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RESEARCH ARTICLE

Compactness of the set of trajectories of the control system described by a Urysohn type integral equation

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ABSTRACT

The control system with integral constraint on the controls is studied, where the behavior of the system by a Urysohn type integral equation is described. It is assumed that the system is nonlinear with respect to the state vector, affine with respect to the control vector. The closed ball of the space $L_p(E;\mathbb{R}^m)$ ($p > 1$) with radius r and centered at the origin, is chosen as the set of admissible control functions, where $E \subset \mathbb{R}^k$ is a compact set. It is proved that the set of trajectories generated by all admissible control functions is a compact subset of the space of continuous functions.

1. Introduction

Nonlinear integral equations appear in many problems of contemporary physics and mechanics (see., e.g. [1] - [7]). Integral constraint on the control functions is inevitable if the control effort is exhausted by consumption. Such controls arise in various problems of economics, medicine, biology, mechanics and physics (see, [8] - [11]). Note that control system with integral constraint on the control functions, where the behavior of the system is given by a nonlinear differential equation is investigated in [8, 9].

In this paper the control system described by a Urysohn type integral equation is considered. It is assumed that integral equation is nonlinear with respect to the state vector and is affine with respect to the control vector. The closed ball of the space $L_p(E;\mathbb{R}^m)$ ($p > 1$) with radius r and centered at the origin is chosen as the set of admissible control functions. The compactness of the set of trajectories of the system generated by all admissible control functions is studied. Note that compactness of the set of trajectories guaranties the existence of the optimal trajectories in the optimal control problem with continuous payoff functional. Compactness of the set of trajectories of control systems described by the Volterra type integral equations is studied in [12, 13].

The paper is organized as follows: In Section 2 the conditions which satisfy the system are formulated (Conditions A, B and C). In Section 3 it is proved that every admissible control function generates a unique trajectory of the system (Theorem 1). In Section 4 it is shown that the set of trajectories is bounded (Theorem 2). Precompactness of the set of trajectories is specified in Section 5 (Theorem 3). In Section 6 the closedness of the set of trajectories is shown (Theorem 4), and hence compactness of the set of trajectories is obtained (Theorem 5).

2. Preliminaries

The control system described by an integral equation

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$$x(\xi) = f(\xi, x(\xi)) + \lambda \int_E \big[K_1(\xi, s, x(s)) + K_2(\xi, s, x(s))\, u(s)\big] ds \qquad (1)$$

is considered, where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector, $\xi \in E$, $E \subset \mathbb{R}^k$ is a compact set.

Let $p > 1$ and $r > 0$ be given numbers. The function $u(\cdot) \in L_p(E; \mathbb{R}^m)$ such that $\|u(\cdot)\|_p \le r$ is said to be an admissible control function, where $\|u(\cdot)\|_p = \left(\int\limits_E \|u(s)\|^p ds \right)^{\frac{1}{p}}$, $\|\cdot\|$ denotes the Euclidean norm.

The set of all admissible control functions is denoted by symbol $U_{p,r}$, i.e.

$$U_{p,r} = \{u(\cdot) \in L_p(E; \mathbb{R}^m) : \|u(\cdot)\|_p \le r\}.$$

If $u(\cdot) \in U_{p,r}$, then Hölder's inequality yields that

$$\int_E \|u(s)\|\, ds \le [\mu(E)]^{\frac{p-1}{p}}\, r \ , \qquad (2)$$

where $\mu(E)$ denotes the Lebesgue measure of the set E.

It is assumed that the functions and a number $\lambda \in \mathbb{R}^1$ given in system (1) satisfy the following conditions:

A. The functions $f(\cdot) : E \times \mathbb{R}^n \to \mathbb{R}^n$, $K_1(\cdot) : E \times E \times \mathbb{R}^n \to \mathbb{R}^n$ and $K_2(\cdot) : E \times E \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous;

B. There exist $l_0 \in [0, 1)$, $l_1 \ge 0$ and $l_2 \ge 0$ such that

$$\|f(\xi, x_1) - f(\xi, x_2)\| \le l_0 \|x_1 - x_2\|$$

for every $(\xi, x_1) \in E \times \mathbb{R}^n$, $(\xi, x_2) \in E \times \mathbb{R}^n$ and

$$\|K_1(\xi, s, x_1) - K_1(\xi, s, x_2)\| \le l_1 \|x_1 - x_2\| ,$$

$$\|K_2(\xi, s, x_1) - K_2(\xi, s, x_2)\| \le l_2 \|x_1 - x_2\|$$

for every $(\xi, s, x_1) \in E \times E \times \mathbb{R}^n$, $(\xi, s, x_2) \in E \times E \times \mathbb{R}^n$;

C. The inequality

$$0 \le \lambda \left[l_1 \mu(E) + l_2 [\mu(E)]^{\frac{p-1}{p}} r \right] < 1 - l_0$$

is satisfied.

