An International Journal of Optimization and Control: Theories & Applications
ISSN:2146-0957 eISSN:2146-5703
Vol.7, No.1, pp.83-89 (2017)
http://doi.org/10.11121/ijocta.01.2017.00338

RESEARCH ARTICLE

Canal surfaces in 4-dimensional Euclidean space

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ARTICLE INFO

Article History:
Received 26 April 2016
Accepted 22 November 2016
Available 13 December 2016

Keywords:
Canal surface
Curvature ellipse
Superconformal surface

AMS Classification 2010:
53C40, 53C42

ABSTRACT

In this paper, we study canal surfaces imbedded in 4-dimensional Euclidean space $\mathbb{E}^4$. We investigate these surface curvature properties with respect to the variation of the normal vectors and ellipse of curvature. Some special canal surface examples are constructed in $\mathbb{E}^4$. Furthermore, we obtain necessary and sufficient condition for canal surfaces to become superconformal in $\mathbb{E}^4$. At the end, we present the graphs of projections of canal surfaces in $\mathbb{E}^3$.

Given a space curve $\gamma(u)$ called spine curve, a canal surface associated to this curve is defined as a surface swept by a family of spheres of varying radius $r(u)$. If $r(u)$ is constant, the canal surface is called a tube or a pipe surface. Apart from being used in pure mathematics, canal surfaces are widely used in many areas especially in CAGD, e.g. construction of blending surfaces, i.e. canal surface with a rational radius, shape reconstruction or robotic path planning (see, [5], [11], [12]). Greater part of the studies on canal surfaces within the CAGD context is related to the search of canal surfaces with rational spine curve and rational radius function. Canal surfaces are also useful in visualising long thin objects such as poles, 3D fonts, brass instruments or internal organs of the body in solid/surface modeling and CG/CAD. A national question is when the canal surface is developable. It is well known that, at regular points, the Gaussian curvature of a developable surface is identically zero. In [14] it has been proved that developable canal surface is either a cylinder or a cone.

This study consists of 5 sections: In section 2, we explain some well-known properties of the surfaces in $\mathbb{E}^4$. In section 3, we give the canal surfaces in $\mathbb{E}^4$ and some examples are presented. Section 4 investigates the ellipse of curvature of canal surfaces in $\mathbb{E}^4$. Additionally we prove necessary and sufficient condition of canal surfaces to become superconformal in $\mathbb{E}^4$. In Section 5, the visualization of canal surfaces are given with using Maple programme.

1. Basic concepts

Let M be a regular surface in $\mathbb{E}^4$ given with the parametrization $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space of M at an arbitrary point $p = X(u,v)$ is spanned by the vectors X_u and X_v. The first fundamental form coefficients of M are computed by

$$E = \langle X_u, X_u \rangle, F = \langle X_u, X_v \rangle, G = \langle X_v, X_v \rangle, \quad (1)$$

where $\langle , \rangle$ is the scalar product of the Euclidean space. We consider the surface patch $X(u,v)$ is regular, which implies that $W^2 = EG - F^2 \neq 0$. For the point $p \in M$, we can take the decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^{\perp}M$, where $T_p^{\perp}M$ is the

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orthogonal component of T_pM in $\mathbb{E}^4$ with the Riemannian connection $\widetilde{\nabla}$.

The induced Riemannian connection ∇ on M for any given local vector fields X_1, X_2 tangent to M, is given by

$$\nabla_{X_1} X_2 = (\widetilde{\nabla}_{X_1} X_2)^T, \tag{2}$$

where T expresses the tangential part.

Let us consider the spaces of the smooth vector fields $\chi(M)$ and $\chi^{\perp}(M)$ which are tangent and normal to M, respectively. The second fundamental map is defined as follows:

$$\begin{aligned} h &: \chi(M) \times \chi(M) \to \chi^{\perp}(M) \\ h(X_i, X_j) &= \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j \quad 1 \leq i, j \leq 2. \end{aligned} \tag{3}$$

This map is well-defined, symmetric and bilinear.

If we take the orthonormal frame field $\{N_1, N_2\}$ of M, then the shape operator which is self-adjoint and bilinear can be given by

$$\begin{aligned} A &: \chi^{\perp}(M) \times \chi(M) \to \chi(M) \\ A_{N_i} X_i &= -(\widetilde{\nabla}_{X_i} N_i)^T, \quad X_i \in \chi(M) \end{aligned} \tag{4}$$

which satisfies the equation:

$$\langle A_{N_k} X_j, X_i \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \ 1 \leq i, j, k \leq 2 \tag{5}$$

for any X_1, $X_2 \in T_pM$.

