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RESEARCH ARTICLE

Approximate solution of generalized pantograph equations with variable coefficients by operational method

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ABSTRACT

In this paper, we present an efficient direct solver for solving the generalized pantograph equations with variable coefficients. An approach is based on the second kind Chebyshev polynomials together with operational method. The main characteristic behind this approach is that it reduces such problem to ones of solving systems of algebraic equations. Only a small number of Chebyshev polynomials are needed to obtain a satisfactory result. Numerical results with comparisons are given to confirm the reliability of the proposed method for solving generalized pantograph equations with variable coefficients.

1. Introduction

Functional-differential equations with proportional delays are usually referred to as pantograph equations or generalized pantograph equations. Pantograph equations have gained more interest in many application fields such a biology, physics, engineering, economy, electrodynamics [1–7]. In recent years, there has been a growing interest in the numerical treatment of pantograph equations of the retarded and advanced type. A special feature of this type of equation is the existence of compactly supported solutions [6]. Pantograph equations are characterized by the presence of a linear functional argument and play an important role in explaining many different phenomena. In particular they turn out to be fundamental when ODEs-based model fail. In the literature, special attention has been given to applications of Taylor polynomials method, variation iteration method, Adomian decomposition method etc. [8–21,25–28]

Consider the generalized linear pantograph equations of the form

$$\sum_{k=0}^{m} P_k(x)y^{(k)}(x) + \sum_{j=0}^{J}\sum_{s=0}^{n} H_{js}(x)y^{(s)}(\alpha_j x - \beta_j) = g(x) \quad (1)$$

for $x \in [-1,1]$, under the mixed condition, for $1 \leq c_j \leq 1,\ i = 0,1,2,...,m-1$

$$\sum_{k=0}^{m-1}\sum_{j=0}^{r} c_{ij}^{k} y^{(k)}(c_j) = \lambda_i \quad (2)$$

which is the $y(x)$ an unknown function, the known function $P_k(x),\ H_{js}(x),\ g(x)$ are defined on an interval and also c_{ij}^{k} are appropriate constant.

Our aim is to find an approximate solution expressed in terms of polynomial of degree N in the form

$$y_N(x) = \sum_{r=0}^{N} a_r U_r(x) \quad (3)$$

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where a_r unknown coefficients and N is chosen any positive integer such that $N \geq m$.

2. Chebyshev polynomial

Orthogonal functions, often used to represent an arbitrary time function, have received considerable attention in dealing with various problems of dynamical system. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem.

Definition 1. *The Chebyshev polynomial of the second kind $U_n(x)$ is a polynomial in x of degree n, defined by the relation*

$$U_n(x) = \frac{sin(n+1)\theta}{sin\theta}, \quad when \quad x = cos(\theta).$$

If the range of the variable x is the interval $[-1, 1]$, the range the corresponding variable θ can be taken $[0, \pi]$. We suppose without lose of generality that the interval of Eq.(1) is $[-1, 1]$ which domain of the Chebyshev polynomial of the second kind, since any finite $[a, b]$ can be transformed to interval $[-1, 1]$ by linear maps [23, 24]. Using Moivre's Theorem we obtained the fundamental recurrence relation [22, 23]

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x), \quad n = 2, 3, ...$$

which together with the initial conditions

$$U_0(x) = 1, \quad U_2(x) = 2x$$

These polynomials have the following properties:

i) $U_{n+1}(x)$ has exactly $n + 1$ real zeroes on the interval $[-1, 1]$. The m-th zero $x_{n,m}$ of $U_n(x)$ is located at

$$x_{n,m} = cos(\frac{m\pi}{n+2})$$

ii) These polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $\omega(x) = (1 - x^2)^{1/2}$

$$\int_{-1}^{1} U_r(x)U_s(x)\omega(x)dx = \begin{cases} \pi, & \text{r=s=0;} \\ \frac{\pi}{2}, & r = s \neq 0; \\ 0, & r \neq s. \end{cases}$$

iii) It is well known that [23] the relation between the powers x^n and the Chebyshev polynomials $U_n(x)$ is

$$x^n = 2^{-n} \sum_{j=0}^{||\frac{n}{2}||} \left(\binom{n}{j} - \binom{n}{j-1} \right) U_{n-2j}(x) \quad (4)$$

iv) Any function $y(x) \in L^2[-1, 1]$ can be approximated as a sum of the second kind Chebyshev polynomials as:

$$y(x) = \sum_{n=0}^{\infty} c_n U_n(x) \quad (5)$$

where, for $n = 0, 1, ...$

$$c_n = \left\langle y(x), U_n(x) \right\rangle = \int_{-1}^{1} y(x)U_n(x)dx. \quad (6)$$

