

RESEARCH ARTICLE

## Brezzi-Pitkaranta stabilization and a priori error analysis for the Stokes control

Aytekin Cıvık\*, Fikriye Yılmaz

*Department of Mathematics, Gazi University, Turkey  
aytechin@gmail.com, yfikriye@gmail.com*

---

### ARTICLE INFO

#### Article History:

*Received 5 March 2016*

*Accepted 22 August 2016*

*Available 12 December 2016*

#### Keywords:

*Finite element*

*Brezzi-Pitkaranta stabilization*

*Optimal control*

*Stokes equations*

AMS Classification 2010:

*65K10, 65K15, 76D07*

### ABSTRACT

In this study, we consider a Brezzi-Pitkaranta stabilization scheme for the optimal control problem governed by stationary Stokes equation, using a P1-P1 interpolation for velocity and pressure. We express the stabilization as extra terms added to the discrete variational form of the problem. We first prove the stability of the finite element discretization of the problem. Then, we derive a priori error bounds for each variable and present a numerical example to show the effectiveness of the stabilization clearly.



## 1. Introduction

This study deals with the optimal control problem of the stationary Stokes equation. Numerical solution of the Stokes equation needs some extra caution due to the coupling of velocity and pressure. Finite element methods are mostly used for the solution of Stokes equation but inaccurate pressure singularities are encountered unless some stable finite element pairs are used for the standard Galerkin finite element approach. Stable finite element pairs are chosen as they satisfy the so called inf-sup condition to overcome such problems. This condition, in particular, does not allow the use of simple interpolations like equal order ones, which are desirable from a computational view point [6]. Thus, if one uses such simple finite element pairs, a pressure stabilization mechanism must be cast to the system in order to avoid pressure singularities.

In most of these stabilized methods, one adds some extra terms to the discrete variational form of the problems to ensure stability. The first

and most well-known stabilization technique applied on a Stokes system is the Brezzi-Pitkaranta method [8], which is considered in this study to stabilize the optimal control problem. This method adds a weighted Laplace operator on the pressure space, which results in an optimally convergent scheme for equal order finite element approximations [5]. Some other popular methods are GLS method [13], SUPG method along with PSPG method [9], the Douglas-Wang method [12], bubble function method [4], Pressure Gradient Projection methods [11] and VMS methods [20]. Some of these stabilization methods are transferred to the Navier-Stokes systems as tested in Stokes flows and demonstrate a good success. Thus, Stokes flow problems bears a great importance as it plays a role as a test bed for more complicated and convective problems such as uncoupled and coupled Navier-Stokes systems. Making use of the Brezzi-Pitkaranta stabilization for the control of the Stokes equation is advantageous

---

\*Corresponding Author

since it doesn't require numerical residual explicitly as in GLS methods and thus, strong differential formulation of the problem is not needed. The Brezzi-Pitkaranta stabilization technique also will not require extra regularity conditions as in some well-known stabilization procedures [19].

There are various studies in the literature concerning the optimal control of flow problems. In [25], a discontinuous galerkin finite element method (DG) with interior penalties for the optimal control problem of the convection-diffusion equation was studied and in [18], an edge stabilized galerkin finite element method for the same optimal control system was considered. Moreover, local error estimates for SUPG solutions of advection-dominated elliptic linear-quadratic optimal control problems was studied in [17]. Similarly, the local (DG) for optimal control problem governed by convection-diffusion equations was analyzed in [27]. In [14], authors presented an analysis concerning the optimal control of fully discretized Stokes equations and a priori error analysis of the same optimal control system are presented in [23]. In most similar studies, researchers choose inf-sup stable finite element pairs for velocity and pressure approximations. The originality of this study comes from the idea of combining the pressure stabilization technique and optimal control problem of Stokes problem with unstable and lower order finite element pairs. To the best of the authors knowledge, this is the first study on optimal control problem of Stokes equations including the Brezzi-Pitkaranta stabilization applied on an equal order finite element interpolations.

In this work, we use Lagrange approach to get the first order optimality conditions. Then, we formulate the discrete optimal control problem. Stabilization terms are added to both weak formulations of the discrete state and adjoint variables. In order to solve the optimal control problem, we use a gradient descent type algorithm. In the numerical example, one can easily see the efficiency of the stabilization for both the state and the adjoint variables.

The organization of the paper is as follows: We first give some notational notes and mathematical preliminaries in order to define the problem and its variational form. Then, we give the finite element discretization of the optimal control problem and prove the stability properties. A priori error analysis of the control problem is proceeded in the following section. We conclude our study with a numerical example.