The equality (3) is known as the Gaussian equation, where

$$\nabla_{X_i} X_j = \sum_{k=1}^{2} \Gamma_{ij}^k X_k \ , \quad 1 \leq i, j \leq 2 \tag{6}$$

and

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k \qquad 1 \leq i, j \leq 2. \tag{7}$$

Here Γ_{ij}^k are Christoffel symbols and c_{ij}^k are the coefficients of the second fundamental form.

The Gaussian curvature are given by

$$K = \frac{\langle h(X_1, X_1), h(X_2, X_2) \rangle - \| h(X_1, X_2) \|^2}{g} \tag{8}$$

and the mean curvature are given by

$$\| H \| = \frac{1}{4g^2} \langle h(X_1, X_1) + h(X_2, X_2), h(X_1, X_1) + h(X_2, X_2) \rangle \tag{9}$$

where

$$g = \| X_1 \|^2 \| X_2 \|^2 - \langle X_1, X_2 \rangle^2 .$$

If the mean curvature of M vanishes identically in $\mathbb{E}^n$, then M is said to be minimal [3]. See also [1].

2. Canal surfaces in $\mathbb{E}^4$

Let $\gamma(u) = (f_1(u), f_2(u), f_3(u), 0)$ be a curve given with arclength parameter. Then the Frenet formulae have the following form:

$$\begin{aligned} \gamma'(u) &= e_1(u), \\ e_1{}'(u) &= \kappa(u) e_2(u), \\ e_2{}'(u) &= -\kappa(u) e_1(u) + \tau(u) e_3(u), \\ e_3{}'(u) &= -\tau(u) e_2(u), \\ e_4{}'(u) &= 0, \end{aligned} \tag{10}$$

where $\{e_1(u), e_2(u), e_3(u), e_4(u)\}$ is the Frenet orthonormal basis of γ. The canal surface in $\mathbb{E}^4$ has the following parametrization (see [6]):

$$M : X(u, v) = \gamma(u) + r(u) \left(e_3(u) \cos v + e_4(u) \sin v \right). \tag{11}$$

Example 1. *Consider the helix* $\gamma(u) = (a \cos \frac{u}{c}, a \sin \frac{u}{c}, \frac{bu}{c})$ *in* $\mathbb{E}^3$*. Then the canal surface of* γ *in* $\mathbb{E}^4$ *has the following parametrization*

$$\begin{aligned} X(u, v) = \ & (a \cos \frac{u}{c} + \frac{b}{c} r(u) \sin \frac{u}{c} \cos v, \\ & a \sin \frac{u}{c} - \frac{b}{c} r(u) \cos \frac{u}{c} \cos v, \\ & \frac{bu}{c} + \frac{a}{c} r(u) \cos v, r(u) \sin v). \end{aligned} \tag{12}$$

Example 2. *Consider the generalized helix* $\gamma(u) = (\frac{(1+u)^{\frac{3}{2}}}{3}, \frac{(1-u)^{\frac{3}{2}}}{3}, \frac{u}{\sqrt{2}})$ *in* $\mathbb{E}^3$*. Then the canal surface of* γ *in* $\mathbb{E}^4$ *has the following parametrization*

$$\begin{aligned} X(u, v) = \ & (\frac{(1+u)^{\frac{3}{2}}}{3} - r(u) \frac{(1+u)^{\frac{1}{2}}}{2} \cos v, \\ & \frac{(1-u)^{\frac{3}{2}}}{3} + r(u) \frac{(1-u)^{\frac{1}{2}}}{2} \cos v, \\ & \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} r(u) \cos v, r(u) \sin v). \end{aligned} \tag{13}$$

The space which is tangent to M is spanned by

$$\begin{aligned} X_u &= e_1(u) - r\tau\cos v e_2 \\ &\quad + r'\cos v e_3 + r'\sin v e_4, \\ X_v &= -r\sin v e_3 + r\cos v e_4. \end{aligned} \tag{14}$$

The first fundamental form coefficients become

$$\begin{aligned} E &= 1 + (r')^2 + r^2\tau^2\cos^2 v, \\ F &= 0, \\ G &= r^2. \end{aligned} \tag{15}$$

