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RESEARCH ARTICLE

On solutions of variable-order fractional differential equations

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ABSTRACT

Numerical calculation of the fractional integrals and derivatives is the code to search fractional calculus and solve fractional differential equations. The exact solutions to fractional differential equations are compelling to get in real applications, due to the nonlocality and complexity of the fractional differential operators, especially for variable-order fractional differential equations. Therefore, it is significant to enhance numerical methods for fractional differential equations. In this work, we consider variable-order fractional differential equations by reproducing kernel method. There has been much attention in the use of reproducing kernels for the solutions to many problems in the recent years. We give an example to demonstrate how efficiently our theory can be implemented in practice.

1. Introduction

Fractional differential equations have been studied by many investigators in recent years. The notion of a variable order operator is a much more recent improvement. Different authors have presented different definitions of variable order differential operators. The kernel of the variable order operators is too complex for having a variable-exponent. Therefore, to get the numerical solutions of variable order fractional differential equations is quite compelling. There are few studies of variable order fractional differential equations. Coimbra [1] applied a consistent approximation with first-order accurate for the solution of variable order differential equations. Lin et al. [2] worked the stability and the convergence of an explicit finite-difference approximation for the variable-order fractional diffusion equation with a nonlinear source term. Zhuang et al. [3] acquired explicit and implicit Euler approximations for the fractional advection-diffusion nonlinear equation of variable-order. For more details see [4–6]. No one had tried to find the numerical solutions of the variable order fractional differential equations by the reproducing kernel method (RKM).

The aim of our work is to investigate the efficiency of RKM to solve variable-order fractional differential equations. Let us consider

$$ {}_{C}D_{0,\nu}^{\alpha(\nu)}u(\nu) = f(\nu), \quad 0 \le \nu \le T, \tag{1}$$

and subjected to the initial condition

$$u(0) = 0, \tag{2}$$

where ${}_{C}D_{0,\nu}^{\alpha(\nu)}$ is variable order fractional derivative of Caputo sense, $f(\nu)$ is the known continuous function, $u(\nu)$ is the unknown function, $0 < \alpha_{min} \le \alpha(\nu) \le \alpha_{max} < 1$.

The theory of reproducing kernels was used for the first time at the beginning of the 20th century by Zaremba [7]. Reproducing kernel theory has

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considerable implementations in numerical analysis, differential equations, probability and statistics [8–11]. Some authors discussed fractional differential equations, nonlinear oscillators with discontinuity, singular nonlinear two-point periodic boundary value problems, integral equations and nonlinear partial differential equations [7, 12–21].

This paper is arranged as follows. Some definitions and properties of the variable order fractional integrals and derivatives are presented in Section 2. Section 3 shows some useful reproducing kernel functions. The representation in $W_2^2[0,1]$ and a related linear operator are given in Section 4. Section 5 gives the main results. Numerical experiments are demonstrated in Section 6. Some conclusions are given in the last section.

2. Some useful definitions

(i) Riemann-Liouville fractional integral of the first kind with order $\alpha(\nu)$ is given as [22]:

$$I_{a^+}^{\alpha(\nu)}u(\nu) = \frac{1}{\Gamma(\alpha(\nu))}\int_{a^+}^{\nu}(\nu-T)^{\alpha(\nu)-1}u(T)\mathrm{d}T, \quad x>0\ [Re(\alpha(\nu))>0].$$

(ii) Riemann-Liouville fractional derivative of the first kind with order $\alpha(\nu)$ is presented by [22]:

$$D_{a^+}^{\alpha(\nu)}u(\nu) = \frac{\mathrm{d}^m}{\Gamma(m-\alpha(\nu))\mathrm{d}\nu^m}\int_{a^+}^{\nu}\frac{u(\tau)}{(\nu-\tau)^{\alpha(\nu)-m+1}}\mathrm{d}\tau,$$

but $D_{a^+}^{\alpha(\nu)}I_{a^+}^{\alpha(\nu)}u \neq u$, $(m-1\le\alpha(\nu)<m)$.