We set

$$l(\lambda) = l_0 + \lambda \left[l_1 \mu(E) + l_2 [\mu(E)]^{\frac{p-1}{p}} r \right]. \qquad (3)$$

If $u(\cdot) \in U_{p,r}$, then (2) and condition C yield

$$\frac{\lambda}{1 - l_0} \int_E (l_1 + l_2 \|u(s)\|)\, ds \le \frac{\lambda}{1 - l_0} \left(l_1 \mu(E) + l_2 [\mu(E)]^{\frac{p-1}{p}} r \right) < 1. \qquad (4)$$

Let us define a trajectory of the system (1) generated by an admissible control function $u(\cdot) \in U_{p,r}$. A continuous function $x(\cdot) : E \to \mathbb{R}^n$ satisfying the integral equation (1) for every $\xi \in E$ is said to be a trajectory of the system (1) generated by the admissible control function $u(\cdot) \in U_{p,r}$. The set of trajectories of the system (1) generated by all control functions $u(\cdot) \in U_{p,r}$ is denoted by $\mathbf{X}_{p,r}$.

For $\xi \in E$ we denote

$$\mathbf{X}_{p,r}(\xi) = \{x(\xi) \in \mathbb{R}^n : x(\cdot) \in \mathbf{X}_{p,r}\}. \qquad (5)$$

The set $\mathbf{X}_{p,r}(\xi)$ is useful for visualization of the set of trajectories.

Now, let us give an auxiliary proposition, which will be used in following arguments.

Proposition 1. *Let $E \subset \mathbb{R}^k$ be a compact set, $v(\cdot) : E \to \mathbb{R}$ and $h(\cdot) : E \to \mathbb{R}$ be continuous functions, $\psi(\cdot) : E \to [0, +\infty)$ be a Lebesgue integrable function, $\int_E \psi(s)ds < 1$ and*

$$v(\xi) \le h(\xi) + \int_E \psi(s)v(s)ds \qquad (6)$$

for every $\xi \in E$. Then the inequality

$$v(\xi) \le h(\xi) + \frac{\int_E h(s)\psi(s)ds}{1 - \int_E \psi(s)ds} \qquad (7)$$

holds for every $\xi \in E$.

Moreover, if $h(\xi) = h_*$ for every $\xi \in E$, then it follows from (7) that

$$v(\xi) \le \frac{h_*}{1 - \int_E \psi(s)ds} \qquad (8)$$

for every $\xi \in E$.

Proof. Since $\psi(\cdot)$ is nonnegative function, we have from (6)

$$v(\xi)\psi(\xi) \le h(\xi)\psi(\xi) + \psi(\xi) \int_E \psi(s)v(s)ds$$

for every $\xi \in E$, and hence

$$\int_E v(s)\psi(s)ds \le \int_E h(s)\psi(s)ds + \int_E \psi(s)ds \cdot \int_E \psi(s)v(s)ds.$$

Since $\int_E \psi(s)ds < 1$, then the last inequality implies

$$\int_E v(s)\psi(s)ds \le \frac{\int_E h(s)\psi(s)ds}{1-\int_E \psi(s)ds}. \qquad (9)$$

(6) and (9) yield the validity of (7). □

3. Existence and Uniqueness of Trajectories

Conditions A - C guarantee that every admissible control function generates a unique trajectory.