The Christoffel symbols Γ_{ij}^k are given by

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2E}\partial_u(E) = \frac{1}{E}\langle X_{uu}, X_u\rangle, \\ \Gamma_{11}^2 &= -\frac{1}{2G}\partial_v(E) = -\frac{1}{G}\langle X_{vu}, X_u\rangle, \\ \Gamma_{12}^1 &= \frac{1}{2E}\partial_v(E) = \frac{1}{E}\langle X_{vu}, X_u\rangle, \\ \Gamma_{12}^2 &= \frac{1}{2G}\partial_u(G) = \frac{1}{G}\langle X_{vu}, X_v\rangle, \\ \Gamma_{22}^1 &= -\frac{1}{2E}\partial_u(G) = -\frac{1}{E}\langle X_{vu}, X_v\rangle, \\ \Gamma_{22}^2 &= \frac{1}{2G}\partial_v(G) = \frac{1}{G}\langle X_{vv}, X_v\rangle = 0. \end{aligned} \tag{16}$$

and they are symmetric according to the covariant indices ([7], p.398).
If we take the second partial derivatives of $X(u,v)$, we find:

$$\begin{aligned} X_{uu} &= \kappa r\tau\cos v e_1 + (\kappa - (r\tau)'\cos v - r'\tau\cos v)e_2 \\ &\quad + \cos v(r'' - r\tau^2)e_3 + r''\sin v e_4, \\ X_{uv} &= r\tau\sin v e_2 - r'\sin v e_3 + r'\cos v e_4, \\ X_{vv} &= -r\cos v e_3 - r\sin v e_4, \end{aligned} \tag{17}$$

Hence, by using (3), we find the Gaussian equations;

$$\begin{aligned} \widetilde{\nabla}_{X_u}X_u &= X_{uu} = \nabla_{X_u}X_u + h(X_u, X_u), \\ \widetilde{\nabla}_{X_u}X_v &= X_{uv} = \nabla_{X_u}X_v + h(X_u, X_v), \\ \widetilde{\nabla}_{X_v}X_v &= X_{vv} = \nabla_{X_v}X_v + h(X_v, X_v), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \nabla_{X_u}X_u &= \Gamma_{11}^1 X_u + \Gamma_{11}^2 X_v, \\ \nabla_{X_u}X_v &= \Gamma_{12}^1 X_u + \Gamma_{12}^2 X_v, \\ \nabla_{X_v}X_v &= \Gamma_{22}^1 X_u + \Gamma_{22}^2 X_v. \end{aligned} \tag{19}$$

Substituting (16) and (18) in (19), we obtain

$$\begin{aligned} h(X_u, X_u) &= X_{uu} - \frac{1}{E}\langle X_{uu}, X_u\rangle X_u \\ &\quad + \frac{1}{G}\langle X_{uv}, X_u\rangle X_v, \\ h(X_u, X_v) &= X_{uv} - \frac{1}{E}\langle X_{uv}, X_u\rangle X_u \\ &\quad - \frac{1}{G}\langle X_{uv}, X_v\rangle X_v, \\ h(X_v, X_v) &= X_{vv} + \frac{1}{E}\langle X_{uv}, X_v\rangle X_u. \end{aligned} \tag{20}$$

Further using (20)

$$\begin{aligned} \langle h(X_u,X_u), h(X_v,X_v)\rangle &= \langle X_{uu}, X_{vv}\rangle \\ &\quad - \frac{1}{E}\langle X_{uu}, X_u\rangle\langle X_{vv}, X_u\rangle, \\ \langle h(X_u,X_v), h(X_u,X_v)\rangle &= \langle X_{uv}, X_{uv}\rangle \\ &\quad - \frac{1}{E}\langle X_{uv}, X_u\rangle^2 \\ &\quad - \frac{1}{G}\langle X_{vu}, X_v\rangle^2, \\ \langle h(X_u,X_v), h(X_v,X_v)\rangle &= \langle X_{uv}, X_{vv}\rangle \\ &\quad - \frac{1}{E}\langle X_{uv}, X_u\rangle\langle X_{vv}, X_u\rangle, \\ \langle h(X_u,X_u), h(X_u,X_u)\rangle &= \langle X_{uu}, X_{uu}\rangle \\ &\quad - \frac{1}{E}\langle X_{uu}, X_u\rangle^2 \\ &\quad + \frac{\langle X_{uv}, X_u\rangle}{G}(2\langle X_{uu}, X_v\rangle \\ &\quad + \langle X_{uv}, X_u\rangle), \\ \langle h(X_v,X_v), h(X_v,X_v)\rangle &= \langle X_{vv}, X_{vv}\rangle \\ &\quad + \frac{1}{E}\langle X_{uv}, X_v\rangle(1 + 2\langle X_{vv}, X_u\rangle), \\ \langle h(X_u,X_u), h(X_u,X_v)\rangle &= \langle X_{uu}, X_{uv}\rangle - \frac{1}{E}\langle X_{uv}, X_u\rangle \\ &\quad - \frac{1}{G}\langle X_{uu}, X_v\rangle\langle X_{uv}, X_v\rangle \end{aligned} \tag{21}$$