(iii) Caputo's fractional derivative with order $\alpha(\nu)$ is introduced with [22]:

$$D^{\alpha(\nu)}u(\nu) = \frac{1}{\Gamma(1-\alpha(\nu))}\int_{0^+}^{\nu}(\nu-\tau)^{-\alpha(\nu)}u'(\tau)\mathrm{d}\tau + \frac{(u(0^+)-u(0^-))\nu^{-\alpha(\nu)}}{\Gamma(1-\alpha(\nu))},$$

where $0<\alpha(\nu)\le 1$. If the starting time is in a perfect situation, we obtain the definition as follows [22]:

$$D^{\alpha(\nu)}u(\nu) = \frac{1}{\Gamma(1-\alpha(\nu))}\int_{0^+}^{\nu}(\nu-\tau)^{-\alpha(\nu)}u'(\tau)\mathrm{d}\tau, \quad (0<\alpha(\nu)<1),$$

with the definition above, we obtain the following formula $(0<\alpha(\nu)\le 1)$:

$$D_*^{\alpha(\nu)}x^\beta = \begin{cases} 0, & \beta=0,\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha(\nu))}x^{\beta-\alpha(\nu)}, & \beta=1,2,3,\dots. \end{cases} \tag{3}$$

3. Reproducing kernel functions

Definition 1. *We define the space $G_2^1[0,1]$ by*

$$G_2^1[0,1] = \{u\in AC[0,1]:\ u'\in L^2[0,1]\},$$

where AC denotes the space of absolutely continuous functions. The inner product and the norm in $G_2^1[0,1]$ are defined by

$$\langle u,h\rangle_{G_2^1} = u(0)h(0)+\int_0^1 u'(\nu)h'(\nu)\mathrm{d}\nu, \quad u,h\in G_2^1[0,1]$$

and

$$\|u\|_{G_2^1} = \sqrt{\langle u,u\rangle_{G_2^1}}, \quad u\in G_2^1[0,1].$$

Lemma 1 (See [23, page 17])**.** *The space $G_2^1[0,1]$ is a reproducing kernel space, and its reproducing kernel function Q_y is given by*

$$Q_y(\nu) = \begin{cases} 1+\nu, & 0\le\nu\le y\le 1,\\ 1+y, & 0\le y<\nu\le 1. \end{cases}$$

Definition 2. *We describe the space $W_2^2[0,1]$ by*

$$W_2^2[0,1] = \{u\in AC[0,1]:\ u'\in AC[0,1],\ u''\in L^2[0,1],\ u(0)=0\}.$$

The inner product and the norm in $W_2^2[0,1]$ are defined by

$$\langle u,h\rangle_{W_2^2} = \sum_{i=0}^{1}u^{(i)}(0)h^{(i)}(0)+\int_0^1 u''(\nu)h''(\nu)\mathrm{d}\nu, \quad u,h\in W_2^2[0,1]$$

and

$$\|u\|_{W_2^2} = \sqrt{\langle u,u\rangle_{W_2^2}}, \quad u\in W_2^2[0,1].$$

Lemma 2 (See [23, page 148])**.** *The space $W_2^2[0,1]$ is a reproducing kernel space, and its reproducing kernel function R_y is given by*

$$R_y(\nu) = \begin{cases} \nu y + \frac{1}{2}\nu^2 y - \frac{1}{6}\nu^3, & 0 \le \nu \le y \le 1, \\ y\nu + \frac{1}{2}y^2\nu - \frac{1}{6}y^3, & 0 \le \nu < y \le 1. \end{cases}$$