Theorem 1. *Let the conditions A - C be satisfied and $u_*(\cdot) \in U_{p,r}$. Then the system (1) has unique trajectory $x_*(\cdot)$ generated by the admissible control function $u_*(\cdot)$.*

Proof. Define a map $x(\cdot) \to A(x(\cdot))$, $x(\cdot) \in C(E;\mathbb{R}^n)$, setting

$$A(x(\cdot))|(\xi) = f(\xi, x(\xi)) + \lambda \int_E \big[K_1(\xi,s,x(s)) + K_2(\xi,s,x(s))\,u_*(s)\big]ds,\ \xi \in E, \qquad (10)$$

where $C(E;\mathbb{R}^n)$ is the space of continuous functions $x(\cdot) : E \to \mathbb{R}^n$ with norm $\|x(\cdot)\|_C = \max\{\|x(\xi)\| : \xi \in E\}$. Since $u_*(\cdot) \in U_{p,r}$, $x(\cdot) \in C(E;\mathbb{R}^n)$ then by virtue of condition A we have that the map $\xi \to A(x(\cdot))|(\xi)$, $\xi \in E$, is continuous, and hence $A(x(\cdot)) \in C(E;\mathbb{R}^n)$.

Let $x_1(\cdot) \in C(E;\mathbb{R}^n)$ and $x_2(\cdot) \in C(E;\mathbb{R}^n)$ be arbitrarily chosen functions. From condition B, (2) and (3) it follows that the inequality

$$\begin{aligned}
&\big\|A(x_2(\cdot))|(\xi) - A(x_1(\cdot))|(\xi)\big\| \\
&\le l_0\|x_2(\xi) - x_1(\xi)\| \\
&\quad + \lambda l_1 \int_E \|x_2(s) - x_1(s)\|\,ds \\
&\quad + \lambda l_2 \int_E \|x_2(s) - x_1(s)\|\,\|u_*(s)\|\,ds \\
&\le \Big[l_0 + \lambda l_1 \mu(E) + \lambda l_2 \int_E \|u_*(s)\|\,ds\Big] \\
&\quad \cdot \|x_2(\cdot) - x_1(\cdot)\|_C \\
&\le \Big[l_0 + \lambda l_1 \mu(E) + \lambda l_2 [\mu(E)]^{\frac{p-1}{p}} r\Big] \\
&\quad \cdot \|x_2(\cdot) - x_1(\cdot)\|_C \\
&= l(\lambda)\|x_2(\cdot) - x_1(\cdot)\|_C
\end{aligned}$$

holds for every $\xi \in E$, and consequently

$$\begin{aligned}
&\big\|A(x_2(\cdot))|(\cdot) - A(x_1(\cdot))|(\cdot)\big\|_C \\
&\quad \le l(\lambda)\|x_2(\cdot) - x_1(\cdot)\|_C.
\end{aligned} \qquad (11)$$

According to the condition C and (3) we have $l(\lambda) < 1$. (11) implies that the map $A(\cdot) : C(E;\mathbb{R}^n) \to C(E;\mathbb{R}^n)$ defined by (10) is contractive, and hence it has a unique fixed point $x_*(\cdot) \in C(E;\mathbb{R}^n)$ which is unique solution of the equation

$$\begin{aligned}
x_*(\xi) &= f(\xi, x_*(\xi)) + \lambda \int_E \big[K_1(\xi,s,x_*(s)) \\
&+ K_2(\xi,s,x_*(s))\,u_*(s)\big]ds,\ \xi \in E.
\end{aligned}$$

□

4. Boundedness

In this section the boundedness of the set of trajectories $\mathbf{X}_{p,r}$ is proved. Denote

$$\gamma_0 = \max\{\|f(\xi,0)\| : \xi \in E\}, \qquad (12)$$

$$\gamma_1 = \max\{\|K_1(\xi,s,0)\| : (\xi,s) \in E \times E\}, \qquad (13)$$

$$\gamma_2 = \max\{\|K_2(\xi,s,0)\| : (\xi,s) \in E \times E\}. \qquad (14)$$

Proposition 2. *Let the functions $f(\cdot) : E\times\mathbb{R}^n \to \mathbb{R}^n$, $K_1(\cdot) : E \times E \times \mathbb{R}^n \to \mathbb{R}^n$ and $K_2(\cdot) : E \times E \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ satisfy the conditions A and B. Then*

$$\|f(\xi,x)\| \le \gamma_0 + l_0\|x\|,$$
$$\|K_1(\xi,s,x)\| \le \gamma_1 + l_1\|x\|,$$
$$\|K_2(\xi,s,x)\| \le \gamma_2 + l_2\|x\|$$

for every $(\xi,s,x) \in E \times E \times \mathbb{R}^n$, where the constants l_0, l_1 and l_2 are given in condition B.

