

RESEARCH ARTICLE

## New travelling wave solutions for fractional regularized long-wave equation and fractional coupled Nizhnik-Novikov-Veselov equation

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ABSTRACT

In this paper, solitary-wave ansatz and the  $(G'/G)$  – expansion methods have been used to obtain exact solutions of the fractional regularized long-wave (RLW) and coupled Nizhnik-Novikov-Veselov (NNV) equation. As a result, three types of exact analytical solutions such as rational function solutions, trigonometric function solutions, hyperbolic function solutions are formally derived from these equations. Proposed methods are more powerful and can be applied to other fractional differential equations arising in mathematical physics.



### 1. Introduction

Fractional differential equations (FDEs) are the generalized form of classical differential equations of integer order. Researchers especially in applied mathematician and physicist became highly interested in obtaining exact solutions for nonlinear FDEs in recent decades. Nonlinear FDEs are frequently used to describe many problems of physical phenomena that may arise in various fields such as biology, physics, chemistry, engineering, heat transfer, applied mathematics, control theory, mechanics, signal processing, seismic wave analysis, finance, and many other fractional dynamical systems [1-3].

In the past several decades, new exact solutions may help to find new phenomena. So, variety of powerful analytical and numerical methods for solving differential equations of fractional order have been suggested such as the adomian

decomposition method, the homotopy perturbation method, the variational iteration method, the finite difference method, the differential transform method, homotopy perturbation method, the homotopy analysis method, the sub-equation method, the first integral method, the  $(G'/G)$ -expansion method, the modified trial equation method, the functional variable method, the exp-function method, the simplest equation method, the exponential rational function method, ansatz method and others [4-31].

To solve mathematical problems, the transforms are an important methods. A variety of useful transforms for solving different problems appeared in the literature, such as the traveling wave transform, the Fourier transform and the others [32-41]. Recently, Li and He [42] suggested a fractional complex transform to convert FDEs into ordinary differential equations (ODEs).

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There are different kinds of fractional derivative operators. The most famous one is the Caputo definition that the function should be differentiable [43]. Recently, Jumarie derived definitions for the fractional derivative called modified Riemann–Liouville, which are suitable for continuous and non-differentiable functions. The order  $\alpha$  of Jumarie's derivative is defined by [44]:

$$D_w^\alpha f(w) = \begin{cases} \frac{1}{\Gamma(1-\alpha)} \frac{d}{dw} \int_0^w \frac{(f(\theta)-f(0))}{(w-\theta)^\alpha} d\theta, 0 < \alpha < 1 \\ (f^{(\rho)}(w))^{(\alpha-\rho)}, \rho \leq \alpha < \rho + 1, \rho \geq 1. \end{cases} \quad (1)$$

Some properties of the fractional modified RL derivative are [45]

$$D_w^\alpha w^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} w^{r-\alpha}, \quad (2)$$

$$D_w^\alpha(c) = 0, \quad (3)$$

$$D_w^\alpha \{af(w) + bg(w)\} = aD_w^\alpha f(w) + bD_w^\alpha g(w), \quad (4)$$

where  $a$ ,  $b$  and  $c$  are constants.

We take into consideration the following general nonlinear FDE of the type

$$H(u, D_t^\alpha u, D_x^\beta u, D_t^{2\alpha} u, D_t^\alpha D_x^\beta u, D_x^{2\beta} u, \dots) = 0, \quad (5)$$

where  $0 < \alpha, \beta < 1$ ,  $H$  is a polynomial of  $u$ ,  $u$  is an unknown function and  $D^\alpha$  partial fractional derivatives of  $u$ .

The traveling wave variable

$$\theta = \frac{u(x, t) = U(\theta),}{\varepsilon x^\alpha} - \frac{\tau t^\alpha}{\Gamma(1+\alpha)}, \quad (6)$$

where  $\tau \neq 0$  and  $\varepsilon \neq 0$  are constants. Applying the fractional chain rule

$$\begin{aligned} D_t^\alpha u &= \sigma_t \frac{dU}{d\xi} D_t^\alpha \theta \\ D_x^\alpha u &= \sigma_x \frac{dU}{d\xi} D_x^\alpha \theta \end{aligned} \quad (7)$$

where  $\sigma_t$  and  $\sigma_x$  are called the sigma indexes [46,28] and we can choose  $\sigma_t = \sigma_x = L$ , where  $L$  is a constant.