3. Fundamental matrix relations

Let us write Eq. (1) in the form

$$D(x) + H(x) = g(x) \quad (7)$$

where

$$D(x) = \sum_{k=0}^{m} P_k(x)y^{(k)}(x),$$

and

$$H(x) = \sum_{j=0}^{J} \sum_{s=0}^{n} H_{js}(x)y^{(s)}(\alpha_j x - \beta_j).$$

We convert these parts and the mixed conditions in to the matrix form. Let us consider the Eq. (1) and find the matrix forms of each term of the equation. We first consider the solution $y_N(x)$ and its derivative $y_N^{(k)}(x)$ defined by a truncated Chebyshev series. Then we can put series in the matrix form

$$y_N(x) = U(x)A, \; y_N^{(k)}(x) = U^{(k)}(x)A \quad (8)$$

where

$$\begin{aligned} U(x) &= \begin{bmatrix} U_0(x) & U_1(x) & \cdots & U_N(x) \end{bmatrix} \\ U^{(k)}(x) &= \begin{bmatrix} U_0^{(k)}(x) & U_1^{(k)}(x) & \cdots & U_N^{(k)}(x) \end{bmatrix} \\ A &= \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T \end{aligned}$$

By using (4), we obtained the corresponding matrix relation as follows:

$$X^T(x) = DU^T(x) \text{ and } X(x) = U(x)D^T$$
$$\text{and so } U(x) = X(x)(D^T)^{-1} \quad (9)$$

where

$$X(x) = \begin{bmatrix} 1 & x & ... & x^N \end{bmatrix}.$$

for odd N,

$$D = \begin{bmatrix} \frac{1}{2^0}\binom{0}{0} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2^1}\binom{1}{0} & 0 & \dots & 0 \\ \frac{1}{2^2}(\binom{2}{1} - \binom{2}{0}) & 0 & \frac{1}{2^2}\binom{2}{0} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1}{2^N}(\binom{N}{\frac{N-1}{2}} - \binom{N}{\frac{N-3}{2}}) & 0 & \dots & \frac{1}{2^N}\binom{N}{0} \end{bmatrix}$$

for even N,

$$D = \begin{bmatrix} \frac{1}{2^0}\binom{0}{0} & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{2^1}\binom{1}{0} & 0 & \dots & 0 \\ \frac{1}{2^2}(\binom{2}{1} - \binom{2}{0}) & 0 & \frac{1}{2^2}\binom{2}{0} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2^N}(\binom{N}{\frac{N}{2}} - \binom{N}{\frac{N-2}{2}}) & 0 & \frac{1}{2^N}(\binom{N}{\frac{N-2}{2}} - \binom{N}{\frac{N-4}{2}}) & \dots & \frac{1}{2^N}\binom{N}{0} \end{bmatrix}$$

Moreover it is clearly seen that the relation between the matrix $X(x)$ and its derivative $X^{(k)}(x)$,

$$X^{(k)}(x) = X(x)B^k \tag{10}$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$B^k = \underbrace{BB...B}_{k-times}.$$

The derivative of the matrix $U(x)$ defined in (8), by using the relation (9), can expressed as

$$\begin{aligned} U^{(k)}(x) &= X^{(k)}(x)(D^T)^{-1} \\ &= X(x)B^k(D^T)^{-1}. \end{aligned} \tag{11}$$

By substituting (10) into (8), we obtain, for $k = 0, 1, ..., N$

$$y_N^{(k)}(x) = X(x)B^k(D^T)^{-1}A. \tag{12}$$

Now, the matrix representation of differential part can be given by

$$D(x) = \sum_{k=0}^{m} P_k(x)X(x)B^k(D^T)^{-1}A. \tag{13}$$

We know that;

$$X(\alpha_j x - \beta_j) = X(x)B_j \tag{14}$$

where

$$B_j = \begin{bmatrix} \binom{0}{0}\alpha_j^0(-\beta_j)^0 & \binom{1}{1}\alpha_j^0(-\beta_j)^1 & \binom{2}{2}\alpha_j^0(-\beta_j)^2 & \dots & \binom{N}{N}\alpha_j^0(-\beta_j)^N \\ 0 & \binom{1}{0}\alpha_j^1(-\beta_j^0) & \binom{2}{1}\alpha_j^1(-\beta_j)^1 & \dots & \binom{N}{N-1}\alpha_j^1(-\beta_j)^{N-1} \\ 0 & 0 & \binom{2}{0}\alpha_j^2(-\beta_j)^0 & \dots & \binom{N}{N-2}\alpha_j^2(-\beta_j)^{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \binom{N}{0}\alpha_j^N(-\beta_j)^0 \end{bmatrix}$$