## 2. Problem Formulation and Optimal Control Problem

In this work, we consider the optimal control problems governed by the Stokes equations. Let  $\Omega$  be a bounded polygonal domain in  $\mathbb{R}^d$ , with  $d = 2$  or  $3$ , and its Lipschitz boundary be  $\Sigma = \partial\Omega$ . Then, we state the distributed control problem as:

$$\begin{aligned} \min J(y, u) &= \frac{1}{2} \|y - y_d\|_{\Omega}^2 + \frac{\beta}{2} \|u\|_{\Omega}^2 & (1) \\ \text{subject to } -\nu\Delta y + \nabla p &= u \text{ in } \Omega, \\ \nabla \cdot y &= 0 \text{ in } \Omega, & (2) \\ y &= 0 \text{ on } \Sigma, \end{aligned}$$

where  $y : \Omega \mapsto \mathbb{R}^d$  is the fluid velocity,  $p : \Omega \mapsto \mathbb{R}$  denotes the pressure and  $u$  is the control variable. The kinematic viscosity is denoted by  $\nu > 0$ . Here,  $\beta > 0$  stands for the regularization parameter and  $y_d$  is the desired state.

We follow the well-known Lagrange approach [21] to get the first order optimality conditions. We let  $\lambda$  denote the adjoint variable that satisfies

$$\begin{aligned} \text{subject to } -\nu\Delta\lambda + \nabla\xi &= y - y_d \text{ in } \Omega, \\ \nabla \cdot \lambda &= 0 \text{ in } \Omega, & (3) \\ \lambda &= 0 \text{ on } \Sigma, \end{aligned}$$

and

$$\beta u + \lambda = 0 \text{ in } \Omega, \quad (4)$$

where  $\xi : \Omega \mapsto \mathbb{R}$ .

We use the standard notations for Sobolev and Lebesgue spaces as in Adams [3] throughout the paper. We denote the velocity space by  $Y = H_0^1(\Omega)$ , the pressure space by  $Q = L_0^2(\Omega)$  and the control space  $U = L^2(\Omega)$ . The usual norm in  $L^2(\Omega)$  is denoted with  $\|\cdot\|$  and the norm of  $H^1(\Omega)$  space is shown with  $\|\cdot\|_1$ . We would like to recall here the dual space of  $Y = H_0^1(\Omega)$ , namely the space  $H^{-1}(\Omega)$  equipped with the  $-1$ -norm

$$\|z\|_{-1} = \sup_{y \in Y} \frac{|\langle z, y \rangle|}{\|y\|_1}. \quad (5)$$

Here,  $\langle \cdot, \cdot \rangle$  denotes the duality pairing. We introduce the following bilinear forms in order to define the variational form of the problem, for  $y, v \in Y$  and  $q \in Q$ :

$$\begin{aligned} a(y, v) &= \nu \int_{\Omega} \nabla y : \nabla v dx, \\ d(v, q) &= \int_{\Omega} q \nabla \cdot v dx. \end{aligned}$$

Now, the weak forms of the equations (2) and (3) read as: Find  $y \in Y$ ,  $u \in U$  and  $q \in Q$  satisfying

$$\begin{aligned} a(y, v) - d(v, p) + d(y, q) \\ = (u, v), \quad \forall (v, q) \in Y \times Q, \end{aligned} \quad (6)$$

for the state part. For the adjoint equation, the problem is: Find  $\lambda \in Y$  and  $\varphi \in Q$  satisfying

$$\begin{aligned} a(\lambda, w) - d(w, \xi) + d(\lambda, \varphi) \\ = (y - y_d, w) \quad \forall (w, \varphi) \in Y \times Q. \end{aligned} \quad (7)$$

Assuming the solution operator  $S : U \mapsto H_0^1(\Omega) \cap H^2(\Omega)$ , we define the reduced cost function:

$$J(y, u) = J(S(u), u) := j(u),$$

where  $S(u)$  solves the auxiliary problem

$$\begin{aligned} a(y(u), v) - d(v, p) + d(y(u), q) \\ = (u, v) \quad \forall (v, q) \in Y \times Q. \end{aligned} \quad (8)$$

Optimality conditions give the gradient equation as

$$j'(u)(\tilde{u} - u) = (\lambda(u) + \beta u, \tilde{u} - u), \quad \forall \tilde{u} \in U, \quad (9)$$

with  $\lambda(u)$  solves the following system:

$$\begin{aligned} a(\lambda(u), w) - d(w, \xi) + d(\lambda(u), \varphi) \\ = (y(u) - y_d, w) \quad \forall (w, \varphi) \in Y \times Q. \end{aligned} \quad (10)$$

We can use the second order sufficient optimality condition to get the positive definiteness of the reduced hessian [21, 26]:

$$j''(u)(\delta u, \delta u) \geq \alpha \|\delta u\|_{L^2(\Omega)}^2 \quad \forall \delta u \in U. \quad (11)$$

We would like to note here that, unless stated otherwise, the letter  $C$  will stand for a generic constant, which is independent from the mesh size  $h$  throughout the entire paper.