Thus, using (14) with (17) we get

$$\begin{aligned} \langle X_{uu}, X_{vv}\rangle &= r^2\tau^2\cos^2 v - rr'', \\ \langle X_{uv}, X_{uv}\rangle &= r^2\tau^2\sin^2 v + (r')^2, \\ \langle X_{uu}, X_{uu}\rangle &= (\kappa r\tau\cos v)^2 \\ &\quad + (\kappa - (r\tau)'\cos v - r'\tau\cos v)^2 + \\ &\quad + \cos^2 v(r'' - r\tau^2)^2 + (r'')^2\sin^2 v, \\ \langle X_{vv}, X_{vv}\rangle &= r^2, \\ \langle X_{uu}, X_u\rangle &= r\tau(r\tau)'\cos^2 v + r'r'', \\ \langle X_{uu}, X_v\rangle &= r^2\tau^2\cos v\sin v, \\ \langle X_{vv}, X_u\rangle &= -rr', \\ \langle X_{uv}, X_u\rangle &= -r^2\tau^2\cos v\sin v, \\ \langle X_{uv}, X_v\rangle &= rr', \\ \langle X_{uv}, X_{vv}\rangle &= 0 \\ \langle X_{uu}, X_{uv}\rangle &= r\tau\sin v(\kappa - (r\tau)'\cos v) \end{aligned} \tag{22}$$

Proposition 1. *The Gaussian curvature of the canal surface M with the parametrization (11) in $\mathbb{E}^4$ is given by*

$$\begin{aligned} K &= \frac{1}{g}(\langle X_{uu}, X_{vv}\rangle - \frac{1}{E}\langle X_{uu}, X_u\rangle\langle X_{vv}, X_u\rangle \\ &\quad - \langle X_{uv}, X_{uv}\rangle + \frac{1}{E}\langle X_{uv}, X_u\rangle^2 + \frac{1}{G}\langle X_{uv}, X_v\rangle^2) \end{aligned} \tag{23}$$

where $g = EG - F^2$.

Proof. By using the equation (8), we find

$$K = \frac{1}{g}\left(\langle h(X_u,X_u), h(X_v,X_v)\rangle - \langle h(X_u,X_v), h(X_u,X_v)\rangle\right), \tag{24}$$

which is the Gaussian curvature of the canal surface M. Taking into account (21) and (24) we obtain (23). □

From the equations (22) with (23) we obtain;

Corollary 1. *The Gaussian curvature of the canal surface M with the parametrization (11) in $\mathbb{E}^4$ is given by*

$$\begin{aligned} K &= \frac{r}{gE}\{r\cos^2 v(2\tau^2+2(r')^2\tau^2-rr''\tau^2+r'\tau(r\tau)') \\ &\quad +r^3\tau^4\cos^4 v-r''-r\tau^2(1+(r')^2)\}, \end{aligned} \tag{25}$$

where

$$\begin{aligned} E &= 1+(r')^2+r^2\tau^2\cos^2 v, \\ g &= r^2(1+(r')^2+r^2\tau^2\cos^2 v). \end{aligned}$$

Proposition 2. *The mean curvature of the canal surface M with the parametrization (11) in $\mathbb{E}^4$ is given by*

$$\begin{aligned} 4\|H\|^2 &= \frac{\langle X_{uu},X_{uu}\rangle}{E^2}+2\frac{\langle X_{uu},X_{vv}\rangle}{EG}+\frac{\langle X_{vv},X_{vv}\rangle}{G^2} \\ &\quad +\frac{\langle X_{uv},X_v\rangle}{EG^2}(2\langle X_{vv},X_u\rangle+\langle X_{uv},X_v\rangle) \\ &\quad +\frac{\langle X_{uv},X_u\rangle}{E^2G}(2\langle X_{uu},X_v\rangle+\langle X_{uv},X_u\rangle) \\ &\quad -\frac{2}{E^2G}\langle X_{uu},X_u\rangle\langle X_{vv},X_u\rangle-\frac{\langle X_{uu},X_u\rangle^2}{E^3}. \end{aligned} \tag{26}$$