When we substitute, (6) with (2) and (7) into (5), we can get Eq.(5) in the following NODE;

$$\Psi(U, U', U'', U''', \dots, U^{(n)}, \dots) = 0, \quad (8)$$

where  $U^{(n)}$  is the  $n^{th}$  derivative of  $U$  with respect to  $\theta$ .

## 2. Description of the ansatz method for solving FDEs

For bright solitons, the starting hypothesis is in the form [47,48]

$$u(x, t) = A \operatorname{sech}^p \theta \quad (9)$$

and

$$\theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)} \quad (10)$$

where  $A$ ,  $k$  and  $c$  are nonzero constants. From the ansatz given above with two equalities, it is possible to obtain necessary derivatives. Then, the obtained derivatives are substituted in the Eq.(5) and we collect all terms with the same order of necessary terms. Then by equating each coefficient of the resulting polynomial to zero, we obtain a set of algebraic equations for;  $A$ ,  $k$  and  $c$ . Finally solving the system of equations we can get exact solution of Eq.(5) [49-52].

### 2.1. Applications of the proposed method

**Example 1:** The space-time fractional RLW equation has the form [53]

$$D_t^\alpha u + v D_x^\alpha u + au D_x^\alpha u - \tau D_t^\alpha D_x^{2\alpha} u = 0, \quad (11)$$

where  $\alpha$  describing the order of the fractional derivatives  $0 < \alpha \leq 1$  and  $v$ ,  $a$  and  $\tau$  are all constants that describe the behavior of the undular bore [54]. The RLW equation is modeled to govern a large number of physical phenomena such as nonlinear transverse waves in shallow water, ion acoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals. Eq.(11) was first put forward as a model for small amplitude long waves on the surface of water in channel by Peregrine [55], and later by Benjamin et al. [56]. This equation is considered as an alternative to the KdV equation. Abdel-Salam and Hassan solved Eq.(11) by the fractional auxiliary sub-equation expansion method [53]. Abdel-Salam and Yousif, have obtained abundant types of exact analytical solutions including generalized trigonometric and hyperbolic functions solutions of this equation with the fractional Riccati expansion method in [57]. Analytical solutions of fractional RLW equation is very low. When  $\alpha = 1$ , equation (11) is called the RLW equation. Conversely, many researchers focus on numerical

methods to obtain approximate solutions of RLW equation. For example, Esen and Kutluay solved the equation by a lumped Galerkin method in [58]. Dag et al. have applied least square quadratic B-spline and cubic B-spline finite element method to obtain new analytical solutions of RLW equation in [59,60]. Saka et al. [61,62] solved this equation by quintic B-spline collocation and B-spline collocation algorithms methods. In [63], the variational iteration method successfully applied to finding the solution of the RLW equation by Yusufoglu and Bekir.

In order to solve Eq.(11), we use the traveling wave transformation

$$u(x, t) = U(\theta), \quad \theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}, \quad (12)$$

where  $k \neq 0$  and  $c \neq 0$  are constants. When we substitute (12) with (2) and (6) into (11) and by integrating once and setting the constants of integration to be zero, the Eq. (11) can carry to an ODE

$$(kv - c)U + \frac{ak}{2}U^2 + \tau ck^2 L^2 U'' = 0, \quad (13)$$

where  $U' = \frac{dU}{d\theta}$ . The solitary wave ansatz for the bright soliton solution, the hypothesis is (9) and (10). From (9) and (10), it is possible to get

$$\frac{d^2 U(\theta)}{d\theta^2} = Ap^2 \operatorname{sech}^p \theta - Ap(p+1) \operatorname{sech}^{p+2} \theta, \quad (14)$$

and

$$U^2(\theta) = A^2 \operatorname{sech}^{2p} \theta. \quad (15)$$

Thus, substituting the ansatz (14) and (15) into Eq.(13), yields to

$$(kv - c)A \operatorname{sech}^p \theta + \frac{ak}{2} A^2 \operatorname{sech}^{2p} \theta + \tau ck^2 L^2 Ap^2 \operatorname{sech}^p \theta - \tau ck^2 L^2 Ap(p+1) \operatorname{sech}^{p+2} \theta = 0. \quad (16)$$