Using relation (10), we can write

$$X^{(s)}(\alpha_j x - \beta_j) = X(x)B^s B_j \tag{15}$$

In a similarly way as (12) , we obtain

$$\begin{aligned} y^{(s)}(\alpha_j x - \beta_j) &= U^{(s)}(\alpha_j x - \beta_j)A \\ &= X(x)B^s B_j(D^T)^{-1}A. \end{aligned} \tag{16}$$

So that, the matrix representation of $H(x)$ part can be given by

$$H(x) = \sum_{j=0}^{J}\sum_{s=0}^{n} H_s(x)X(x)B^s B_j(D^T)^{-1}A. \tag{17}$$

4. Method of solution

In this section, we presents the method for solving Eq.(1) with conditions Eq.(2). Firstly, we can write the Eq.(1) follow as:

$$\Big(\sum_{k=0}^{m} P_k(x)X(x)B^k(D^T)^{-1} + \sum_{j=0}^{J}\sum_{s=0}^{n} H_{js}(x)X(x)B^s B_j(D^T)^{-1}\Big)A = g(x). \tag{18}$$

Then, residual $R_N(x)$ can be written as

$$\begin{aligned} R_N(x) \approx & \Big(\sum_{k=0}^{m} P_k(x)X(x)B^k(D^T)^{-1} \\ & + \sum_{j=0}^{J}\sum_{s=0}^{n} H_{js}(x)X(x)B^s B_j(D^T)^{-1}\Big)A \\ & -G^T X(x)(D^T)^{-1}. \end{aligned} \tag{19}$$

Applying typical tau method [29–33], Eq.(19) can be converted in $(N-m)$ linear or nonlinear equations by applying

$$\Big\langle R_N(x), U_n(x)\Big\rangle = \int_{-1}^{1} R_N(x)U_n(x)dx = 0 \qquad (20)$$

for $n = 0, 1, ..., N-m$. The initial conditions are given by

$$\sum_{k=0}^{m-1}\sum_{j=0}^{r} c_{ij}^{k} X(c_j)B^k(D^T)^{-1}A = \lambda_i \qquad (21)$$

where $-1 \leq c_j \leq 1$, $i = 0, 1, 2, ..., m-1$

$$X(c_j) = [\ c_j^0 \quad c_j^1 \quad \cdots \quad c_j^N\ ].$$

Hence, we obtain the $(N+1)$ sets of linear or nonlinear algebraic equation with $(N+1)$ unknowns by Eq.(20) and Eq.(21). Using the Maple program, we solve the $(N+1)$ sets of linear or nonlinear algebraic equations with $(N+1)$ unknowns and so approximate solution $y_N(x)$ can be calculated.

4.1. Checking of Solution

Likewise we can easily check the accuracy of the obtained solutions as follows: Since the obtained the Chebyshev polynomial of the second kind expansion is an approximate solution of Eq.(1), when the function $y_N(x)$ and its derivatives are substituted in Eq.(1), the resulting equation must be satisfied approximately; that is for [24]

$$E_N(x) = \Big|\sum_{k=0}^{m} P_k(x)y_N^{(k)}(x) + \sum_{j=0}^{J}\sum_{s=0}^{n} H_{js}(x)y_N^{(s)}(\alpha_j x - \beta_j) - g(x)\Big| \cong 0$$

5. Illustrative example

In this section, several numerical examples are given to illustrate the accuracy and effectiveness of the properties of the method and all of them were performed on the computer using a program written in Maple 13. The absolute errors in tables are the values of $N_e = |y(x) - y_N(x)|$ at selected points.

