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Optimization of cereal output in presence of locusts

Nacima Moussouni[a] and Mohamed Aidene[b]

University Mouloud Mammeri of Tizi-Ouzou, Faculty of Science,
Department of Mathematics, L2CSP Laboratory

Email: [a]nmoussouni@yahoo.fr, [b]aidene@mail.ummto.dz

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Abstract. In this paper, we study a modelization of the evolution of cereal output production, controlled by adding fertilizers and in presence of locusts, then by adding insecticides. The aim is to maximize the cereal output and meanwhile minimize pollution caused by adding fertilizers and insecticides. The optimal control problem obtained is solved theoretically by using the Pontryagin Maximum Principle, and then numerically with shooting method.

Keywords: Optimal control; optimization; Pontryagin maximum principal.
AMS Classification: 49J15.

1. Statement of the problem

Consider $x(t)$, $t \in [0, T]$ the rate of pollution at time t. In cereal field, If the farmer does not put fertilizers and insecticides then the evolution of pollution satisfies

$$\dot{x}(t) = -\alpha x(t), \quad t \in [0, T];$$

where α is the natural decreasing rate. Note that the rate $x(t)$, $t \in [0, T]$ is decreasing.
In order to increase the cereal output, we add fertilizers and insecticides to protect the crop harm locusts. Denoting by $u(t)$ and $v(t)$, $t \in [0, T]$ the quantities of fertilizers and insecticides respectively, in this case $x(t)$ is evolving as

$$\begin{aligned} \dot{x}(t) &= -\alpha x(t) + u(t) + v(t), \qquad (1) \\ x(0) &= x_0 > 0, t \in [0, T] \end{aligned}$$

Our goal is to minimize the pollution generated by fertilizers and insecticides, and optimize the cereal output from the seed to the harvest.

In practice, we choose typically $T = 10\, months$, which corresponds to a cycle of cereal production from September to July.
Let be $y(t)$, $t \in [0, T]$ the quantity of the cereal production. Adding fertilizers, the production increases, this production decreases with the presence of locusts and by adding large quantities of fertilizers.

Denoting by $z(t), t \in [0, T]$, the quantity of locusts presents in cereal field. In this case the evolution of cereal output is given for $t \in [0, T]$ by:

$$\begin{aligned} \dot{y}(t) &= -by(t)z(t) + \sqrt{(M - u(t))(m + u(t))} \qquad (2) \\ y(0) &= 0. \end{aligned}$$

and $z(t)$ verify the following equation

$$\begin{aligned} \dot{z}(t) &= z(t)(c(t)y(t) - d(t)) - v(t), \qquad (3) \\ z(0) &= z_0 > 0, \quad t \in [0, T] \end{aligned}$$

where $m > 0, M > 0$ are real numbers, b is the rate of reproduction of cereal, $c(t), t \in [0, T]$ is the rate of reproduction of locusts and $d(t), t \in [0, T]$, is the rate of extinction of locusts. All those parameters will be identified subsequently.

Note that if we add a too large quantity of fertilizers and insecticides, this causes the death of locusts but also the death of cereals.

The functions $u(.)$ and $v(.)$ are considered as controls. Those controls $u(.)$ and $v(.)$ considered are submitted to constraints. They are such that

$$0 \leq v(.) \leq V, \ (*)$$

and

$$0 \leq u(.) \leq M. \ (**)$$

Note that $V > 0$ will be by identified in the following section.

In a cereal field, the aim is to maximize the production of cereals and to minimize the bad effects of pollution given by insecticides and fertilizers. For this our criterion is:

$$J(u) = \beta x(T) - y(T) \to \min_{u,\ v},$$

where $\beta > 0$ is a real number to be chosen, $x(.)$ is solution of (1) and $y(.)$ is solution of (2). Minimizing J corresponds to realizing a compromise between maximizing the cereal output and minimizing the bad effects of pollution given by insecticides and fertilizers.

Finally, our problem is as follows

$$\begin{cases} \dot{x}(t) = -\alpha x(t) + u(t) + v(t), \\ x(0) = x_0 > 0 \\ \\ \dot{y}(t) = -by(t)z(t) \\ +\sqrt{(M - u(t))(m + u(t))}, \\ y(0) = 0 \\ \\ \dot{z}(t) = z(t)(c(t)y(t) - d(t)) - v(t), \\ z(0) = z_0 > 0 \\ \\ 0 \leq v(t) \leq V \\ \\ 0 \leq u(t) \leq M. \quad t \in [0, T]. \end{cases}$$

Here we consider that the final time T is fixed.