Now, from (16), equating the exponents  $p+2$  and  $2p$  leads to

$$p = 2. \quad (17)$$

From (16), setting the coefficients of  $\operatorname{sech}^{p+2} \theta$  and  $\operatorname{sech}^{2p} \theta$  terms to zero, we get

$$\frac{ak}{2} A^2 - \tau ck^2 L^2 Ap(p+1) = 0, \quad (18)$$

by use (17) and after some calculations, we have

$$A = \frac{12\tau kcL^2}{a}, \quad a \neq 0. \quad (19)$$

We find, from setting the coefficients of  $\operatorname{sech}^p \theta$  terms in Eq.(16) to zero

$$(kv - c)A + \tau ck^2 L^2 Ap^2 = 0, \quad (20)$$

also we obtain

$$c = \frac{vk}{1 - 4\tau k^2 L^2}. \quad (21)$$

From (21) it is important to note that

$$4\tau k^2 L^2 \neq 1. \quad (22)$$

Thus finally, bright soliton solution of (11) is given by:

$$u(x, t) = \frac{12\tau kcL^2}{a} \operatorname{sech}^2 \left( \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{vkt^\alpha}{(1-4\tau k^2 L^2)\Gamma(1+\alpha)} \right). \quad (23)$$

**Example 2:** Secondly, we consider the following the space-time fractional coupled Nizhnik-Novikov-Veselov (NNV) equation [64]

$$\begin{aligned} D_t^\alpha u - AD_x^{3\alpha} u - BD_y^{3\alpha} u + 3AuD_x^\alpha v \\ + 3AvD_x^\alpha u + 3BuD_y^\alpha w + 3BwD_y^\alpha u = 0, \\ D_x^\alpha u - D_y^\alpha v = 0, \\ D_y^\alpha u - D_x^\alpha w = 0, \end{aligned} \quad (24)$$

where  $0 < \alpha \leq 1$ ,  $A$  and  $B$  are given constants satisfying  $A+B \neq 0$ , and  $u$ ,  $v$  and  $w$  are the functions of  $(x, y, t)$ . Yan has found three types of travelling wave solutions of equation (24) by using the fractional sub-equation method [64]. The (2+1)-dimensional NNV equation is an isotropic extension of the well-known (1+1)-dimensional KdV equation. In recent years, NNV equation have been studied several areas of physics including condense matter physics, optics, fluid mechanics, and plasma physics when  $\alpha \rightarrow 1$  [65-67]. Darvishi et al. have applied exp-function method to obtain exact traveling wave solutions of classical NNV equation in [68]. Deng solved the equation by use of the extended hyperbolic function method in

[69]. In [70], Wazwaz et al. have investigated the bright soliton solutions with wave ansatz method. For our goal, we present the following transformation

$$\begin{aligned} u(x, y, t) &= U(\theta), \theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}, \\ v(x, y, t) &= V(\theta), \theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}, \\ w(x, y, t) &= W(\theta), \theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}, \end{aligned} \tag{25}$$

where  $k \neq 0$ ,  $m \neq 0$  and  $n \neq 0$  are constants.

Then by using of Eq. (25) with (2) and (7), Eq.(24) can be turned into an ODEs and by integrating once and setting the constants of integration to be zero, we obtain

$$\begin{aligned} (Ak^3L^2 + Bm^3L^2)U'' - 3kA(UV) \\ - 3mB(UW) + nU &= 0, \\ kU - mV &= 0, \\ mU - kW &= 0, \end{aligned} \tag{26}$$

where  $U' = \frac{dU}{d\theta}$  and  $V' = \frac{dV}{d\theta}$ . In order the start off with the solution hypothesis, the following ansatz is assumed

$$u(x, y, t) = \lambda_1 \operatorname{sech}^p \theta, \tag{27}$$

and

$$v(x, y, t) = \lambda_2 \operatorname{sech}^s \theta, \tag{28}$$

and

$$w(x, y, t) = \lambda_3 \operatorname{sech}^r \theta, \tag{29}$$

where

$$\theta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{my^\alpha}{\Gamma(1+\alpha)} - \frac{nt^\alpha}{\Gamma(1+\alpha)}. \tag{30}$$