### 3. Discretization

In this section, we will discretize our continuous problems using a finite element approach and the Brezzi-Pitkaranta stabilization term will appear in discrete variational problem. We let  $Y^h \subset Y$ ,  $Q^h \subset Q$  and  $U^h \subset U$  be the finite element spaces with a quasiuniform triangulation  $\tau^h$  of  $\Omega$ . The corresponding triangles of the domain are denoted by  $K_1, K_2, \dots, K_n$ . We let  $h_i = \text{diam}(K_i)$  and  $h = \max\{h_1, h_2, \dots, h_n\}$ . We consider  $Y^h$  and  $Q^h$  to be the spaces of continuous piecewise linears

(P1-P1 pair), which is an unstable pair known not to satisfy discrete inf-sup condition. We also make the standard assumptions that the finite element spaces satisfy the following approximation properties:

$$\begin{aligned} \inf_{y^h \in Y^h, q^h \in Q^h} \left\{ \|(y - y^h)\| + h \|\nabla(y - y^h)\| \right. \\ \left. + h \|p - q^h\| \right\} \leq Ch^2 (\|y\|_2 + \|p\|_1), \end{aligned} \quad (12)$$

for  $(y, p \in (Y \cap H^2(\Omega), Q \cap H^1(\Omega)))$ . We also assume that the control variable  $u$  satisfies

$$\|u - \tilde{u}\| \leq Ch^2 \|u\|_2 \quad \text{for } u \in U \cap H^2(\Omega), \quad (13)$$

where  $\tilde{u}$  is the  $L^2$  projection from  $U$  to  $U^h$ . Now, the finite element scheme considered for the optimal control problem here reads as follows: Find  $y^h, \lambda^h \in Y^h$ ,  $u^h \in U^h$  and  $q^h, \xi^h \in Q^h$  such that

$$\min J(y^h, u^h) = \frac{1}{2} \|y^h - y_d\|^2 + \frac{\alpha}{2} \|u^h\|^2 \quad (14)$$

subject to

$$\begin{aligned} a(y^h, v^h) - d(v^h, p^h) + d(y^h, q^h) + c(p^h, q^h) \\ = (u^h, v^h), \quad \forall (v^h, q^h) \in Y^h \times Q^h, \end{aligned} \quad (15)$$

$$\begin{aligned} a(\lambda^h, w^h) - d(w^h, \xi^h) + d(\lambda^h, \varphi^h) + c(\xi^h, \varphi^h) \\ = (y^h - y_d, w^h) \quad \forall (w^h, \varphi^h) \in Y^h \times Q^h. \end{aligned} \quad (16)$$

Here, the terms  $c(p^h, q^h)$  and  $c(\xi^h, \varphi^h)$  stand for the Brezzi-Pitkaranta stabilization.  $\alpha$  is a positive parameter and  $c(\cdot, \cdot)$  is a mesh dependent bilinear form, which is defined by

$$c(p^h, q^h) = \alpha \sum_{i=1}^n h_i^2 \int_{K_i} \nabla p^h \cdot \nabla q^h dx \quad \forall p^h, q^h \in Q^h,$$

and assumed to satisfy following properties [19]:

- i.  $c(p^h, q^h)$  is defined for all  $p^h, q^h \in Q^h$ .
- ii.  $c(q^h, q^h) = \alpha [q^h]^2$ ,  $\forall q^h \in Q^h$  is a mesh dependent norm.
- iii.  $c(p^h, q^h)$  is continuous in the sense that,  $c(p^h, q^h) \leq [p^h][q^h]$ .
- iv. For  $\forall y^h \in Y^h, q^h \in Q^h$ ;  $\exists$  a positive constant  $\gamma$ , which is independent from  $h$  and satisfies

$$d(y^h, q^h) \leq \gamma \frac{1}{h^k} \|y^h\| \|q^h\|, \quad k = 1, 1/2.$$

- v.  $\exists c_0$ , a positive constant independent from  $h$ , such that

$$\forall q^h \in Q^h \quad [q^h] \leq c_0 h^k \|q^h\|_1, \quad k = 1, 1/2.$$

Similar to continuous case, we can define the discrete solution operator  $S_h$  such that  $S_h(u) = y_h(u)$ . Then, there hold

$$j'_h(u)(\tilde{u} - u) = (\lambda^h(u) + \alpha u, \tilde{u} - u), \quad \forall \tilde{u} \in U, \quad (17)$$

and

$$j_h''(u)(\delta u, \delta u) \geq \alpha \|\delta u\|_{L^2(\Omega)}^2, \quad \forall \delta u \in U. \quad (18)$$

The discrete auxiliary problem in variational formulation follows as:

$$a(y^h(u), v^h) - d(v^h, p^h) + d(y^h(u), q^h) + c(p^h, q^h) = (u, v^h) \quad \forall (v^h, q^h) \in Y^h \times Q^h, \quad (19)$$

$$\begin{aligned} a(\lambda^h(u), w^h) - d(w^h, \xi^h) + d(\lambda^h(u), \psi^h) \\ + c(\xi^h, \psi^h) = (y^h(u) - y_d, w^h) \\ \forall (w^h, \psi^h) \in Y^h \times Q^h. \end{aligned} \quad (20)$$

Now, we will obtain the stability results for both state and adjoint state variables in the following lemma.