4. Solution representation in $W_2^2[0,1]$

In this section, the solution of (1)–(2) is presented in the $W_2^2[0,1]$. On defining the linear operator $L : W_2^2[0,1] \to G_2^1[0,1]$ by

$$Lu =_C D_{0,\nu}^{\alpha(\nu)} u(\nu), \quad 0 \le \nu \le T, \quad u \in W_2^2[0,1], \tag{4}$$

model problem (1)–(2) changes to the problem

$$\begin{cases} Lu = f(\nu), & \nu \in [0,T], \\ u(0) = 0. \end{cases} \tag{5}$$

Theorem 1. *The linear operator L is a bounded linear operator.*

Proof. We need to show $\|Lu\|^2_{G_2^1} \le M \|u\|^2_{W_2^2}$, where $M > 0$ is a positive constant. We get

$$\|Lu\|^2_{G_2^1} = \langle Lu, Lu\rangle_{G_2^1} = [Lu(0)]^2 + \int_0^1 \left[Lu'(\nu)\right]^2 \mathrm{d}\nu.$$

We obtain

$$u(\nu) = \langle u(\cdot), R_\nu(\cdot)\rangle_{W_2^2},$$

and

$$Lu(\nu) = \langle u(\cdot), LR_\nu(\cdot)\rangle_{W_2^2},$$

by reproducing property. Therefore, we get

$$|Lu(\nu)| \le \|u\|_{W_2^2} \|LR_\nu\|_{W_2^2} = M_1 \|u\|_{W_2^2},$$

where $M_1 > 0$. Therefore, we get

$$[(Lu)(0)]^2 \le M_1^2 \|u\|^2_{W_2^2}.$$

Since

$$(Lu)'(\nu) = \left\langle u(\cdot), (LR_\nu)'(\cdot)\right\rangle_{W_2^2},$$

then

$$\left|(Lu)'(\nu)\right| \le \|u\|_{W_2^2} \left\|(LR_\nu)'\right\|_{W_2^2} = M_2 \|u\|_{W_2^2},$$

where $M_2 > 0$. Therefore, we obtain

$$\left[(Lu)'(x)\right]^2 \le M_2^2 \|u\|^2_{W_2^2},$$

and

$$\int_0^1 \left[(Lu)'(\nu)\right]^2 \mathrm{d}\nu \le M_2^2 \|u\|^2_{W_2^2}.$$

Thus, we get

$$\begin{aligned} \|Lu\|^2_{G_2^1} &\le [(Lu)(0)]^2 + \int_0^1 \left(\left[(Lu)'(\nu)\right]^2\right) \mathrm{d}\nu \\ &\le \left(M_1^2 + M_2^2\right) \|u\|^2_{W_2^2} = M \|u\|^2_{W_2^2}, \end{aligned}$$

where $M = M_1^2 + M_2^2 > 0$ is a positive constant. □

5. The main results

Put $\varphi_i(\nu) = Q_{\nu_i}(\nu)$ and $\psi_i(\nu) = L^*\varphi_i(\nu)$, where L^* is conjugate operator of L. The orthonormal system $\left\{\widehat{\Psi}_i(\nu)\right\}_{i=1}^\infty$ of $W_2^2[0,1]$ can be obtained from Gram-Schmidt orthogonalization process of $\{\psi_i(\nu)\}_{i=1}^\infty$,

$$\widehat{\psi}_i(\nu) = \sum_{k=1}^{i} \beta_{ik}\psi_k(\nu), \quad (\beta_{ii} > 0, \quad i = 1, 2, \ldots). \tag{6}$$

Theorem 2. *Let $\{\nu_i\}_{i=1}^\infty$ be dense in $[0,1]$ and $\psi_i(\nu) = L_y R_\nu(y)|_{y=\nu_i}$. Then the sequence $\{\psi_i(\nu)\}_{i=1}^\infty$ is a complete system in $W_2^2[0,1]$.*

Proof. We obtain

$$\begin{aligned} \psi_i(\nu) &= (L^*\varphi_i)(\nu) = \langle (L^*\varphi_i)(y), R_\nu(y)\rangle \\ &= \langle (\varphi_i)(y), L_y R_\nu(y)\rangle = L_y R_\nu(y)|_{y=\nu_i}. \end{aligned}$$