Here in (27)-(30),  $\lambda_1$ ,  $\lambda_2$ ,  $\lambda_3$ ,  $k$  and  $m$  are the free parameters of the solitons and  $n$  is the velocity of the soliton. The exponents  $p$ ,  $s$  and  $r$  are unknown values will be find later. Now, from (27)-(29) and (30) it is possible to obtain

$$\begin{aligned} (Ak^3L^2 + Bm^3L^2)\lambda_1 p^2 \operatorname{sech}^p \theta \\ - (Ak^3L^2 + Bm^3L^2)\lambda_1 p(p+1) \operatorname{sech}^{p+2} \theta \\ - 3kA\lambda_1\lambda_2 \operatorname{sech}^{p+s} \theta - 3mB\lambda_1\lambda_3 \operatorname{sech}^{p+r} \theta \\ + n\lambda_1 \operatorname{sech}^p \theta &= 0, \end{aligned} \tag{31}$$

and

$$k\lambda_1 \operatorname{sech}^p \theta - m\lambda_2 \operatorname{sech}^s \theta = 0, \tag{32}$$

and

$$m\lambda_1 \operatorname{sech}^p \theta - k\lambda_3 \operatorname{sech}^r \theta = 0. \tag{33}$$

Now from (32) and (33) equating the exponents of  $\operatorname{sech} \theta$  functions we have  $p = s = r$ . In (32) we obtain,

$$\lambda_2 = \frac{k\lambda_1}{m}. \tag{34}$$

Similarly in (34) that gives

$$\lambda_3 = \frac{m\lambda_1}{k}. \tag{35}$$

Now, equating the exponents of  $\operatorname{sech}^{p+2} \theta$  or  $\operatorname{sech}^{p+s} \theta$  and  $\operatorname{sech}^{p+r} \theta$  functions in (31) with  $p = s = r$ , one gets

$$p + 2 = p + s = p + r, \tag{36}$$

so that

$$p = s = r = 2. \tag{37}$$

Setting the coefficients of  $\operatorname{sech}^{p+2} \theta$  in (31), to zero gives

$$\begin{aligned} (Ak^3L^2 + Bm^3L^2)\lambda_1 p(p+1) + 3kA\lambda_1\lambda_2 \\ + 3mB\lambda_1\lambda_3 &= 0, \end{aligned} \tag{38}$$

using Eqs. (34), (35),  $p = 2$  and some calculations

$$\lambda_1 = -2kmL^2. \tag{39}$$

Again from (31), setting the coefficients of  $\operatorname{sech}^p \theta$  terms to zero one obtains,

$$(Ak^3L^2 + Bm^3L^2)\lambda_1 p^2 + n\lambda_1 = 0, \tag{40}$$

which gives

$$n = -4L^2(Ak^3 + Bm^3). \tag{41}$$

Lastly, the bright soliton solution for space-time fractional coupled NNV equation is given by

$$u(x, y, t) = \lambda_1 \operatorname{sech}^2 \theta, \quad (42)$$

and

$$v(x, y, t) = \lambda_2 \operatorname{sech}^2 \theta, \quad (43)$$

and

$$w(x, y, t) = \lambda_3 \operatorname{sech}^2 \theta, \quad (44)$$

where the velocity of the solitons  $n$  is given in (41), free parameters  $\lambda_1$ ,  $\lambda_2$  and  $\lambda_3$  are given by (39), (34) and (35) respectively.