This problem is inspired by a model used in [9], where the authors formulated a model without presence of locusts. They calculated the quantities of fertilizers to put in cereal field to get a better production. The reader can refer also to [11].

This article is structured as follows. In Section 2, we provide an identification of the parameters considered in the model with real life measures used in Algeria see [4,7]. In this section, we calculated the rate of reproduction of cereals using a simple dechotomy method. we calculated also the durations of maturity of locusts, then the reproduction rate and the rate of extinction of locusts, in hot and cold seasons.

Section 3 is devoted to the study of necessary condition of optimality based on the Pontryagin Maximum Principle see[10,12]. We make a rigorous mathematical analysis of the extremal equations leading to a precise expression of the optimal control. In Section 4, we provide numerical simulations based on the rigorous mathematical analysis, using the shooting method and we comment these results. Note that these numerical results describe the best possible way for a farmer to realize a good compromise between maximizing the cereal output and minimizing pollution effects consequences of fertilizers and insecticides.

2. Identification of the parameters of the model

In what follows, the time t is given in months, and T corresponds to a cycle of cereal production, $T = 10$ months.

The quantities of fertilizers used in Algeria are given by [7]:

$$u(t) = \begin{cases} 100kg/ha \ if \\ t \in [0,1] = [September, October] \\ \\ \frac{100}{3}kg/ha \ if \\ t \in [2,3] = [November, December] \\ \\ \frac{200}{3}kg/ha \ if \\ t \in [6,7] = [March, April] \\ \\ 0 \\ otherwise([1,2] \cup [3,6] \cup [7,10]) \end{cases} \tag{4}$$

According to [7], the quantity of cereal output without fertilizers is equal to 500 *kilograms* per *hectare*.

From this in our model corresponds to $u(t) = 0$, and hence, using equation (2); $\dot{y}(t) = \sqrt{Mm}$, then we obtain

$$10\sqrt{Mm} = 500. \tag{5}$$

From [7], the cereal output with the addition of fertilizers and in absence of locusts as described

by (2) is equal to 4500 *kilograms* per *hectare*.
In our model, this leads to

$$y(T) = 4500 = \int_0^1 \sqrt{(M-100)(m+100)}dt + \int_2^3 \sqrt{(M-100/3)(m+100/3)}dt + \int_6^7 \sqrt{(M-200/3)(m+200/3)}dt + 7\sqrt{Mm}.$$

solving the system

$$\begin{cases} 10\sqrt{Mm} = 500 \\ \sqrt{(M-100)(m+100)} \\ +\sqrt{(M-100/3)(m+100/3)} \\ +\sqrt{(M-200/3)(m+200/3)} \\ +7\sqrt{Mm} = 4500 \end{cases}.$$

and leads to

$$M = 300, 83, \quad m = 0.00083.$$

Let us now compute the value of the decreasing rate α of pollution, according to real-life data. In the absence of fertilizers and insecticides, $u(t) = 0$, we have $x(0) = x_0 = 119mg/l$ at $t = 0$, and $x(T) = 28mg/l$ at $T = 10$ *months*. From this by using formula (1), we obtain:

$$x(T) = x_0 e^{-\alpha T}$$
$$\Leftrightarrow 28 = 119e^{-10\alpha},$$

then

$$\alpha = 0.12.$$

Note that the locust attack held in May. From equation (2), and in absence of fertilizers ($u(t) = 0$),

$$\dot{y}(t) = -b\ y(t)\ z(t) + \sqrt{mM},\ t \in [0, T].$$

the larval density causing damage is 5000 locusts per hectare [4], they consume 80 % of cereal a day.

For $t_1 = \frac{1}{30}$ *month* = 1 day, we will have:

$$y(\frac{1}{30}) = 0.2y_0.$$

Note that $y_0 = y(8)$ is cereal production at the time of the attack of locusts. the value of b is determined by solving the following differential equation:

$$\dot{y}(t) = -5000\ b\ y(t) + \sqrt{Mm}$$

under the initial conditions:

$$y(8) = y_0, y(\frac{1}{30}) = 0.2y_0.$$

Using these data, we will have

$$\frac{d}{dt}(y(t) - \frac{\sqrt{Mm}}{bz(t)}) = -bz(t)(y(t) - \frac{\sqrt{Mm}}{bz(t)})$$

$$\Rightarrow \quad y(t) - \frac{\sqrt{Mm}}{bz(t)} = cste \times e^{-bz(t)t};$$

then

$$y(t) = \frac{\sqrt{Mm}}{bz(t)} + (y_0 - \frac{\sqrt{Mm}}{bz(t)})e^{-bz(t)t},\ t \in [0, T].$$

For $t = t_1$, we will have:

$$\frac{\sqrt{Mm}}{bz(t)} + (y_0 - \frac{\sqrt{Mm}}{bz(t)})e^{-bz(t)t_1} = 0.2y_0.$$

To determine the value of y_0, we set the following assumptions:

- The locust come in May .
- Insects attack a fraught field of cereal.
- The field has not been attacked before May.