Proof. By considering (9) the mean curvature of the canal surface M becomes

$$\|H\| = \frac{1}{4g^2}\left(\langle h(X_u,X_u)+h(X_v,X_v), h(X_u,X_u)+h(X_v,X_v)\rangle\right), \tag{27}$$

Taking into account (21) and (27) we get the result. □

By the use of (22) and Proposition 2, we have the following results:

Corollary 2. *The mean curvature of the canal surface M with the parametrization (11) in $\mathbb{E}^4$ is given by*

$$\begin{aligned} \|H\|^2 &= \frac{1}{4E^2r^2}[-\frac{r^2}{E}(r\tau(r\tau)'\cos^2 v+r'r'')^2 \\ &\quad +r^2\cos^2 v((\tau kr)^2+ \\ &\quad ((r\tau)'+r'\tau)^2-r^2\tau^4+4\tau^2+ \\ &\quad +3(r')^2\tau^2-2r\tau^2r''+2r'\tau(r\tau)') \\ &\quad +4r^4\tau^4\cos^4 v-2kr^2\cos v((r\tau)'+r'\tau)+ \\ &\quad +k^2r^2-2rr''+1+(r')^2]. \end{aligned}$$

Corollary 3. *If the base curve γ of the canal surface M is a straight line, then the Gaussian and mean curvatures of M are*

$$K = \frac{-r''}{r(1+(r')^2)^2},$$

and

$$\begin{aligned} \|H\|^2 &= \frac{-1}{4r^2(1+(r')^2)^3}\{(rr'r'')^2 \\ &\quad +(2rr''-1)(1+(r')^2)\}, \end{aligned}$$

respectively.

3. Ellipse of curvature of the canal surfaces in $\mathbb{E}^4$

Let M be a regular surface given with the parametrization $X(u,v):(u,v)\in\mathbb{D}\subseteq\mathbb{E}^2$. Consider a circle given with the angle $\theta\in[0,2\pi]$ in the tangent space T_pM. The intersection of the direct sum of the tangent direction of $X=\cos\theta X_1+\sin\theta X_2$ and the normal space $T_p^{\perp}M$ with the surface M forms a curve. Such a curve is called as a normal section curve in the direction θ. Denote this curve by γ_θ. Normal curvature vector η_θ of γ_θ lies in $T_p^{\perp}M$. When θ changes from 0 to 2π, the normal curvature vector constitutes an ellipse called as a ellipse of curvature of M at p in $T_p^{\perp}M$. Thus, the curvature ellipse of M at point p is given as follows with the second fundamental form h:

$$E(p)=\{h(X,X)\mid X\in T_pM,\ \|X\|=1\}.$$

To see that this shows an ellipse, it is enough to have a look at the formulas

$$X=\cos\theta X_1+\sin\theta X_2$$

and

$$h(X,X)=\overrightarrow{H}+\cos 2\theta\overrightarrow{B}+\sin 2\theta\overrightarrow{C}. \tag{28}$$

Here,

$$\overrightarrow{B}=\frac{1}{2}(h(X_1,X_1)-h(X_2,X_2)), \overrightarrow{C}=h(X_1,X_2), \tag{29}$$

are normal vectors and $\overrightarrow{H}=\frac{1}{2}(h(X_1,X_1)+h(X_2,X_2))$ is the mean curvature vector. This implies that, the vector $h(X,X)$ goes twice around the ellipse of curvature centered at $\overrightarrow{H}$, while X goes once around the unit tangent circle [9].