### 3. Description of the $\left(\frac{G'}{G}\right)$ expansion method for solving FDEs

Suppose that the solution of ODE (8) can be expressed by a polynomial in  $(G'/G)$  as:

$$U = \sum_{i=0}^z a_i \left(\frac{G'}{G}\right)^i, \quad a_z \neq 0, \quad (45)$$

where  $G = G(\xi)$  satisfies the second order LODE in the form [71]

$$\frac{d^2 G(\xi)}{d\xi^2} + \lambda \frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0, \quad (46)$$

where  $a_1, \dots, a_z$ ,  $\lambda$  and  $\mu$  are constants will be determined later,  $z$  is the positive integer which can be determined by the homogeneous balance with the highest order derivatives and highest order nonlinear appearing in ODE (8). When we substitute (45) into (8) and use Eq.(46), we collect all terms with the same order of  $(G'/G)$  together. When we equate all coefficient of this polynomial to zero, it gives us a set of algebraic equations for  $a_1, \dots, a_z$ ,  $\lambda$ ,  $\tau$ ,  $\varepsilon$  and  $\mu$  by using Maple. Then substituting  $a_1, \dots, a_z$ ,  $\lambda$ ,  $\mu$ ,  $\varepsilon$ ,  $\tau$  and general solutions of Eq. (46) into (8) we can get a variety of exact solutions of the FDEs (5).

#### 3.1. Applications of the proposed method

##### Example 1:

In order to solve Eq.(11) by the  $(G'/G)$  –expansion method, we use the traveling wave transformation (12) and with a similar approach in section 2, we get

$$(kv - c)U + \frac{ak}{2}U^2 + \tau ck^2 L^2 U'' + \xi_0 = 0, \quad (47)$$

where " $U''$ " =  $\frac{dU}{d\xi}$  and  $\xi_0$  is an integral constant. Balancing  $U''$  with  $U^2$  in (47) gives

$$\begin{aligned} 2z &= z + 2, \\ z &= 2. \end{aligned} \quad (48)$$

Assume that it is possible to express solution of (47) by a polynomial in  $\left(\frac{G'}{G}\right)$  as:

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \quad (49)$$

By using Eq.(46), from Eq.(49) we have

$$\begin{aligned} U''(\xi) &= 6a_2 \left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda) \left(\frac{G'}{G}\right)^3 \\ &+ (8a_2\mu + 3a_1\lambda + 4a_2\lambda^2) \left(\frac{G'}{G}\right)^2 \\ &+ (6a_2\lambda\mu + 2a_1\mu + a_1\lambda^2) \left(\frac{G'}{G}\right) \\ &+ 2a_2\mu^2 + a_1\lambda\mu. \end{aligned} \quad (50)$$

When we substitute Eqs.(49) and (50) into Eq.(47), collecting the coefficients of  $\left(\frac{G'}{G}\right)^i$  ( $i = 0, \dots, 4$ ) and set them to zero we get a system. The solutions of this algebraic equations can be done by Maple which gives

$$\begin{aligned} a_0 &= \frac{c-vk-\tau ck^2 L^2 \lambda^2 - 8\tau ck^2 L^2 \mu}{ak}, \\ a_1 &= -\frac{12\lambda\tau ck L^2}{a}, \\ a_2 &= -\frac{12\tau ck L^2}{a}, \\ \xi_0 &= \frac{c^2 k^4 \tau^2 L^4 (8\lambda^2 \mu - 16\mu^2 - \lambda^4) + c^2 - 2vkc + v^2 k^2}{2ak}. \end{aligned} \quad (51)$$

where  $\lambda$  and  $\mu$  are arbitrary constants. By using Eq.(51), expression (49) can be written as

$$\begin{aligned} U(\xi) &= \frac{c-vk-\tau ck^2 L^2 \lambda^2 - 8\tau ck^2 L^2 \mu}{ak} \\ &- \frac{12\lambda\tau ck L^2}{a} \left(\frac{G'}{G}\right) - \frac{12\tau ck L^2}{a} \left(\frac{G'}{G}\right)^2. \end{aligned} \quad (52)$$

When we substitute general solutions of Eq. (46) into Eq.(52) we have below travelling wave solutions of the equation as follows:

When  $\lambda^2 - 4\mu > 0$ ,

$$U_1(\xi) = \frac{c-vk+2\tau ck^2L^2(\lambda^2-4\mu)}{ak} - \frac{3\tau ck^2L^2(\lambda^2-4\mu)}{ak} \times \left( \frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2-4\mu}\xi} \right)^2, \tag{53}$$

where  $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}$ .