Proof. Let us prove the validity of 3rd inequality. The proofs of 1st and 2nd inequalities are similar. According to the conditions A and B we have

$$\|K_2(\xi,s,x) - K_2(\xi,s,0)\| \le l_2\|x\|$$

for every $(\xi,s,x) \in E \times E \times \mathbb{R}^n$, and hence

$$\begin{aligned}
&\|K_2(\xi,s,x)\| \le l_2\|x\| \\
&\quad + \max\{\|K_2(\xi,s,0)\| : (\xi,s) \in E \times E\}
\end{aligned}$$

The last inequality and (14) complete the proof. □

Denote

$$\gamma_* = \frac{\gamma_0 + \lambda\gamma_1\mu(E) + \lambda\gamma_2[\mu(E)]^{\frac{p-1}{p}} r}{1 - l(\lambda)}, \qquad (15)$$

where $l(\lambda)$ is defined by (3), $\gamma_0 \ge 0$, $\gamma_1 \ge 0$ and $\gamma_2 \ge 0$ are defined by (12), (13) and (14) respectively.

Theorem 2. *Let the conditions A - C be satisfied. Then for every $x(\cdot) \in \mathbf{X}_{p,r}$ the inequality*

$$\|x(\cdot)\|_C \le \gamma_*$$

holds.

Proof. Let $x(\cdot) \in \mathbf{X}_{p,r}$ be an arbitrary trajectory, generated by the admissible control function

$u(\cdot) \in U_{p,r}$. Proposition 2 and (2) imply

$$\begin{aligned} \|x(\xi)\| &\leq \gamma_0 + l_0 \|x(\xi)\| \\ &+\lambda \int_E (\gamma_1 + l_1 \|x(s)\|)\, ds \\ &+\lambda \int_E (\gamma_2 + l_2 \|x(s)\|) \|u(s)\|\, ds \\ &\leq l_0 \|x(\xi)\| + \gamma_0 + \lambda\gamma_1 \mu(E) \\ &+\lambda\gamma_2 [\mu(E)]^{\frac{p-1}{p}} r \\ &+\lambda \int_E (l_1 + l_2 \|u(s)\|) \|x(s)\|\, ds \end{aligned}$$

for every $\xi \in E$. Since $l_0 \in [0,1)$, then we obtain from the last inequality

$$\|x(\xi)\| \leq \frac{\gamma_0 + \lambda\gamma_1\mu(E) + \lambda\gamma_2 [\mu(E)]^{\frac{p-1}{p}} r}{1-l_0} + \frac{\lambda}{1-l_0} \int_E (l_1 + l_2 \|u(s)\|) \|x(s)\|\, ds \tag{16}$$

for every $\xi \in E$. Since $u(\cdot) \in U_{p,r}$, then from (3), (4), (15), (16) and Proposition 1 it follows

$$\begin{aligned} \|x(\xi)\| &\leq \frac{\gamma_0 + \lambda\gamma_1\mu(E) + \lambda\gamma_2 [\mu(E)]^{\frac{p-1}{p}} r}{1-l_0} \cdot \frac{1}{1 - \frac{\lambda}{1-l_0}\int_E (l_1 + l_2\|u(s)\|)\, ds} \\ &\leq \frac{\gamma_0 + \lambda\gamma_1\mu(E) + \lambda\gamma_2 [\mu(E)]^{\frac{p-1}{p}} r}{1-l_0} \cdot \frac{1}{1 - \frac{\lambda}{1-l_0}\left[l_1\mu(E) + l_2 [\mu(E)]^{\frac{p-1}{p}} r\right]} \\ &= \frac{\gamma_0 + \lambda\gamma_1\mu(E) + \lambda\gamma_2 [\mu(E)]^{\frac{p-1}{p}} r}{1-l(\lambda)} = \gamma_* \end{aligned}$$

for every $\xi \in E$, and hence $\|x(\cdot)\|_C \leq \gamma_*$. □

5. Precompactness

Let $\Delta > 0$ be a given number, $\gamma_* > 0$ be defined by (15), $B_n(\gamma_*) = \{x \in \mathbb{R}^n : \|x\| \leq \gamma_*\}$. Denote

$$G_1 = E \times B_n(\gamma_*), \quad G_2 = E \times E \times B_n(\gamma_*), \tag{17}$$

$$\omega_0(\Delta) = \max\{ \|f(\xi_2, x) - f(\xi_1, x)\| : \|\xi_2 - \xi_1\| \leq \Delta, (\xi_1, x) \in G_1, (\xi_2, x) \in G_1 \}, \tag{18}$$