**Lemma 1.** *The discrete state and adjoint state variables (15)-(16) are stable and there hold*

$$\nu \|\nabla y^h\|^2 \leq C\nu^{-1} \|u^h\|^2, \quad (21)$$

and

$$\nu \|\nabla \lambda^h\|^2 \leq C\nu^{-1} \|y^h - y_d\|^2. \quad (22)$$

**Proof.** We let  $v^h = y^h$  in (15) to get

$$a(y^h, y^h) - d(y^h, p) + d(y^h, q) = (u^h, y^h).$$

We choose  $q = p$  and applying the Cauchy-Schwartz and Young's inequalities we get the desired result for the state part. We proceed the similar argument for the stability of the adjoint state variable.

□

#### 4. Finite element error analysis

In this section, we derive the error estimates for the control, state and adjoint state variables.

**Lemma 2.** *Let  $(y(u), p)$  and  $(y^h(u), p^h)$  be solutions of (6) and (19), respectively. Then, we have*

$$\begin{aligned} & \nu \|\nabla(y(u) - y^h(u))\|^2 + [p - p^h]^2 \\ & \leq \inf_{\tilde{y} \in Y^h, \tilde{p} \in Q^h} C \\ & \left\{ \nu^{-1} \|\nabla(y(u) - \tilde{y})\|^2 + \nu^{-1} \|(p - \tilde{p})\|^2 \right. \\ & \left. + h^{-1} \|\nabla(y(u) - \tilde{y})\|^2 + [p - \tilde{p}]^2 + [p]^2 \right\}. \end{aligned}$$

**Proof.** We subtract (19) from (6) via the same test functions  $v^h, q^h$ . Thus, we get the error equation

$$\begin{aligned} & a(y(u) - y^h(u), v^h) - d(v^h, p - p^h) + \\ & d(y(u) - y^h(u), q^h) - c(p^h, q^h) \\ & = 0, \quad \forall (v^h, q^h) \in Y^h \times Q^h. \end{aligned}$$

Now we split the error term  $y - y^h(u)$  as  $y - y^h(u) = y - \tilde{y} - (y^h(u) - \tilde{y}) = \eta - \phi^h$ , where  $\tilde{y}$  is the best approximation of  $y$  in  $Y^h$ . So the error equation becomes:

$$\begin{aligned} & a(\eta - \phi^h, v^h) - d(v^h, p - p^h) \\ & + d(\eta - \phi^h, q^h) - c(p^h, q^h) = 0. \end{aligned}$$

Rearranging the new error equation and adding and subtracting  $c(p, q^h)$  with the test function choice  $v^h = \phi^h$  yield:

$$\begin{aligned} a(\phi^h, \phi^h) &= a(\eta, \phi^h) - d(\phi^h, p - p^h) - d(\phi^h, q^h) \\ & + d(\eta, q^h) + c(p - p^h, q^h) - c(p, q^h). \end{aligned}$$

Splitting the error in the pressure in a similar manner gives  $p - p^h = p - \tilde{p} - (p^h - \tilde{p}) = \zeta - \psi^h$ , where  $\tilde{p}$  is the best approximation of  $p$  in  $Q^h$ . Then, we have

$$\begin{aligned} a(\phi^h, \phi^h) &= a(\eta, \phi^h) - d(\phi^h, \zeta - \psi^h) - d(\phi^h, q^h) \\ & + d(\eta, q^h) + c(\zeta - \psi^h, q^h) - c(p, q^h). \end{aligned}$$

Picking  $q^h = \psi^h$  gives:

$$\begin{aligned} & a(\phi^h, \phi^h) + c(\psi^h, \psi^h) = a(\eta, \phi^h) - d(\phi^h, \zeta) \\ & + d(\phi^h, \psi^h) - d(\phi^h, \psi^h) + d(\eta, \psi^h) \\ & + c(\zeta, \psi^h) - c(p, \psi^h) = a(\eta, \phi^h) - d(\phi^h, \zeta) \\ & + d(\eta, \psi^h) + c(\zeta, \psi^h) - c(p, \psi^h). \end{aligned} \quad (23)$$