When  $\lambda^2 - 4\mu < 0$ ,

$$U_2(\xi) = \frac{c-vk+2\tau ck^2L^2(\lambda^2-4\mu)}{ak} + \frac{3\tau ck^2L^2(\lambda^2-4\mu)}{ak} \left( \frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu-\lambda^2}\xi} \right)^2, \tag{54}$$

where  $\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)}$ .

In particular, if  $C_1 \neq 0, C_2 = 0, \lambda > 0, \mu = 0$ , then  $U_1$  and  $U_2$  become

$$u_1(x, t) = \frac{c-vk+2\tau ck^2L^2\lambda^2}{ak} + \frac{3\tau ckL^2\lambda^2}{a} \operatorname{sech}^2 \left( \frac{\lambda kx^\alpha}{2\Gamma(1+\alpha)} - \frac{\lambda ct^\alpha}{2\Gamma(1+\alpha)} \right). \tag{55}$$

When  $\lambda^2 - 4\mu = 0$ , we obtain rational function solution of Eq. (45)

$$u_2(x, t) = \frac{c-vk}{ak} - \frac{12\tau ckL^2}{a} \left( \frac{C_1}{C_1 \left( \frac{kx^\alpha}{\Gamma(1+\alpha)} - \frac{ct^\alpha}{\Gamma(1+\alpha)} \right) + C_2} \right)^2. \tag{56}$$

**Example 2:**

Similarly, in order to solve Eq. (24) by the proposed method, suppose that the solutions of the Eq. (26) can be expressed by a polynomial in  $\left(\frac{G'}{G}\right)$  as follows:

$$U(\xi) = \sum_{i=0}^z a_i \left(\frac{G'}{G}\right)^i, \quad a_z \neq 0, \tag{57}$$

$$V(\xi) = \sum_{i=0}^r b_i \left(\frac{G'}{G}\right)^i, \quad b_r \neq 0, \tag{58}$$

$$W(\xi) = \sum_{i=0}^p c_i \left(\frac{G'}{G}\right)^i, \quad c_p \neq 0. \tag{59}$$

By the same procedure as illustrated in example 1, the homogeneous balance between highest order derivatives and non-linear terms in (26) we get positive integers  $z = 2, r = 2$  and  $p = 2$ . Consequently, we have:

$$U(\xi) = a_0 + a_1 \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0, \tag{60}$$

$$V(\xi) = b_0 + b_1 \left(\frac{G'}{G}\right) + b_2 \left(\frac{G'}{G}\right)^2, \quad b_2 \neq 0, \tag{61}$$

$$W(\xi) = c_0 + c_1 \left(\frac{G'}{G}\right) + c_2 \left(\frac{G'}{G}\right)^2, \quad c_2 \neq 0. \tag{62}$$

When we substitute Eqs. (50), (60)-(62) into Eq.(26), collecting the coefficients of  $\left(\frac{G'}{G}\right)^i$  ( $i = 0, \dots, 3$ ) and apply the same procedure of example 1, we have

**Case 1:**

$$\begin{aligned} a_0 &= a_2\mu, & a_1 &= a_2\lambda, & a_2 &= a_2, \\ b_0 &= \frac{a_2^2\mu}{2m^2L^2}, & b_1 &= \frac{a_2^2\lambda}{2m^2L^2}, & b_2 &= \frac{a_2^2}{2m^2L^2}, \\ c_0 &= 2m^2L^2\mu, & c_1 &= 2m^2L^2\lambda, & c_2 &= 2m^2L^2, \\ k &= \frac{a_2}{2mL^2}, & n &= \frac{(4\mu-\lambda^2)(Aa_2^3+8Bm^6L^6)}{8m^3L^4}, \end{aligned} \tag{63}$$

where  $\lambda$  and  $\mu$  are arbitrary constants. Substituting Eq. (63) into Eqs.(60)-(62) yields

$$U(\xi) = a_2\mu + a_2\lambda \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \tag{64}$$

$$V(\xi) = \frac{a_2^2\mu}{2m^2L^2} + \frac{a_2^2\lambda}{2m^2L^2} \left(\frac{G'}{G}\right) + \frac{a_2^2}{2m^2L^2} \left(\frac{G'}{G}\right)^2, \tag{65}$$

$$W(\xi) = 2m^2L^2\mu + 2m^2L^2\lambda \left(\frac{G'}{G}\right) + 2m^2L^2 \left(\frac{G'}{G}\right)^2 \tag{66}$$

