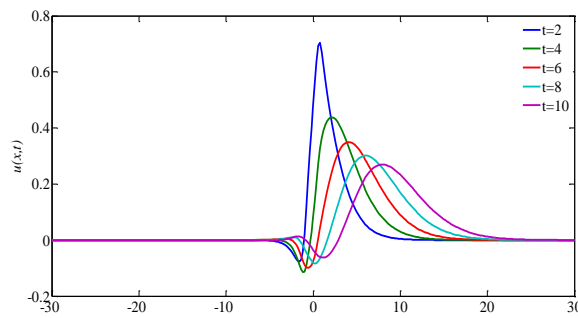
A hybrid approach based on two different types of finite difference scheme has been introduced and applied for the solutions of some physical problems constructed with the RLW-Burgers equation. To reveal the accuracy of the proposed scheme, five test problems are considered for various values of parameters taken part in the RLW-Burgers equation, and some error norms, such as absolute,  $L_2$  and  $L_\infty$ , are presented. The computed results revealed that the suggested method highly accurate, computationally powerful and user-friendly. The present approach is also believed to be easier in producing computer codes for applications. Therefore, it is seen to be a strongly advisable alternative to discover both qualitative and quantitative behaviors of similar processes for further studies.



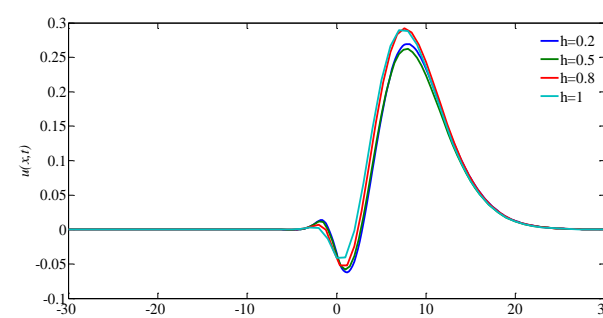
(a)  $g(u) = u^3/3$



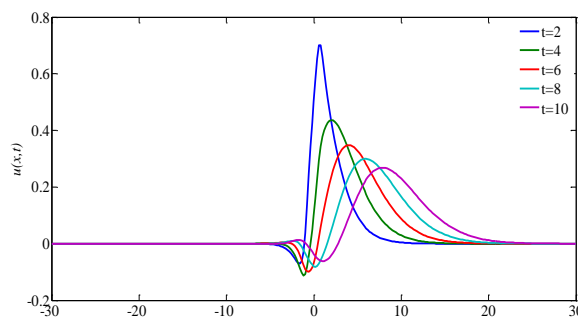
(a)  $g(u) = u^3/3$



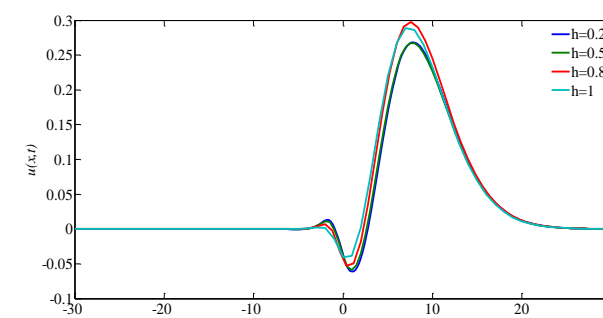
(b)  $g(u) = u^5/5$



(b)  $g(u) = u^5/5$



(c)  $g(u) = u^9/9$



(c)  $g(u) = u^9/9$

**Figure 5.** The behaviors of the wave in Example 5 using  $h = 0.2$ ,  $\Delta t = 0.1$  for various  $g(u)$

**Figure 6.** Numerical solutions for Example 5 at  $t = 10$  using  $\Delta t = 0.1$  and various values of  $h$  with various  $g(u)$

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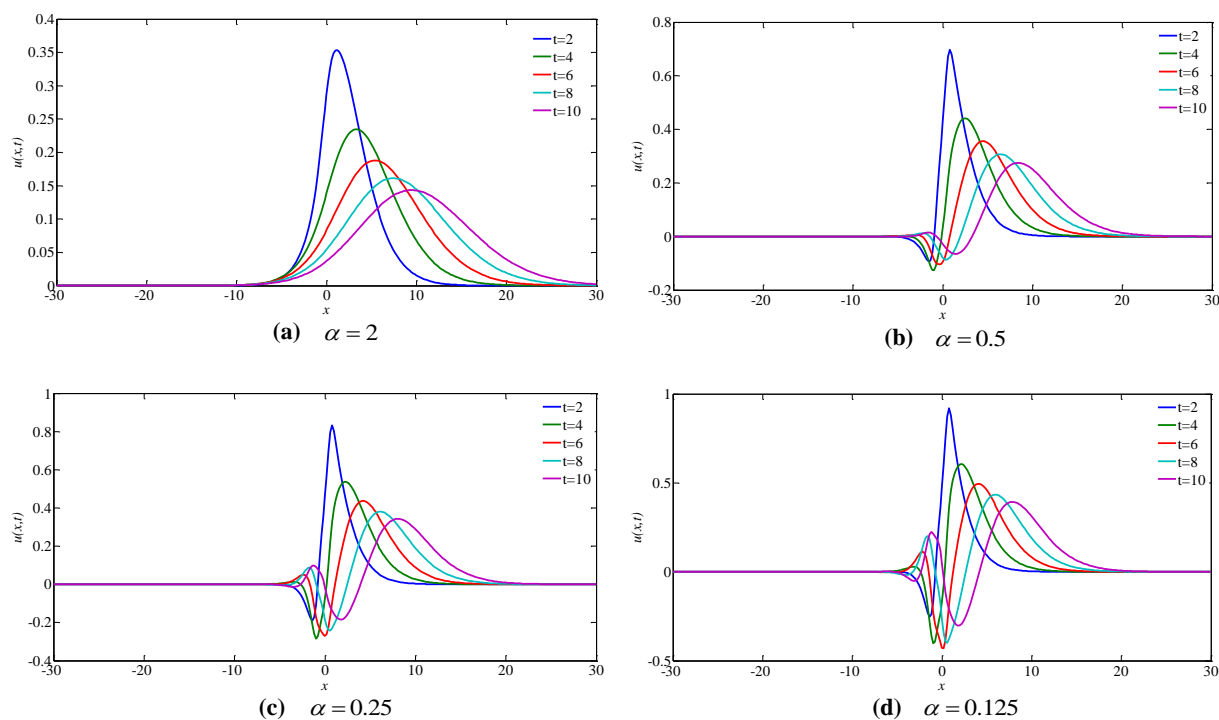
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**Figure 7.** The behaviors of the wave in Example 5 using  $h = 0.2$ ,  $\Delta t = 0.1$  and  $g(u) = u^3/3$  for various values of  $\alpha$ .

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