We now estimate absolute value of each term at right-hand side of (23) separately. By using Cauchy-Schwartz and Young's inequalities we have:

$$\begin{aligned} |a(\eta, \phi^h)| & \leq \nu \|\nabla \eta\| \|\nabla \phi^h\| \\ & \leq \frac{\nu}{4} \|\nabla \phi^h\|^2 + C\nu^{-1} \|\nabla \eta\|^2, \end{aligned}$$

$$\begin{aligned} |-d(\phi^h, \zeta)| & \leq C \|\zeta\| \|\nabla \phi^h\| \\ & \leq \frac{\nu}{4} \|\nabla \phi^h\|^2 + C\nu^{-1} \|\zeta\|^2. \end{aligned}$$

Making use of Poincare-Friedrich's inequality and property (iv) of the bilinear form  $c(\cdot, \cdot)$ , we have

$$\begin{aligned} |-d(\eta, \psi^h)| & \leq Ch^{-1/2} \|\nabla \eta\| [\psi^h] \\ & \leq \frac{1}{6} [\psi^h]^2 + Ch^{-1} \|\nabla \eta\|^2. \end{aligned}$$

For stabilization terms, we use the properties along with the usual inequalities to get:

$$|c(\zeta, \psi^h)| \leq [\zeta][\psi^h] \leq \frac{1}{6}[\psi^h]^2 + C[\zeta]^2,$$

and

$$|c(p, \psi^h)| \leq [p][\psi^h] \leq \frac{1}{6}[\psi^h]^2 + C[p]^2.$$

Rearranging the error equation with obtained bounds will result:

$$\begin{aligned} \frac{\nu}{2} \|\nabla \phi^h\|^2 + \frac{1}{2}[\psi^h]^2 \leq \\ C \{ \nu^{-1} \|\nabla \eta\|^2 + \nu^{-1} \|\zeta\|^2 + h^{-1} \|\nabla \eta\|^2 + [\zeta]^2 + [p]^2 \}. \end{aligned}$$

Making a final use of triangle inequality gives the desired result now.  $\square$

**Lemma 3.** Let  $(\lambda, \xi)$  and  $(\lambda^h(u), \xi^h)$  be solutions of (10) and (20), respectively. Then, we have

$$\begin{aligned} \nu \|\nabla(\lambda(u) - \lambda^h(u))\|^2 + [\xi - \xi^h]^2 \leq \inf_{\tilde{\lambda} \in Y^h, \tilde{\xi} \in Q^h} \\ C \left\{ \nu^{-1} \|\nabla(\lambda(u) - \tilde{\lambda})\|^2 + \nu^{-1} \|(\xi - \tilde{\xi})\|^2 \right. \\ \left. + h^{-1} \|\nabla(\lambda(u) - \tilde{\lambda})\|^2 + [\xi - \tilde{\xi}]^2 \right. \\ \left. + [\xi]^2 + \nu^{-1} \|\nabla(y(u) - y^h(u))\|^2 \right\}. \end{aligned}$$

**Proof.** We omit the proof since it is very similar to the previous case.  $\square$

In order to get an estimate for the control variable, we need a relation between the discrete continuous and auxiliary solutions for both state and adjoint state equations.

**Lemma 4.** If  $(y^h, p^h)$  and  $(y^h(u), p^h)$  be solutions of (15) and (19), respectively. Then, there holds

$$\nu \|\nabla(y^h - y^h(u))\|^2 \leq \frac{C}{\nu} \|u - u^h\|^2. \quad (24)$$

Similarly, if  $(\lambda^h, \xi^h)$  and  $(\lambda^h(u), \xi^h)$  be solutions of (16) and (20), respectively. Then, there holds

$$\nu \|\nabla(\lambda^h - \lambda^h(u))\|^2 \leq \frac{C}{\nu} \|y^h - y^h(u)\|^2. \quad (25)$$

**Proof.** For the state part, we subtract (19) from (15). Since the pressure terms are independent of the control, the proof is trivial.  $\square$

**Lemma 5.** The first derivative of the reduced cost function for the continuous and the discrete cases satisfy

$$\|j'(u)(\delta) - j'_h(u)(\delta)\| \leq \|\lambda(u) - \lambda^h(u)\| \|\delta\| \quad \forall u, \delta \in U. \quad (26)$$

**Proof.** The result is obtained by using Eqns. (9) and (17) directly.  $\square$

The following lemma gives the error estimate for the control variable  $u$  [5].

**Lemma 6.** Let  $(u, y)$  and  $(u^h, y^h)$  be solutions to (6) and (15), respectively. Then, we have

$$\|u - u^h\| \leq \|u - \tilde{u}\| + \frac{1}{\alpha} \|\lambda(u) - \lambda^h(u)\|, \quad \tilde{u} \in U, \quad (27)$$

where  $\lambda(u)$  and  $\lambda^h(u)$  are solutions to (7) and (20), respectively.