To calculate y_0, we solve the following equation:

$$y(8) = \int_0^1 \sqrt{(M-100)(m+100)}dt + \int_2^3 \sqrt{(M-100/3)(m+100/3)}dt + \int_6^7 \sqrt{(M-200/3)(m+200/3)}dt + 5\sqrt{Mm}.$$

Such that $M = 300$ and $m = 0.00083$, then $y(8) = 4379kg/ha$. To determine the value of b, plot the graph of the following function:

$$b \mapsto \frac{\sqrt{Mm}}{bz} + (y_0 - \frac{\sqrt{Mm}}{bz})e^{-bzt_1} - 0.2y_0$$

where $t_1 = \frac{1}{30}$ month, $y_0 = 4379kg/ha$ and z = $5000locusts/ha$.

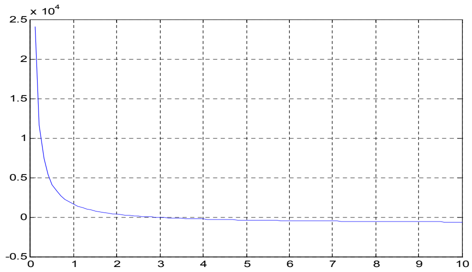

Figure 1. $t \mapsto b(t)$.

By Dichotomy, we obtain $b = 2.85$ (see Figure 1).

Population of locusts on which is based our work is 55000 larvae and 375 adults. A female lays 140 eggs in two generations [4]. Average losses of eggs are about 33% [4]. The Table 1 shows the duration of maturity of locusts. For method for calculating incubation periods see [4].

Table 1. Durations of maturity of locusts [4].

Locusts temperatures	high	bass
Eggs	IT: 11 days	IT: 41 days
Larvae	DT: 80 days	DT: 21 days
Adults	AM : 20 days	AM: 6 months

Indications: IT: Incubation time, DT: Development time and AM: Adults maturity.

After hatching of eggs, the larvae pass from five larval stages L_1, L_2, L_3,L_4, L_5. The percentages of mortality in different stages of larvae are given in Table 2. For more informations see[4]

Table 2. Larval mortality [4].

Stages	L_1	L_2	L_3	L_4	L_5
percentages	70%	20%	10%	10%	10%

There, and using these data, we calculate the number of locusts that can produce a viable female in hot and cold seasons.

Hot seasons:

$$N_1 = 140 \times 0.66 \times 0.3 \times 0.8 \times (0.9)^3 = 16.16$$
$$\simeq 17 locusts$$

Cold seasons:

$$N_2 = 140 \times 0.35 \times 0.3 \times 0.8 \times (0.9)^3 = 8.57$$
$$\simeq 9\ locusts$$

In other words: On 55375 Locust (larvae, immature adults , mature adults) we assume that 100 females lay their eggs in two generations. They will generate 17 locusts viable in the hot season and 9 locusts viable in the cold season. The rate of reproduction of locusts $c(t)$, $t \in [0, T]$ is represented in Figure 2 and calculated as follows:

$$c(t) = \begin{cases} \frac{200}{55375} * 17, & in\ hot\ season \\ \frac{200}{55375} * 9, & in\ cold\ season \end{cases}.$$

Then

$$c(t) = \begin{cases} 0.0613, & in\ hot\ season \\ 0.0288, & in\ cold\ season \end{cases}.$$

Analytic expression of $c(t), t \in [0, T]$ is

$$c(t) = 0.0288 + (0.0613 - 0.0288)\frac{(t-5)^2}{25}$$

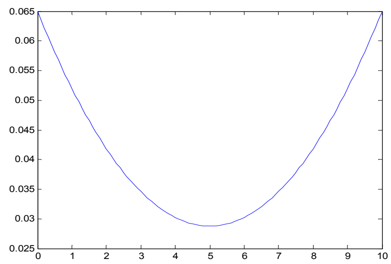

Figure 2. $t \mapsto c(t)$.