Example 1. *Let us consider the first order pantograph equation [11, 20, 21]*

$$y'(x) - \frac{1}{2}y(x) - \frac{1}{2}e^{\frac{x}{2}}y(\frac{x}{2}) = 0 \qquad (22)$$

with $y(0) = 1$ *and the exact solution* $y = e^x$. *Then* $P_0(x) = -\frac{1}{2}$, $P_1(x) = 1$, $H_{00}(x) = -\frac{1}{2}e^{\frac{x}{2}}$, $g(x) = 0$, $\alpha_0 = \frac{1}{2}$, $\beta_0 = 0$. *We seek the approximate solution for* $N = 4$. *Then, we have residual*

$$R_4(x) \approx \Big(P_1(x)X(x)B(D^T)^{-1} + P_0(x)X(x)(D^T)^{-1} + H_{00}(x)X(x)B_0(D^T)^{-1}\Big)A$$

where

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} B_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix}$$

If the residual $R_4(x)$ *are substituted (19) for* $n = 0, 1, 2, 3$ *and with initial condition, we obtain a linear algebraic equations system. Solving this linear equations system, we obtain the Chebyshev coefficients follows as:*

$$a_0 = 1.130077, a_1 = 0.542776$$
$$a_2 = 0.132856, a_3 = 0.223357E-1$$
$$a_4 = 0.277975E-2$$

then so, we get the approximate solution for $N = 4$

$$y_4(x) = 0.999999 + 0.996210x + 0.498070x^2 + 0.178686x^3 + 0.044476x^4$$

Table 1 shows approximate solutions of the Eq.(22) for $N = 4, 6, 8$ *by the above mentioned method. Figure 1 display the exact solution and numerical solutions for* $N = 6, 8$. *Figure 2 displays error function* $N = 6$ *and Figure 3 displays error function* $N = 8$ *Figure 3 compare the error functions and* $E_N(x)$ *for* $N = 6, 8$.

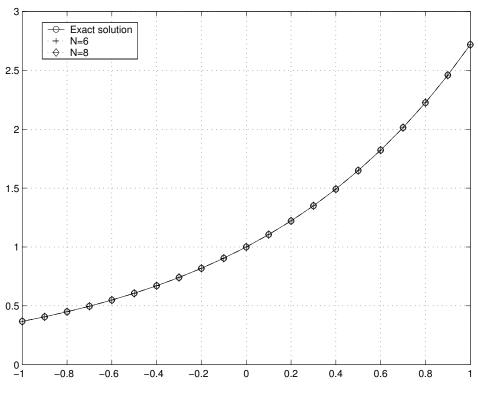


Figure 1. Comparison of exact solution and approximate solutions of Example 1 for various N.

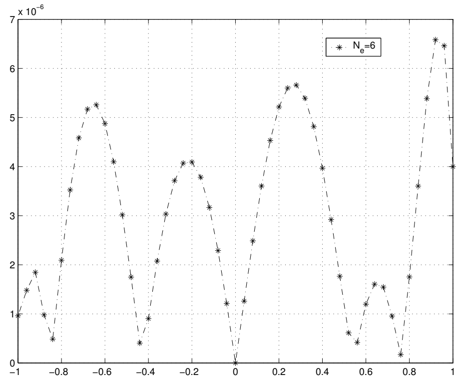


Figure 2. Error functions of Example 1 for various N.

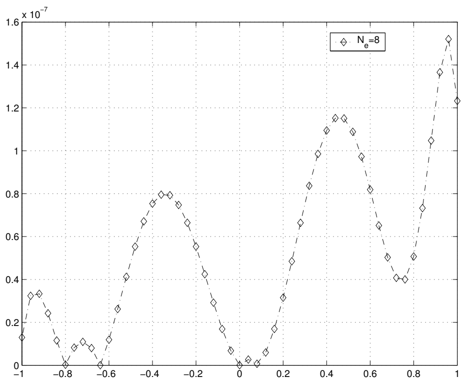
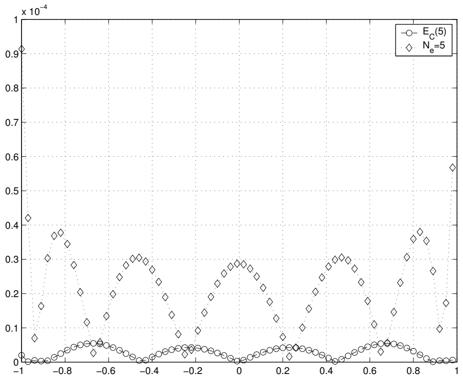


Figure 3. Error functions of Example 1 for various N.