**Corollary 1.** The error in state variable  $y$  satisfies:

$$\begin{aligned} \nu \|\nabla(y - y^h)\|^2 + [p - p^h]^2 \leq \inf_{\tilde{y} \in Y^h, \tilde{p} \in Q^h} \\ C \left\{ \nu^{-1} \|\nabla(y(u) - \tilde{y})\|^2 + \nu^{-1} \|(p - \tilde{p})\|^2 \right. \\ \left. + h^{-1} \|\nabla(y(u) - \tilde{y})\|^2 + [p - \tilde{p}]^2 + [p]^2 \right. \\ \left. + \nu^{-1} \|u - u^h\|^2 \right\}. \end{aligned} \quad (28)$$

**Proof.** The corollary is the combination of Lemma (2) and Lemma (4).  $\square$

**Corollary 2.** The error in adjoint state variable  $\lambda$  satisfies:

$$\begin{aligned} \nu \|\nabla(\lambda - \lambda^h)\|^2 + [\xi - \xi^h]^2 \leq \inf_{\tilde{\lambda}, \tilde{y} \in Y^h, \tilde{\xi}, \tilde{p} \in Q^h} \\ C \left\{ \nu^{-1} \|\nabla(y(u) - \tilde{y})\|^2 + \nu^{-1} \|(p - \tilde{p})\|^2 + [p]^2 \right. \\ \left. + [p - \tilde{p}]^2 + h^{-1} \|\nabla(y(u) - \tilde{y})\|^2 + \nu^{-1} \|u - u^h\|^2 \right. \\ \left. + \nu^{-1} \|\nabla(\lambda(u) - \tilde{\lambda})\|^2 + \nu^{-1} \|(\xi - \tilde{\xi})\|^2 + \right. \\ \left. h^{-1} \|\nabla(\lambda(u) - \tilde{\lambda})\|^2 + [\xi - \tilde{\xi}]^2 + [\xi]^2 \right\}. \end{aligned} \quad (29)$$

**Proof.** The proof is just a combination of the results of Lemma (3) and Lemma(4).  $\square$

We are now in a position to state approximation results. We give corollaries for each variable. We first assume that  $y, u, p, \lambda, \xi$  are sufficiently smooth, before stating the approximation results.

**Corollary 3.** The control variable  $u$  satisfies

$$\|u - u^h\| \cong \mathcal{O}(h^{1/2}).$$

**Proof.** Making use of approximation assumptions (12), (13) and property (v.) of the bilinear form  $c(\cdot, \cdot)$  in Lemma 6, we get

$$\|u - u^h\| \leq C(u)h^2 + C(\nu^{-1}, \alpha^{-1}, y, \lambda, p, \xi)h^{1/2},$$

which completes the proof.  $\square$

**Corollary 4.** *The adjoint state variable  $\lambda$  satisfies*

$$\nu \|\nabla(\lambda - \lambda^h)\|^2 + [\xi - \xi^h]^2 \cong \mathcal{O}(h).$$

**Proof.** The proof is similar to the previous case, which is stated for  $u$ .  $\square$

**Corollary 5.** *The state variable  $y$  satisfies*

$$\nu \|\nabla(y - y^h)\|^2 + [p - p^h]^2 \cong \mathcal{O}(h).$$

**Proof.** The proof is similar to the previous cases, which are stated for  $u$  and  $\lambda$ .  $\square$

**Remark 1.** *By the property (v.) of the bilinear form  $c(\cdot, \cdot)$ , the norm  $[\cdot]$  is approximated as  $h^{1/2} \|\cdot\|_1$ . Thus, the pressure error terms at left-hand side of all error relations does not give any convergence for pressure. Since pressure is not guaranteed to be unique, this situation is expected. However, the discrete pressure remains bounded in any case [19].*

## 5. Numerical Application

In this section, we perform a numerical experiment to verify the effectiveness of the proposed method. We use the finite element software package Freefem++ [16] to carry out all computations. In considered test case, we study in the domain  $(0,1)^2$  with a mesh resolution of  $32 \times 32$ . We choose the parameters as  $\nu = 1$  and  $\beta = 0.1$ . The stabilization parameter  $\alpha$  is calculated as:

$$\alpha = \frac{|K|^2}{5(c_1^2 + c_2^2 + c_3^2)}.$$

Here  $K$  denotes any triangle in  $\tau^h$  and  $|K|^2$  is its area.  $c_1, c_2, c_3$  stands for the lengths of sides of the triangle  $K$ .