$$\omega_1(\Delta) = \max\{ \|K_1(\xi_2, s, x) - K_1(\xi_1, s, x)\| : \|\xi_2 - \xi_1\| \leq \Delta, (\xi_1, s, x) \in G_2, (\xi_2, s, x) \in G_2 \}, \tag{19}$$

$$\omega_2(\Delta) = \max\{ \|K_2(\xi_2, s, x) - K_2(\xi_1, s, x)\| : \|\xi_2 - \xi_1\| \leq \Delta, (\xi_1, s, x) \in G_2, (\xi_2, s, x) \in G_2 \}, \tag{20}$$

$$\varphi(\Delta) = \frac{1}{1-l_0}\Big\{\omega_0(\Delta) + \lambda\mu(E)\omega_1(\Delta) + \lambda\omega_2(\Delta) [\mu(E)]^{\frac{p-1}{p}} r\Big\}. \tag{21}$$

The function $\varphi(\cdot) : (0, +\infty) \to [0, +\infty)$ is not decreasing and $\varphi(\Delta) \to 0^+$ as $\Delta \to 0^+$.

Proposition 3. *Let the conditions A - C be satisfied. Then for every $x(\cdot) \in \mathbf{X}_{p,r}$, $\xi_1 \in E$, $\xi_2 \in E$ the inequality*

$$\|x(\xi_2) - x(\xi_1)\| \leq \varphi(\|\xi_2 - \xi_1\|)$$

holds, where $\varphi(\cdot)$ is defined by (21).

Proof. Let us choose an arbitrary $x(\cdot) \in \mathbf{X}_{p,r}$ and $\xi_1 \in E$, $\xi_2 \in E$. Then there exists $u(\cdot) \in U_{p,r}$ such that

$$x(\xi) = f(\xi, x(\xi)) + \lambda \int_E \big[K_1(\xi, s, x(s)) + K_2(\xi, s, x(s))\, u(s)\big] ds$$

for every $\xi \in E$, and hence

$$\begin{aligned} &\|x(\xi_2) - x(\xi_1)\| \\ &\leq \|f(\xi_2, x(\xi_2)) - f(\xi_1, x(\xi_2))\| \\ &+ \|f(\xi_1, x(\xi_2)) - f(\xi_1, x(\xi_1))\| \\ &+\lambda \int_E \big\|K_1(\xi_2, s, x(s)) - K_1(\xi_1, s, x(s))\big\| ds \\ &+\lambda \int_E \big\|K_2(\xi_2, s, x(s)) - K_2(\xi_1, s, x(s))\big\| \|u(s)\|\, ds . \end{aligned} \tag{22}$$

By virtue of condition B we have

$$\|f(\xi_1, x(\xi_2)) - f(\xi_1, x(\xi_1))\| \leq l_0 \|x(\xi_2) - x(\xi_1)\|, \tag{23}$$

where $l_0 \in [0,1)$. Since $x(\cdot) \in \mathbf{X}_{p,r}$, then it follows from Theorem 2 that

$$x(s) \in B_n(\gamma_*) \tag{24}$$

for every $s \in E$. (17), (18), (19), (20) and (24) imply

$$\|f(\xi_2, x(\xi_2)) - f(\xi_1, x(\xi_2))\| \leq \omega_0(\|\xi_2 - \xi_1\|), \tag{25}$$

$$\|K_1(\xi_2, s, x(s)) - K_1(\xi_1, s, x(s))\| \leq \omega_1(\|\xi_2 - \xi_1\|), \tag{26}$$

$$\|K_2(\xi_2, s, x(s)) - K_2(\xi_1, s, x(s))\| \leq \omega_2(\|\xi_2 - \xi_1\|) \tag{27}$$

for every $s \in E$.

From (2), (21), (22), (23), (25), (26) and (27) we obtain that

$$\begin{aligned}\|x(\xi_2)-x(\xi_1)\| &\le \frac{1}{1-l_0}\Big\{\omega_0\left(\|\xi_2-\xi_1\|\right)\\ &+\lambda\mu(E)\omega_1\left(\|\xi_2-\xi_1\|\right)\\ &+\lambda\omega_2\left(\|\xi_2-\xi_1\|\right)\int_E\|u(s)\|\,ds\Big\}\\ &\le \frac{1}{1-l_0}\Big\{\omega_0\left(\|\xi_2-\xi_1\|\right)\\ &+\lambda\mu(E)\omega_1\left(\|\xi_2-\xi_1\|\right)\\ &+\lambda\omega_2\left(\|\xi_2-\xi_1\|\right)[\mu(E)]^{\frac{p-1}{p}}r\Big\}\\ &=\varphi\left(\|\xi_2-\xi_1\|\right).\end{aligned}$$