Example 2. *Let us consider the following pantograph equation of first-order [21]*

$$y'(x) + 2y^2(\frac{x}{2}) = 1 \tag{23}$$

with $y(0) = 0$ *and exact solution is* $y(x) = sin(x)$. *Table 2 shows numerical solutions Eq.(23) with* $N = 5, 7$ *and* 9 *by present method. We see that the approximation solutions obtained by present method has good agreement with exact solution. In Table 2 compare the absolute errors and* $E_N(x)$ *some selected points. Figure 4 display values of the absolute error and* $E_N(x)$.



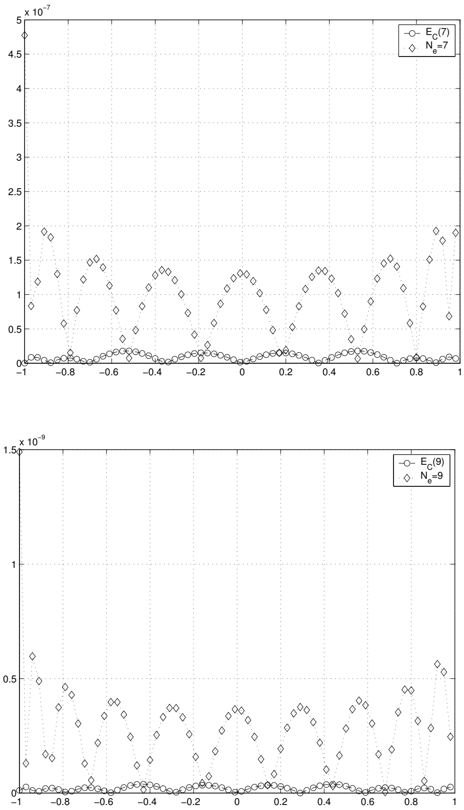


Figure 4. Comparison of error functions and $E_N(x)$of Example 2 for various N.

Example 3. *Let us consider the linear delay differential equation with constant coefficients and proportional delay qx*

$$y'(x) = ay(x) + by(qx), \ 0 < q < 1 \tag{24}$$

with initial condition

$$y(0) = \gamma$$

arose in the mathematical modeling of the wave motion in the supply line to an overhead current collector (pantograph) of an electric locomotive [1-2]. For values of $a = -1$,$b = -1$, $q = 0.8$ *and* $\gamma = 1$ *[8], Table 3 shows solutions of Eq.(24) with* $N = 8$ *by present method. Moreover, the previous results of Walsh series approach (WSA) [34], delayed unit step function series approach (DUSFA) [35], Laguerre series approach (LSA) [36], Taylor series method (TSM) [8] and present method (PM) are also given in Table 3 for comparison. The present method seems more rapidly convergent than Laguerre series and Taylor series and with errors more under control than Walsh or DUSFA series. The truncated errors for Eq.(24) are* $O(9)$ *and* $O(15)$ *for* $N = 8$ *and* $N = 15$ *respectively are also indicated.*

Example 4. *Consider the nonlinear pantograph equation of third order [11, 20],*

$$y'''(x) = -1 + 2y^2(\frac{x}{2}),$$
$$y(0) = 0,\ y'(0) = 1,''(0) = 0 \qquad (25)$$

which has the exact solution $y(x) = sin(x)$. *If we take* $N = 9$, *we get the difference between the exact and numerical solutions given in Table 4. Table 4 shows previous results of HPM [20], Adomian decomposition method (ADM) [11] and Present method (PM) for comparison. This shows that the errors are very small. Then, Figure 5 displays the comparison of error function and* $E_N(x)$ *for* $N = 15$.

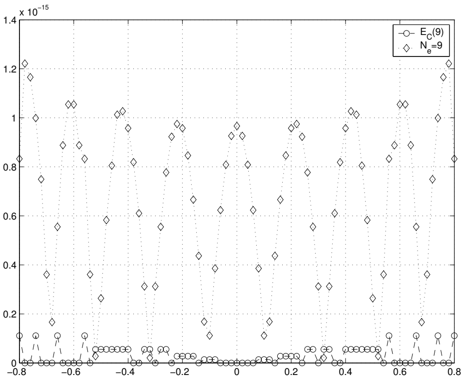


Figure 5. Comparison of error function and $E_{15}(x)$of Example 4.