Let $\langle u(\nu), \psi_i(\nu)\rangle = 0$, $(i = 1, 2, \ldots)$, which means that,

$$\langle u(\nu), (L^*\varphi_i)(\nu)\rangle = \langle Lu(\cdot), \varphi_i(\cdot)\rangle = (Lu)(\nu_i) = 0.$$

$\{\nu_i\}_{i=1}^{\infty}$ is dense in $[0,1]$. Therefore, $(Lu)(\nu) = 0$. $u \equiv 0$ by L^{-1}. □

Theorem 3. *If $u(\nu)$ is the exact solution of (1), then we obtain*

$$u(\nu) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(\nu_k)\widehat{\psi}_i(\nu). \tag{7}$$

where $\{\nu_i\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof. We obtain

$$\begin{aligned}
u(\nu) &= \sum_{i=1}^{\infty} \left\langle u(\nu), \widehat{\psi}_i(\nu) \right\rangle_{W_2^2} \widehat{\psi}_i(\nu) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(\nu), \psi_k(\nu) \right\rangle_{W_2^2} \widehat{\psi}_i(\nu) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle u(\nu), L^*\varphi_k(\nu) \right\rangle_{W_2^2} \widehat{\psi}_i(\nu) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle Lu(\nu), \varphi_k(\nu) \right\rangle_{G_2^1} \widehat{\psi}_i(\nu) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle f(\nu), Q_{\nu_k} \right\rangle_{G_2^1} \widehat{\psi}_i(\nu) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} f(\nu_k)\widehat{\psi}_i(\nu).
\end{aligned}$$

by (6) and uniqueness of solution of (1). This completes the proof. □

The approximate solution $u_n(\nu)$ can be acquired as:

$$u_n(\nu) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} f(\nu_k)\widehat{\Psi}_i(\nu). \tag{8}$$

6. Numerical results

To prove the efficiency and the practicability of the RKM, we give an example and find its solution.

Example 1. *Let us consider Eq. (1) at $T = 1$ with*

$$f(\nu) = \frac{3\nu^{1-\alpha(\nu)}}{\Gamma(2-\alpha(\nu))} + \frac{2\nu^{2-\alpha(\nu)}}{\Gamma(3-\alpha(\nu))} \tag{9}$$

for variable order $0 < \alpha(\nu) < 1$, one can obtain the exact solution as $u(\nu) = 3\nu + \nu^2$. Numerical results are shown in the Table 1.

Table 1. The comparisons between the RKM and the method given in [24] at T = 1 with CPU time(s)=9.469.

$\alpha(\nu)$	ν	[24]	RKM
$\nu/2$	1/4	$5.9851e-003$	$7.36735e-005$
$\nu/2$	1/8	$1.4262e-003$	$3.35160e-006$
$\nu/2$	1/16	$3.4719e-004$	$5.83687e-005$
$\nu/2$	1/32	$8.5572e-005$	$1.44027e-004$
$\nu/2$	1/64	$2.1234e-005$	$6.39753e-004$
$sin(\nu)$	1/4	$2.4701e-002$	$3.34572e-004$
$sin(\nu)$	1/8	$5.6021e-003$	$1.54293e-005$
$sin(\nu)$	1/16	$1.3335e-003$	$9.04701e-005$
$sin(\nu)$	1/32	$3.2530e-004$	$1.70832e-004$
$sin(\nu)$	1/64	$8.0317e-005$	$6.50846e-004$

7. Conclusion

We used the reproducing kernel method to solve a class of the variable order fractional differential equation in this work. We defined the method and used it in the test example in order to prove its applicability and validity in comparison with exact and other numerical solutions. The obtained results are uniformly convergent and the operator that was used is a bounded linear operator.

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