□

Proposition 4. *Let the conditions A - C be satisfied. Then the set of trajectories $\mathbf{X}_{p,r}\subset C\left(E;\mathbb{R}^n\right)$ is a set of equicontinuous functions.*

Proof. Since $\varphi(\Delta)\to 0^+$ as $\Delta\to 0^+$, then for given $\varepsilon>0$ there exists $\Delta_*(\varepsilon)>0$ such that for every $\Delta\in(0,\Delta_*(\varepsilon)]$ the inequality

$$\varphi(\Delta)\le\varepsilon \tag{28}$$

is satisfied, where $\varphi(\cdot)$ is defined by (21).
Now let $x(\cdot)\in\mathbf{X}_{p,r}$ be an arbitrarily chosen trajectory, $\xi_1\in E$, $\xi_2\in E$ be such that $\|\xi_2-\xi_1\|\le\Delta_*(\varepsilon)$. Since the function $\varphi(\cdot):(0,+\infty)\to[0,+\infty)$ is not decreasing, then from (28) and Proposition 3 it follows

$$\|x(\xi_2)-x(\xi_1)\|\le\varphi\left(\|\xi_2-\xi_1\|\right)\le\varphi\left(\Delta_*(\varepsilon)\right)\le\varepsilon,$$

and hence the set of trajectories $\mathbf{X}_{p,r}\subset C\left(E;\mathbb{R}^n\right)$ is a set of equicontinuous functions. □

Theorem 2 and Proposition 4 yield the validity of the following theorem.

Theorem 3. *Let the conditions A - C be satisfied. Then the set of trajectories $\mathbf{X}_{p,r}$ is a precompact subset of the space $C\left(E;\mathbb{R}^n\right)$.*

The Hausdorff distance between the sets $P\subset\mathbb{R}^n$ and $S\subset\mathbb{R}^n$ is denoted by $H(P,S)$ and defined as

$$H(P,S)=\max\{\sup_{p\in P}d(p,S),\sup_{s\in S}d(s,P)\},$$

where $d(p,S)=\inf\left\{\|p-s\|:s\in S\right\}$.
Proposition 3 implies the validity of the following proposition.

Proposition 5. *Let the conditions A - C be satisfied. Then for every $\xi_1\in E$ and $\xi_2\in E$ the inequality*

$$H\left(\mathbf{X}_{p,r}(\xi_2),\mathbf{X}_{p,r}(\xi_1)\right)\le\varphi\left(\|\xi_2-\xi_1\|\right)$$

is satisfied, where the function $\varphi(\cdot):(0,\infty)\to[0,\infty)$ is defined by (21), the sets $\mathbf{X}_{p,r}(\xi_1)$ and $\mathbf{X}_{p,r}(\xi_2)$ are defined by (5).

Since $\varphi(\Delta)\to 0^+$ as $\Delta\to 0^+$ then we conclude the validity of the following corollary.

Corollary 1. *Let the conditions A - C be satisfied. Then the set valued map $\xi\to\mathbf{X}_{p,r}(\xi)$, $\xi\in E$, is continuous.*

6. Closedness

Theorem 4. *Let the conditions A - C be satisfied. Then the set of trajectories $\mathbf{X}_{p,r}$ is a closed subset of the space $C\left(E;\mathbb{R}^n\right)$.*

Proof. Let us choose a sequence of trajectories $\{x_i(\cdot)\}_{i=1}^{\infty}$, where $\|x_i(\cdot)-x_*(\cdot)\|_C\to 0$ as $i\to\infty$ and $x_*(\cdot)\in C\left(E;\mathbb{R}^n\right)$. We have to prove that $x_*(\cdot)\in\mathbf{X}_{p,r}$.
Since $x_i(\cdot)\in\mathbf{X}_{p,r}$, then there exists $u_i(\cdot)\in U_{p,r}$ such that

$$\begin{aligned}x_i(\xi) &= f\left(\xi,x_i\left(\xi\right)\right)+\lambda\int_E\big[K_1\left(\xi,s,x_i\left(s\right)\right)\\ &+ K_2\left(\xi,s,x_i\left(s\right)\right)u_i\left(s\right)\big]ds\end{aligned}\tag{29}$$