**Example** As a numerical test, we consider the driven cavity problem. In this problem, the horizontal velocity on the upper boundary is 1 and the vertical component is 0. We consider a numerical experiment from [14]. We do not have any constraint on the control or the state variable. Let the desired state be

$$y_d = \begin{pmatrix} \sin(\pi x)^2 \sin(\pi y) \cos(\pi y) \\ -\sin(\pi y)^2 \sin(\pi x) \cos(\pi x) \end{pmatrix}.$$

In Figure 1, we compare the pressure terms of the state equation for both stabilized and unstabilized solutions. We observe that unstabilized pressure diverges. Similarly, for the adjoint state, unstabilized pressure blows up in Figure 2. Finally, we compare the first component of the stabilized and unstabilized solutions. One can easily see the efficiency of the stabilization through comparison of these figures.

## 6. Conclusion and Outlook

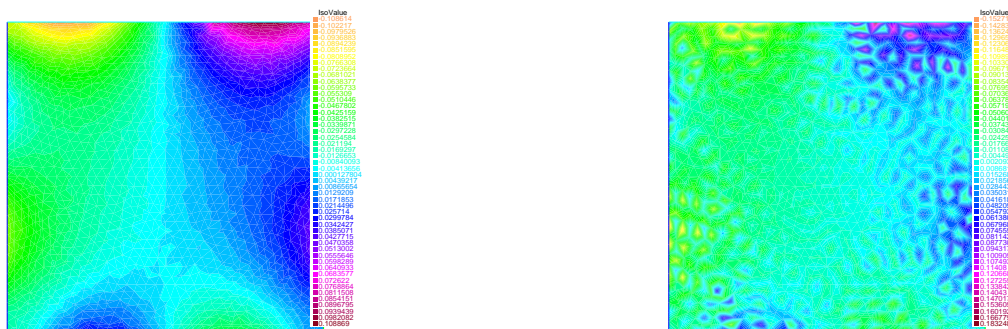
In this work, we have studied Brezzi-Pitkaranta stabilization scheme for the optimal control problems governed by Stokes equations. We have obtained the stability results for both the state and adjoint state variables. We derived a priori error bounds for each variable and proved that the error is of order 1/2. In the numerical example, we have shown the efficiency of the stabilization in both solutions of the state and adjoint state. As future works, we will consider the optimal control of time dependent and nonlinear flow problems.

## References

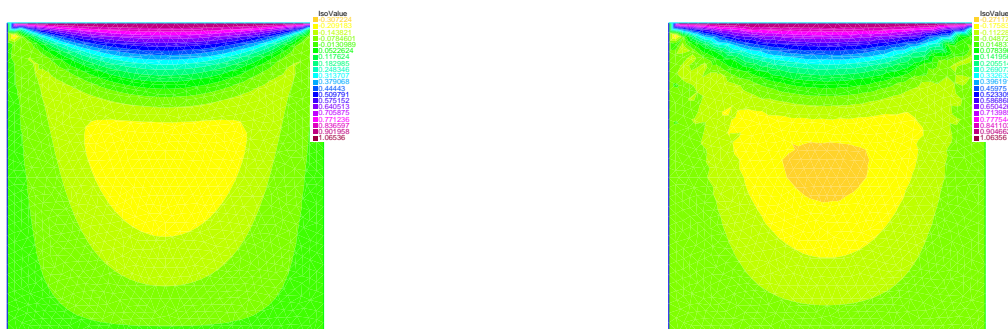
- [1] Abergel, F., Temam, R., On some optimal control problems in fluid mechanics, Theoret. Comput. Fluid Mech., 1 (6), 303-325, 1990.
- [2] Arnold, D., Brezzi, F., Fortin, M., A stable finite element for the Stokes equations, Calcolo 21, 337-344, 1984.
- [3] Adams, R.A., Sobolev Spaces, Academic Press, New York, 1975.
- [4] Baiocchi, C., Brezzi, F. and Franca, L., Virtual bubbles and Galerkin-least-squares type methods (Ga. L. S.), Comput. Methods Appl. Mech. Eng., 105, 125-141, 1993.
- [5] Becker, R. and Hansbo, P., A simple pressure stabilization method for the Stokes equation, Commun. Numer. Meth. Engng, 24, 2008.
- [6] Barrenechea, G.R. and Blasco, J., Pressure stabilization of finite element approximations of time-dependent incompressible flow problems, Computer Methods in Applied Mechanics and Engineering, 197:1-4, 219-231, 2007.
- [7] Bochev, P. and Gunzburger, M., Least-squares finite-element methods for optimization and control for the Stokes equations, Comput. Math. Appl., 48, 1035-1057, 2004.
- [8] Brezzi, F. and Pitkaranta, J., On the stabilization of finite element approximations of the Stokes problem, in Efficient Solution of Elliptic Systems, W. Hackbush, ed., vol. 10 of Notes on Numerical Fluid Mechanics, Vieweg, 11-19, 1984.
- [9] Brooks, A. and Hughes, T., Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations, Computer Methods in Applied Mechanics and Engineering, 32, 199-259, 1982.
- [10] Casas, E., Optimality conditions for some control problems of turbulent flows, in: Flow Control, Minneapolis, MN, 1992, in: IMA Vol. Math. Appl., vol. 68, Springer, New York, 127-147, 1995.
- [11] Codina, R. and Blasco, J., Analysis of a pressure-stabilized finite element approximation of the stationary Navier-Stokes equations, Numer. Math., 87, 59-81, 2000.