Then, when we substitute general solutions of Eq.(46) into Eqs.(64)-(66), we have two types of solutions of the Eqs.(24) as follows:

When  $\lambda^2 - 4\mu > 0$ ,

$$U_1(\xi) = \frac{a_2(4\mu - \lambda^2)}{4} (1 - \Omega_1^2), \quad (67)$$

$$V_1(\xi) = \frac{a_2^2(4\mu - \lambda^2)}{8m^2L^2} (1 - \Omega_1^2), \quad (68)$$

$$W_1(\xi) = \frac{m^2L^2(4\mu - \lambda^2)}{2} (1 - \Omega_1^2) \quad (69)$$

where

$$\Omega_1 = \frac{K_1 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + K_2 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi}{K_1 \cosh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi + K_2 \sinh \frac{1}{2} \sqrt{\lambda^2 - 4\mu} \xi},$$

$$\xi = \frac{a_2 x^\alpha}{2mL^2\Gamma(1+\alpha)} + \frac{\lambda y^\alpha}{\Gamma(1+\alpha)} - \frac{(4\mu - \lambda^2)(Aa_2^3 + 8Bm^6L^6)t^\alpha}{8m^3L^4\Gamma(1+\alpha)}$$

When  $\lambda^2 - 4\mu < 0$ ,

$$U_2(\xi) = \frac{a_2(4\mu - \lambda^2)}{4} (1 + \Omega_2^2), \quad (70)$$

$$V_2(\xi) = \frac{a_2^2(4\mu - \lambda^2)}{8m^2L^2} (1 + \Omega_2^2), \quad (71)$$

$$W_2(\xi) = \frac{m^2L^2(4\mu - \lambda^2)}{2} (1 + \Omega_2^2) \quad (72)$$

where

$$\Omega_2 = \frac{-K_1 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + K_2 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}{K_1 \cos \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi + K_2 \sin \frac{1}{2} \sqrt{4\mu - \lambda^2} \xi}$$

In particular, if  $K_1 \neq 0$ ,  $K_2 = 0$ ,  $\mu = 0$  then  $U_1(\xi)$ ,  $V_1(\xi)$  and  $W_1(\xi)$  become

$$u_1(x, y, t) = -\frac{\lambda^2 a_2}{4} \operatorname{sech}^2(\Phi), \quad (73)$$

$$v_1(x, y, t) = -\frac{\lambda^2 a_2^2}{8m^2L^2} \operatorname{sech}^2(\Phi), \quad (74)$$

$$w_1(x, y, t) = -\frac{\lambda^2 m^2 L^2}{2} \operatorname{sech}^2(\Phi), \quad (75)$$

where

$$\Phi = \frac{\lambda a_2 x^\alpha}{4mL^2\Gamma(1+\alpha)} + \frac{\lambda m y^\alpha}{2\Gamma(1+\alpha)} + \frac{\lambda^3 (Aa_2^3 + 8Bm^6L^6)t^\alpha}{16m^3L^4\Gamma(1+\alpha)}$$

Also if  $K_1 \neq 0$ ,  $K_2 = 0$ ,  $\mu = 0$  then  $U_2(\xi)$ ,  $V_2(\xi)$  and  $W_2(\xi)$  become,  $u_1(x, y, t)$ ,  $v_1(x, y, t)$  and  $w_1(x, y, t)$ .