for every $\xi\in E$. Since the set of admissible control functions $U_{p,r}\subset L_p\left(E;\mathbb{R}^n\right)$ is weakly compact, then without loss of generality, one can assume that the sequence $\{u_i(\cdot)\}_{i=1}^{\infty}$ weakly converges to a $u_*(\cdot)\in U_{p,r}$. Let $y_*(\cdot):E\to\mathbb{R}^n$ be a trajectory of the system (1) generated by the admissible control function $u_*(\cdot)\in U_{p,r}$. Then

$$\begin{aligned}y_*(\xi) &= f\left(\xi,y_*\left(\xi\right)\right)+\lambda\int_E\big[K_1\left(\xi,s,y_*\left(s\right)\right)\\ &+ K_2\left(\xi,s,y_*\left(s\right)\right)u_*\left(s\right)\big]ds\end{aligned}\tag{30}$$

for every $\xi\in E$. (29), (30) and condition B yield that

$$\begin{aligned}&\|x_i\left(\xi\right)-y_*\left(\xi\right)\|\\ &\le\frac{\lambda}{1-l_0}\int_E\left(l_1+l_2\left\|u_i(s)\right\|\right)\\ &\cdot\left\|x_i\left(s\right)-y_*\left(s\right)\right\|ds\\ &+\frac{\lambda}{1-l_0}\bigg\|\int_E K_2\left(\xi,s,y_*\left(s\right)\right)\\ &\cdot\left(u_i(s)-u_*(s)\right)ds\bigg\|\end{aligned}\tag{31}$$

for every $\xi\in E$. Denote $w(\xi,s)=K_2\left(\xi,s,y_*\left(s\right)\right)$. Since the function $w(\cdot):E\times E\to\mathbb{R}^{n\times m}$ is continuous and the sequence $\{u_i(\cdot)\}_{i=1}^{\infty}$ weakly converges to $u_*(\cdot)\in U_{p,r}$ in the space $L_p\left(E;\mathbb{R}^n\right)$, then we have that for each fixed $\xi\in E$

$$\left\|\int_E w(\xi,s)\left[u_i(s)-u_*(s)\right]ds\right\|\to 0 \tag{32}$$

as $i\to\infty$. From (32) we obtain that for $\varepsilon>0$ and fixed $\xi\in E$ there exists $N(\varepsilon,\xi)>0$ such that for every $i > N(\varepsilon, \xi)$ the inequality

$$\left\| \int_E w\left(\xi, s\right)\left[u_i(s) - u_*(s)\right] ds \right\| < \varepsilon \qquad (33)$$

is satisfied.

Now let us prove that for each $\varepsilon > 0$ there exists $N(\varepsilon) > 0$ (which does not depend on ξ) such that for every $i > N(\varepsilon)$ and $\xi \in E$ the inequality

$$\left\| \int_E w\left(\xi, s\right)\left(u_i(s) - u_*(s)\right) ds \right\| < \varepsilon \qquad (34)$$

holds.

Let us assume the contrary, i.e. let there exist $\varepsilon_* > 0$, $i_j > 0$ and $\xi_j \in E$ $(j = 1, 2, \ldots)$ such that $i_j \to \infty$ as $j \to \infty$ and

$$\left\| \int_E w\left(\xi_j, s\right)\left[u_{i_j}(s) - u_*(s)\right] ds \right\| \geq \varepsilon_* \,. \qquad (35)$$

Since $\xi_j \in E$ for every $j = 1, 2, \ldots$ and $E \subset \mathbb{R}^k$ is a compact set, then without loss of generality one can assume that $\xi_j \to \xi_*$ as $j \to \infty$ and $\xi_* \in E$.

(33) implies that for $\varepsilon_* > 0$ and $\xi_* \in E$ there exists $N_1 > 0$ such that for every $j > N_1$ the inequality

$$\left\| \int_E w\left(\xi_*, s\right)\left[u_{i_j}(s) - u_*(s)\right] ds \right\| < \frac{\varepsilon_*}{4} \qquad (36)$$

is verified.