Example 5. *We consider the equation with* $y(0) = 1$

$$y'(x) = -y(x) + \mu_1(x)y(0.5x)$$
$$+\mu_2(x)y(0.25x) \qquad (26)$$

Here $\mu_1(x) = -exp^{-0.5x}sin(0.5x)$, $\mu_2(x) = -2exp^{-0.75x}cos(0.5x)sin(0.25x)$. *It can be seen that the exact solution of Eq.(26) is* $y(x) = e^{-x}cos(x)$. *Using present method, we obtain the numerical solution for* $N = 10$. *In Figure 6 we give the exact solution and numerical solutions corresponding.*

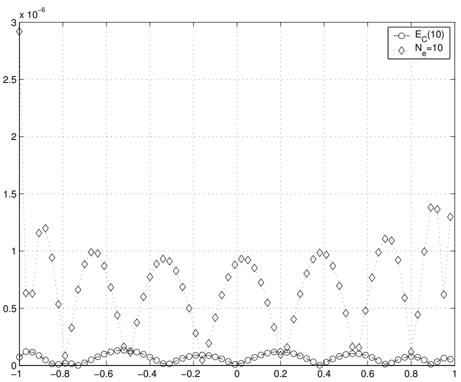


Figure 6. Comparison of error function and $E_{10}(x)$of Example 5.

6. Conclusion

A new method based on the truncated Chebyshev series of the second kind is developed to numerical solve generalized pantograph equations with mixed conditions. Pantograph equations are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solution. For this propose, the present method can be proposed. In this paper, the second kind Chebyshev polynomial approach has been used for the approximate solution of generalized pantograph equations. Thus the proposed method is suggested as an efficient method for generalized pantograph equations. Examples with the satisfactory results are used to demonstrate the application of this method. Suggested approximations make this method rather attractive and contributed to the good agreement between approximate and exact values in the numerical examples for only a few terms. Then examples shows truncated errors, absolute errors and $E_N(x)$ are coherent, and performed on the computer using a program written in Maple 13. Moreover, suggested method is applicable for the approximate solution of the pantograph-type integro-differential equations with variable delays.

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References

[1] Ockendon, J.R. and Tayler, A.B., The dynamics of a current collection system for an electric locomotive. Proc. Roy. Soc. London, Ser. A, 322, 447-468 (1971).

[2] Tayler, A.B., Mathematical Models in Applied Mathematics. Clarendon Press, Oxford, (1986).

[3] Ajello, W.G., Freedman, H.I. and Wu, J., Analysis of a model representing stage-structured population growth with state-dependent time delay. SIAM J. Appl. Math., 52, 855-869 (1992).

[4] Buhmann, M.D. and Iserles, A., Stability of the discretized pantograph differential equation. Math. Comput., 60, 575-589 (1993).

[5] Derfel, G., On compactly supported solutions of a class of functional-differential equations, in Modern Problems of Function Theory and Functional Analysis. Karaganda University Press, Kazakhstan, (1980).

[6] Feldstein, A. and Liu, Y., On neutral functional-differential equations with variable time delays. Math. Proc. Camb. Phil. Soc., 124, 371-384 (1998).

[7] Derfeland, G. and Levin, D., Generalized refinement equation and subdivision process. J. Approx. Theory, 80, 272-297 (1995).

[8] Sezer, M. and Akyuz, A., A Taylor method for numerical solution of generalized pantograph equation with linear functional argument. J. Comp. Appl. Math., 200, 217-225 (2007).

[9] Sezer, M., Yalnbas, S. and Sahin, N., Approximate solution of multi-pantograph equation with variable coefficients. J. Comp. Appl. Math., 214, 406-416 (2008).

[10] Saadatmandi, A. and Dehghan, M., Variational iteration method for solving a generalized pantgoraph equation. Comp. Math. Appl., 58, 2190-2196 (2009).

[11] Evans, D.J. and Raslan, K.R., The Adomian decomposition method for solving delay differential equation. Int. J. Comp. Math., 82(1), 49-54 (2005).

[12] Buhmann, M. and Iserles, A., Stability of the discretized pantograph differential equation. J. Math. Comp., 60, 575-589 (2005).

[13] Li, D. and Liu, M.Z., Runge-Kutta methods for the multi-pantograph delay equation. Appl. Math. Comp., 163, 383-395 (2005).

[14] Shakeri, F. and Dehghan, M., Solution of the delay differential equations via homotopy perturbation method. Math. Comp. Modelling, 48, 486-498 (2008).