**Case 2:**

$$a_0 = \frac{a_2(2\mu + \lambda^2)}{6}, \quad a_1 = a_2\lambda, \quad a_2 = a_2,$$

$$b_0 = \frac{a_2^2(2\mu + \lambda^2)}{12m^2L^2}, \quad b_1 = \frac{a_2^2\lambda}{2m^2L^2}, \quad b_2 = \frac{a_2^2}{2m^2L^2},$$

$$c_0 = \frac{2m^2L^2\mu + m^2L^2\lambda^2}{3}, \quad c_1 = 2m^2L^2\lambda, \quad c_2 = 2m^2L^2,$$

$$k = \frac{a_2}{2mL^2}, \quad n = \frac{(4\mu - \lambda^2)(Aa_2^3 + 8Bm^6L^6)}{8m^3L^4}, \quad (76)$$

where  $\lambda$  and  $\mu$  are arbitrary constants. Substituting Eq.(76) into Eqs.(60)-(62), yields

$$U(\xi) = \frac{a_2(2\mu + \lambda^2)}{6} + a_2\lambda \left(\frac{G'}{G}\right) + a_2 \left(\frac{G'}{G}\right)^2, \quad (77)$$

$$V(\xi) = \frac{a_2^2}{2m^2L^2} \left( \frac{(2\mu + \lambda^2)}{6} + \lambda \left(\frac{G'}{G}\right) + \left(\frac{G'}{G}\right)^2 \right), \quad (78)$$

$$W(\xi) = m^2L^2 \left( \frac{2\mu + \lambda^2}{3} + 2\lambda \left(\frac{G'}{G}\right) + 2 \left(\frac{G'}{G}\right)^2 \right). \quad (79)$$

When we substitute general solutions of Eq.(46) into Eqs.(77)-(79), we deduce the following traveling wave solutions:

When  $\lambda^2 - 4\mu > 0$ ,

$$U_3(\xi) = \frac{a_2(4\mu - \lambda^2)}{4} \left( \frac{1}{3} - \Omega_1^2 \right), \quad (80)$$

$$V_3(\xi) = \frac{a_2^2(4\mu - \lambda^2)}{8m^2L^2} \left( \frac{1}{3} - \Omega_1^2 \right), \quad (81)$$

$$W_3(\xi) = \frac{m^2L^2(4\mu - \lambda^2)}{2} \left( \frac{1}{3} - \Omega_1^2 \right). \quad (82)$$

When  $\lambda^2 - 4\mu < 0$ ,

$$U_4(\xi) = \frac{a_2(4\mu - \lambda^2)}{4} \left( \frac{1}{3} + \Omega_2^2 \right), \quad (83)$$

$$V_4(\xi) = \frac{a_2^2(4\mu - \lambda^2)}{8m^2L^2} \left( \frac{1}{3} + \Omega_2^2 \right), \quad (84)$$

$$W_4(\xi) = \frac{m^2L^2(4\mu - \lambda^2)}{2} \left( \frac{1}{3} + \Omega_2^2 \right). \quad (85)$$

In particular, if  $K_1 \neq 0$ ,  $K_2 = 0$ ,  $\mu = 0$  then  $U_3(\xi)$ ,  $V_3(\xi)$  and  $W_3(\xi)$  become

$$u_2(x, y, t) = -\frac{\lambda^2 a_2}{4} \left( \frac{1}{3} - \tanh^2(\Phi) \right), \quad (86)$$

$$v_2(x, y, t) = -\frac{a_2^2 \lambda^2}{8m^2 L^2} \left( \frac{1}{3} - \tanh^2(\Phi) \right), \quad (87)$$

$$w_2(x, y, t) = -\frac{m^2 L^2 \lambda^2}{2} \left( \frac{1}{3} - \tanh^2(\Phi) \right). \quad (88)$$

Also if  $K_1 \neq 0$ ,  $K_2 = 0$ ,  $\mu = 0$  then  $U_4(\xi)$ ,  $V_4(\xi)$  and  $W_4(\xi)$  become,  $u_2(x, y, t)$ ,  $v_2(x, y, t)$  and  $w_2(x, y, t)$ .

#### 4. Conclusion

The ansatz and the  $(G'/G)$  expansion methods are used in this article to obtain some new exact solutions of the fractional regularized long-wave equation and the fractional coupled Nizhnik-Novikov-Veselov equation. The  $(G'/G)$  expansion method is more effective and more general than the ansatz method because it gives exact solutions in more general forms. These methods are quite proficient methods for obtaining new exact solutions of FDEs. The obtained solutions are new and the methods can be extended to solve problems of nonlinear FDEs arising in the theory of solitons and other areas. To our knowledge, these new solutions have not been reported in former literature.

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