Continuity of the function $w\left(\cdot\right) : E \times E \to \mathbb{R}^{n\times m}$ and compactness of the set E yield that for given $\dfrac{\varepsilon_*}{8\left[\mu(E)\right]^{\frac{p-1}{p}} r}$ there exists $N_2 > 0$ such that for every $j > N_2$ and $s \in E$ the inequality

$$\left\|w(\xi_j, s) - w(\xi_*, s)\right\| < \frac{\varepsilon_*}{8\left[\mu(E)\right]^{\frac{p-1}{p}} r} \qquad (37)$$

holds. Since $u_*(\cdot) \in U_{p,r}$, $u_{i_j}(\cdot) \in U_{p,r}$ for every $j = 1, 2, \ldots$, then from (2) and (37) it follows

$$\begin{aligned}
&\left\| \int_E \left[w(\xi_j, s) - w(\xi_*, s)\right]\left[u_{i_j}(s) - u_*(s)\right] ds \right\| \\
&\leq \frac{\varepsilon_*}{8\left[\mu(E)\right]^{\frac{p-1}{p}} r} \int_E \left[\, \left\|u_{i_j}(s)\right\| + \left\|u_*(s)\right\| \,\right] ds \\
&\leq 2\, \frac{\varepsilon_*}{8\left[\mu(E)\right]^{\frac{p-1}{p}} r} \left[\mu(E)\right]^{\frac{p-1}{p}} r = \frac{\varepsilon_*}{4} \qquad (38)
\end{aligned}$$

for every $j > N_2$.

Denote $N_* = \max\left\{N_1, N_2\right\}$. Then (36) and (38) imply that

$$\begin{aligned}
&\left\| \int_E w(\xi_j, s)\left[u_{i_j}(s) - u_*(s)\right] ds \right\| \\
&\leq \left\| \int_E \left[w\left(\xi_j, s\right) - w(\xi_*, s)\right] \right. \\
&\quad \left. \cdot \left[u_{i_j}(s) - u_*(s)\right] ds \right\| \\
&\quad + \left\| \int_E w\left(\xi_*, s\right)\left[u_{i_j}(s) - u_*(s)\right] ds \right\| \\
&< \frac{\varepsilon_*}{4} + \frac{\varepsilon_*}{4} = \frac{\varepsilon_*}{2} < \varepsilon_* \qquad (39)
\end{aligned}$$

for every $j > N_*$. The inequalities (35) and (39) contradict, and hence the inequality (34) is held. Thus, from (31) and (34) we have that for every $\xi \in E$ and $i > N(\varepsilon)$ the inequality

$$\begin{aligned}
\left\|x_i\left(\xi\right) - y_*\left(\xi\right)\right\| &\leq \frac{\lambda\varepsilon}{1 - l_0} \\
&+ \frac{\lambda}{1 - l_0} \int_E \left(l_1 + l_2 \left\|u_i(s)\right\|\right) \\
&\cdot \left\|x_i\left(s\right) - y_*\left(s\right)\right\| ds \qquad (40)
\end{aligned}$$

is satisfied.

Since $u_i(\cdot) \in U_{p,r}$ for every $i = 1, 2, \ldots$, then from (4), (40) and Proposition 1 we have that for every $i > N(\varepsilon)$ and $\xi \in E$ the inequality

$$\begin{aligned}
&\left\|x_i\left(\xi\right) - y_*\left(\xi\right)\right\| \\
\leq\; & \frac{\lambda\varepsilon}{1 - l_0} \cdot \frac{1}{1 - \dfrac{\lambda}{1 - l_0} \displaystyle\int_E \left(l_1 + l_2 \left\|u_i(s)\right\|\right) ds} \\
\leq\; & \frac{\lambda\varepsilon}{1 - l_0} \cdot \frac{1}{1 - \dfrac{\lambda}{1 - l_0}\left[l_1\mu(E) + l_2\left[\mu(E)\right]^{\frac{p-1}{p}} r\right]} \\
=\; & \frac{\lambda}{1 - l(\lambda)} \cdot \varepsilon
\end{aligned}$$

holds, where $l(\lambda)$ is defined by (3). This means that $x_i(\cdot) \to y_*(\cdot)$ as $i \to +\infty$. From uniqueness of the limit we have $x_*(\cdot) = y_*(\cdot)$ and hence $x_*(\cdot) \in \mathbf{X}_{p,r}$. □

Theorem 3 and Theorem 4 imply compactness of the set of trajectories.

Theorem 5. *Let the conditions A - C be satisfied. Then the set of trajectories* $\mathbf{X}_{p,r}$ *is a compact subset of the space* $C\left(E; \mathbb{R}^n\right)$.

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