From the equation (28), one can get that $E(p)$ is a circle if and only if for some orthonormal basis of $T_p(M)$ it holds that

$$\langle h(X_1,X_2), h(X_1,X_1)-h(X_2,X_2)\rangle=0, \tag{30}$$

and

$$\|h(X_1,X_1)-h(X_2,X_2)\|=2\|h(X_1,X_2)\|. \tag{31}$$

General aspects of the ellipse of curvature for surfaces in $\mathbb{E}^4$ studied by Wong [13]. (See also [2], [8], [9] and [10])

Definition 1. *The surface M with the parametrization $X(u,v)$ in $\mathbb{E}^4$ is superconformal if and only if its ellipse of curvature is a circle, i.e. $\left\langle\overrightarrow{B},\overrightarrow{C}\right\rangle=0$ and $\left\|\overrightarrow{B}\right\|=\left\|\overrightarrow{C}\right\|$ holds* [4]. *If the equality $\left\langle\overrightarrow{B},\overrightarrow{C}\right\rangle=0$, the surface M is called weak superconformal.*

Theorem 1. *The canal surface M with the parametrization (11) in $\mathbb{E}^4$ is superconformal if and only if the equalities*

$$\langle\frac{1}{E}h(X_u,X_u)-\frac{1}{G}h(X_v,X_v), \frac{1}{\sqrt{EG}}h(X_u,X_v)\rangle=0 \tag{32}$$

and

$$2\left\|\frac{1}{\sqrt{EG}}h(X_u,X_v)\right\| = \left\|\frac{1}{E}h(X_u,X_u) - \frac{1}{G}h(X_v,X_v)\right\| \quad (33)$$

hold.

Proof. If we use the orthonormal frame

$$X_1 = \frac{X_u}{\|X_u\|} = \frac{X_u}{\sqrt{E}}, X_2 = \frac{X_v}{\|X_v\|} = \frac{X_v}{\sqrt{G}}, \quad (34)$$

we get

$$\begin{aligned} h(X_1,X_1) &= \frac{1}{E}h(X_u,X_u), \\ h(X_1,X_2) &= \frac{1}{\sqrt{EG}}h(X_u,X_v), \\ h(X_2,X_2) &= \frac{1}{G}h(X_v,X_v)\ . \end{aligned} \quad (35)$$

Therefore, from (29) the normal vectors $\overrightarrow{B}$ and $\overrightarrow{C}$ become

$$\overrightarrow{B} = \frac{1}{2}(\frac{1}{E}h(X_u,X_u) - \frac{1}{G}h(X_v,X_v)) \quad (36)$$

and

$$\overrightarrow{C} = \frac{1}{\sqrt{EG}}h(X_u,X_v). \quad (37)$$

Suppose M is superconformal then by Definition 1 $\langle\overrightarrow{B},\overrightarrow{C}\rangle = 0$ and $\left\|\overrightarrow{B}\right\| = \left\|\overrightarrow{C}\right\|$ hold. Thus by the use of the equalities (36) and (37) we get the result.

Conversely, if the equations (32) and (33) hold then by the use of the equalities (36) and (37) we obtain $\langle\overrightarrow{B},\overrightarrow{C}\rangle = 0$ and $\left\|\overrightarrow{B}\right\| = \left\|\overrightarrow{C}\right\|$, which shows that M is superconformal. □

Substituting (21) and (22) into (32) we obtain the following results.

Corollary 4. *Let M be a canal surface in $\mathbb{E}^4$ given with the parametrization (11). Then M is weak superconformal if and only if the equality*

$$\begin{aligned} 0 &= r^3\tau\sin v((k-(r\tau)'\cos v)(1+(r')^2) \\ &\quad +r\tau\cos v(r'r''+kr\tau\cos v)) \end{aligned}$$

holds.

Corollary 5. *Every canal surface whose spine curve is a straight line of the form $\gamma(u) = (a_1u + b_1, a_2u + b_2, a_3u + b_3, 0)$ is weak superconformal, where a_1, a_2, a_3, b_1, b_2, b_3 are real constants.*