[15] Muroya, Y., Ishiwata, E. and Brunner, E., On the attainable order of collocation methods for pantograph integro-differential equations. J. Comp. Appl. Math., 152, 347-366 (2003).

[16] Huan-Yu, Z., Variational iteration method for solving the multi-pantograph delay equation. Physics Letter A, 372 (43), 6475-6479 (2008).

[17] Sezer, M. and Gulsu, M., A new polynomial approach for solving difference and Fredholm integro-difference equations with mixed argument. Appl. Math. Comp., 171, 332-344 (2005).

[18] Ping Yang, S. and Xiao, A.G., Convergence of the variational iteration method for solving multi-delay differential equaitons. Comp. Math. Appl., 61(8), 2148-2151 (2011).

[19] Liu, M.Z. and Li, D., Properties of analitic solution and numerical solution of multi-pantograph equation. Appl. Math. Comp., 155, 853-871 (2004).

[20] Yusufoglu, E., An efficint algorithm for solving generalized pantograph equation with linear functional argument. Appl. Math. Comp., 217, 3591-3595 (2010).

[21] El-Safty, Salim, M.S. and El-Khatib, M.A., Convergence of the spline function for delay dynamic system. Int. J. Comp. Math., 80(4),509-518 (2004).

[22] Fox, L. and Parker, I.B., Chebyshev Polynomials in Numerical Analysis. Oxford University Press, London, (1968).

[23] Mason, J.C. and Handscomb, D.C., Chebyshev polynomials. Chapman and Hall/CRC, New York, (2003).

[24] Body, J.P., Chebyshev and fourier spectral methods, University of Michigan, New York, (2000).

[25] Saadatmandi, A. and Dehghan, M., Numerical solotion of hyperbolic telegraph equation using the Chebshev tau method. Numer. Meth. Partial Diff. Equations, 26, 239-252 (2010).

[26] Lakestani, M. and Dehghan, M., Numerical solution of Riccati equation using the cubic B spline scaling functions and Chebyshev cardinal functions. Comp. Physics Communications, 181, 957-966 (2010).

[27] Lakestani M. and Dehghan, M., Numerical solution of fourth order integro differential equations using Chebyshev cardinal functions. Inter. J. Com. Math., 87, 1389-1394 (2010).

[28] Saadatmandi, A. and Dehghan, M., The numerical solution of problems in calculus of variation using Chebyshev finite difference method. Physi. Letter A, 22, 4037-4040 (2008).

[29] Pandey, R.K. and Kumar, N., Solution of Lane-Emden type equations using Berstein operational matrix of differentiation. New Astronomy, 17, 303-308 (2012).

[30] Saadatmandi, A. and Dehghan, M., A new aperational matrix for solving fractional-order differential equations. Comp. Math. Appl., 59, 1326-1336 (2010).

[31] Parand, K. and Razzaghi, M., Rational Chebyshev tau method for solving higher-order ordinary differential equations. Int. J. Comput. Math., 81, 73-80 (2004).

[32] Razzaghi, M. and Yousefi, S., Legendre wavelets operational matrix of integration. Int. J. Syst. Sci., 32(4), 495-502 (2001).

[33] Babolian E. and F. Fattahzadeh, Numerical solution of differential equations by using Chebyshev wavelet operational matrix of integration. Appl. Math. Comp., 188, 417-425 (2007).

[34] Rao G.P. and Palanisamy, K.R., Walsh stretch matrices and functional differential equations. IEEE Trans. Autom. Control, 27, 272-276 (1982).

[35] Hwang, C., Solution of a functional differential equation via delayed unit step functions. Int. J. Syst. Sci., 14(9), 1065-1073 (1983).4

[36] Hwang, C. and Shih, Y.P., Laguerre series solution of a functional differential equation. Int. J. Syst. Sci., 13(7), 783-788 (1982).