**Figure 1.** Comparison of pressure solution of the state equation: stabilized(left) and unstabilized(right)



**Figure 2.** Comparison of pressure of the adjoint the state equation: stabilized(left) and unstabilized(right)



**Figure 3.** Comparison of first component of the solutions of the state variable: stabilized(left) and unstabilized(right)

- [12] Douglas, J. and Wang, J., An absolutely stabilized finite element method for the Stokes problem, *Math. Comp.*, 52,495-508, 1989.
- [13] Franca, L. and Hughes, T., Convergence analyses of Galerkin-Least-Squares methods for symmetric advective-diffusive forms of the Stokes and incompressible Navier-Stokes equations, *Comput. Methods. Appl. Mech. Engrg.*, 105 ,285-298.,1993.
- [14] De los Reyes, J.C., Meyer, C. and Vexler, B., Finite element error analysis for state-constrained optimal control of the Stokes equations, *WIAS Preprint No. 1292*, 2008.
- [15] Gunzburger, M., *Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory, Practice and Algorithms*, Academic Press, Boston, 1989.
- [16] Hecht, N., New development in FreeFem++, *J. Numer. Math.* 20, no. 3-4, 251-265, 2012.
- [17] Heinkenschloss, M. and Leykekhman, D., *Local Error Estimates for SUPG Solutions of Advection-Dominated Elliptic Linear-Quadratic Optimal Control Problems*, Rice University, Houston, CAAM Technical Report TR08-30, 2008.
- [18] Hinze, M., Yan, N. and Zhou, Z., Variational discretization for optimal control governed by convection dominated diffusion equations, *J. Comput. Math.*, 27, 237-253, 2009.
- [19] Latche, J.-C. and Vola, D., Analysis of the Brezzi-Pitkaranta Stabilized Galerkin Scheme for Creeping Flows of Bingham Fluids, *Siam J. Numer. Anal.*, 42:3, 1208-1225, 2004.

- [20] Layton, W.J., A connection between subgrid scale eddy viscosity and mixed methods, Appl. Math. Comput., 133, 147-157, 2002.
- [21] Lions, J.L., Optimal control of systems governed by partial differential equations, Springer Verlag, New York, 1971.
- [22] Rees, T. and Wathen, A.J., Preconditioning Iterative Methods for the Optimal Control of the Stokes Equations, SIAM J. Sci. Comput., 33(5), 2903-2926, 2011.
- [23] Röscher, A. and Vexler, B., Optimal Control of the Stokes Equations: A Priori Error Analysis for Finite Element Discretization with Postprocessing, SIAM J. Numer. Anal., 44, 1903-1920, 2006.
- [24] Stoll, M. and Wathen, A., All-at-once solution of time-dependent Stokes control, Journal of Computational Physics, 231(1), 498-515, 2013.
- [25] Sun, T., Discontinuous Galerkin finite element method with interior penalties for convection diffusion optimal control problem. Int. J. Numer. Anal. Model. 7, 87-107, 2010.
- [26] Tröltzsch, F., Optimal Control of Partial Differential Equations: Theory, Methods and Applications, Amer Mathematical Society, 2010.
- [27] Zhou, Z. and Yan, N., The local discontinuous Galerkin method for optimal control problem governed by convection-diffusion equations, International Journal of Numerical Analysis & Modeling 7, 681-699, 2010.

*Aytekin Cıbık* was born in Şanlıurfa. He received his Ph.D. degree from the department of mathematics, Middle East Technical University (METU) in 2011. He works as an assistant professor at the department of mathematics, Gazi University. His research interests are numerical analysis and computational fluid dynamics.

*Fikriye Yılmaz* was born in Ankara. She received his B.D and Ph.D. degrees from the department of mathematics, Middle East Technical University (METU) in 2004 and 2011, respectively. She works as an assistant professor at the department of mathematics, Gazi University. Her research interests are numerical analysis and optimal control problems.

An International Journal of Optimization and Control: Theories & Applications (<http://ijocta.balikesir.edu.tr>)



This work is licensed under a Creative Commons Attribution 4.0 International License. The authors retain ownership of the copyright for their article, but they allow anyone to download, reuse, reprint, modify, distribute, and/or copy articles in IJOCTA, so long as the original authors and source are credited. To see the complete license contents, please visit <http://creativecommons.org/licenses/by/4.0/>.