4. Visualization

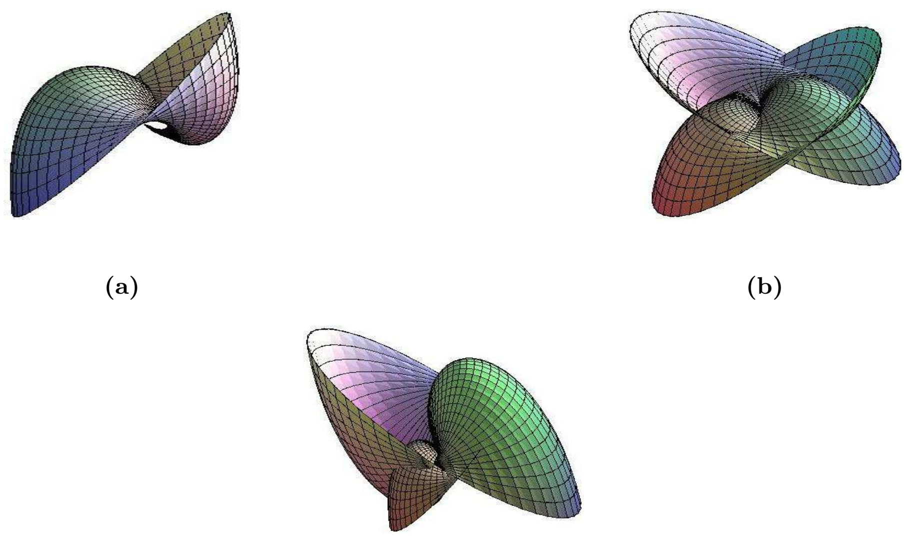
The 3D-surfaces geometric modeling are very important in the surface modeling systems such as; CAD/CAM systems and NC-processing. We give the visualization of the surfaces with the parametrization

$$X(u,v) = (x(u,v), y(u,v), z(u,v), w(u,v))$$

in $\mathbb{E}^4$ by use of Maple Software Program. We plot the graph of the surface with plotting command

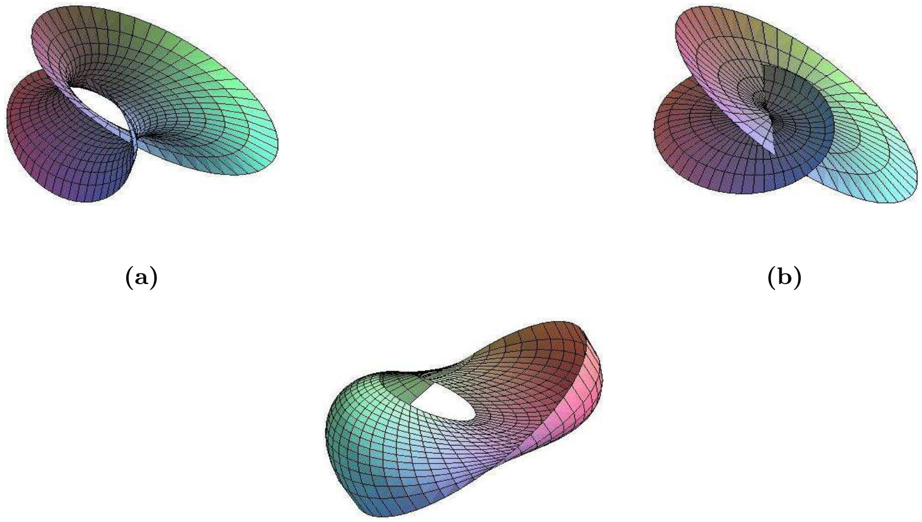
$$plot3d([x,y,z+w], u = a..b, v = c..d). \quad (38)$$

We construct the geometric model of the canal surfaces defined in Example 1 for the following values (see, Figure 1);



(a) **(b)** **(c)**

Figure 1. The projections of canal surfaces of helix in $\mathbb{E}^3$



(a) (b) (c)

Figure 2. The projections of canal surfaces of general helix in $\mathbb{E}^3$

(a) (b)

Figure 3. The projections of canal surfaces of straight line in $\mathbb{E}^3$

$$\begin{aligned} a)\ r(u) &= e^{u/3}, \\ b)\ r(u) &= u^2, \\ c)\ r(u) &= 3u+5. \end{aligned}$$

Further, we construct the geometric model of the canal surfaces defined in Example 2 for the following values (see, Figure 2);

$$\begin{aligned} a)\ r(u) &= e^{u^2}, \\ b)\ r(u) &= 5u^2, \\ c)\ r(u) &= 3u+5. \end{aligned}$$

Additionally, we construct the geometric model of the canal surfaces defined in Corollary 3 for the following values (see, Figure 3);

$$\begin{aligned} a)\ r(u) &= e^{u}, \\ b)\ r(u) &= \sinh u. \end{aligned}$$

5. Conclusion

In this manuscript, we considered canal surfaces in the 4-dimensional Euclidean space $\mathbb{E}^4$. Most of the literature on canal surfaces within the CAGD context has been motivated by the observation that canal surfaces with rational spine curve. We have proved this property mathematically and also illustrated with some nice examples.

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An International Journal of Optimization and Control: Theories & Applications (http://ijocta.balikesir.edu.tr)