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Table 1. Numerical results of Example 1 for different N.

x	Exact Solution	Present Method $N=4$	$N_e=4$	$N=6$	$N_e=6$	$N=8$	$N_e=8$
-1.0	0.367879	0.367649	0.229E-3	0.367878	0.592E-6	0.367879	0.129E-7
-0.8	0.449328	0.448526	0.802E-3	0.449331	0.245E-5	0.449328	0.253E-9
-0.6	0.548811	0.548746	0.646E-4	0.548816	0.481E-5	0.548811	0.119E-7
-0.4	0.670320	0.670909	0.589E-3	0.670318	0.113E-5	0.670320	0.753E-7
-0.2	0.818730	0.818730	0.591E-3	0.818726	0.405E-5	0.818730	0.553E-7
0.0	0.999999	1.000000	0.000E-0	1.000000	0.221E-6	0.999999	0.700E-14
0.2	1.221402	1.221402	0.737E-3	1.221408	0.525E-5	1.221402	0.314E-7
0.4	1.491824	1.491824	0.107E-2	1.491824	0.373E-5	1.491824	0.109E-6
0.6	1.822118	1.822118	0.726E-3	1.822117	0.125E-5	1.822118	0.818E-7
0.8	2.225540	2.225540	0.102E-3	2.228543	0.211E-6	2.225540	0.506E-7
1.0	2.718281	2.718281	0.838E-3	2.718284	0.244E-5	2.718281	0.123E-6

Table 2. Numerical results of Example 2 for different N.

x	Exact Solution	Present Method E_5	$N_e=5$	E_7	$N_e=7$	E_9	$N_e=9$
-1.0	-0.841470	0.913568E-4	0.200902E-5	0.477454E-6	0.224832E-9	0.149014E-8	0.115905E-10
-0.8	-0.717356	0.359449E-4	0.319621E-5	0.840162E-8	0.773644E-8	0.448240E-9	0.833401E-11
-0.6	-0.564642	0.157055E-4	0.469280E-5	0.102028E-6	0.147040E-7	0.363683E-9	0.626964E-11
-0.4	-0.389418	0.269612E-4	0.149996E-5	0.127995E-6	0.697329E-8	0.144294E-9	0.357318E-10
-0.2	-0.198669	0.737766E-5	0.418202E-5	0.191299E-7	0.149003E-7	0.193295E-9	0.290406E-10
0.0	0.000000	0.287867E-4	0.000000E-0	0.131290E-6	0.000000E-0	0.368155E-9	0.000000E-0
0.2	0.198669	0.737766E-5	0.418202E-5	0.191299E-7	0.149003E-7	0.193295E-9	0.290406E-10
0.4	0.389418	0.269612E-4	0.149996E-5	0.127995E-6	0.697329E-8	0.144294E-9	0.357318E-10
0.6	0.564642	0.157055E-4	0.469280E-5	0.102028E-6	0.147040E-7	0.363683E-9	0.626964E-11
0.8	0.717356	0.359449E-4	0.319621E-5	0.840162E-8	0.773644E-8	0.448240E-9	0.833401E-11
1.0	0.814470	0.913568E-4	0.200902E-5	0.477454E-6	0.224832E-9	0.149014E-8	0.115905E-10

Table 3. Comparison of the solution of Eq.(24)

x	WSA	DUSFA $m=100$	LSA $n=20$	TSM $N=8$	TSM E_{19}	PM $N=8$	PM $N=15$	PM E_{15}
0.0	1.000000	1.000000	0.999971	1.000000	0.844E-14	1.000000	1.00000000000	0.200E-18
0.2	0.665621	0.664677	0.664703	0.664691	0.138E-13	0.664691	0.66469100082	0.318E-16
0.4	0.432426	0.433540	0.433555	0.433561	0.322E-3	0.433560	0.43356077877	0.170E-15
0.6	0.275140	0.276460	0.276471	0.276483	0.125E-13	0.276481	0.27648233022	0.972E-15
0.8	0.170320	0.171464	0.171482	0.171494	0.738E-14	0.171484	0.17148411197	0.206E-14
1.0	0.100856	0.102652	0.102679	0.102744	0.155E-13	0.102670	0.10267012657	0.814E-14

Table 4. Comparison of the solution of Eq.(25)

x	Exact solution	ADM	HPM	PM
0.0	0.0	0.0	0.0	0.0
0.2	0.19866933079506122	0.19866933079506122	0.19866933079506122	0.19866933079506121
0.4	0.38941834230865050	0.38941834230865050	0.38941834230865050	0.38941834230865049
0.6	0.56464224733950355	0.56464224733950355	0.56464224733950355	0.56464247339503535
0.8	0.71735609089952280	0.71735609089952270	0.71735609089952280	0.71735609089952276
1.0	0.84147109848078965	0.84147109848078966	0.84147109848078965	0.84141098480789650

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