The average lifespan of a locust is 3 months in hot periods, it is 8 months in cold periods (see [4]). we assume that we are at 0 when 90 % of the population locusts disappeared (this assumption is possible because after elimination of locusts, the solitary locusts do not disappear).

From Constraint (3), in the absence of insecticides and food,

$$\dot{z}(t) = -d\ z(t).$$

This differential equation has the solution:

$$z(t) = z_0 e^{-dt}, \quad t \in [0, T],$$

where $z_0 = z(0)$.

In hot season, $t = 3\ months$,

$$z_0 \; e^{-3d} = 0.1 \; z_0$$

$$\Rightarrow e^{-3d} = 0.1$$

$$d = -\frac{1}{3} ln(0.1) = 0.767$$

In cold season $t = 8$ *months*:

$$z_0 \; e^{-8d} = 0.1 \; z_0$$

$$\Rightarrow e^{-8d} = 0.1$$

$$d = -\frac{1}{8} ln(0.1) = 0.287.$$

In other words:

$$d(t) = \begin{cases} 0.767 & \text{in hot season} \\ 0.287 & \text{in cold season} \end{cases}.$$

The analytical expression for the rate of extinction of locusts $d(t)$ represented in Figure 3 and it is given by:

$$d(t) = 0.287 + (0.767 - 0.287) \frac{(t-5)^2}{25}, \quad t \in [0, T].$$

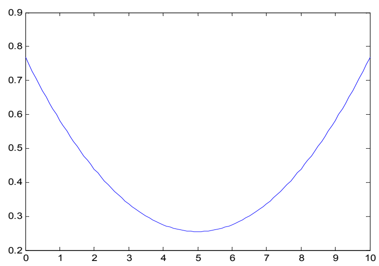

Figure 3. $t \mapsto d(t)$.

For the value of $V = 5l/ha$ see [4].

3. Theoretical solving of the optimal control problem

In this section, we solve the problem (4) theoretically by employing the Pontryagin Maximum Principle.

Let us first recall a version of the Pontryagin Maximum Principle (see [10,12]).

Theorem 1. *We consider the control system on* $\mathbb{R}^n$

$$\dot{x}(t) = f(t, x(t), u(t)), \tag{6}$$

where $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *in* C^1. *Where the controls are the measurable and bounded functions in* $[0, t_e(u)]$ *of* $\mathbb{R}^+$ *in values in* $\Omega \subset \mathbb{R}^m$. *Let* M_0 *and* M_1 *two subsets of* $\mathbb{R}^m$. *Note by* U *the set of an admissible controls* u *whose corresponding trajectories connect one point of* M_0 *to a final point in* M_1 *in time* $t(u) < t_e(u)$. *Note the quality criterion by*

$$C(t, u) = \int_0^t f^0(s, x(s), u(s))ds + g(t, x(t)),$$

where $f^0 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ *and* $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *in* C^1, *and* $x(.)$ *is solution of (7) associated to the control* u.

We consider the following optimal control problem: determine one trajectory connecting M_0 *to* M_1 *and minimize the cost.*

If a control u *is optimal in* $[0, T]$, *then there exist an application p(.) absolutely continuous on [0,T], with values in* $\mathbb{R}^n$, *called adjoint vector, and a real nonpositive number* p^0 *such that* $(p(.), p^0)$ *is nontrivial, and for almost all* $t \in [0, T]$

$$\dot{x}(t) = \frac{\partial H}{\partial p}(t, x(t), p(t), p^0, u(t)), \tag{7}$$

$$\dot{p}(t) = \frac{-\partial H}{\partial x}(t, x(t), p(t), p^0, u(t)). \tag{8}$$

where

$$H(t, x, p, p^0, u) = p'(t) f(t, x, u) + \; p^0 f^0(t, x, u)$$

is the Hamiltonian of the system (7).

Moreover, we have a condition of maximization

$$\begin{aligned} & H(t, x(t), p(t), p^0, u(t)) \\ & = \max_{v \in U} H(t, x(t), p(t), p^0, v), \end{aligned} \tag{9}$$

for all $t \in [0, T]$.

Moreover, if M_0 *and* M_1 *are two submanifolds of* $\mathbb{R}^n$ *having tangent spaces in* $x(0) \in M_0$ *and* $x(T) \in M_1$, *then the adjoint vector satisfies the transversality conditions*

$$p(0) \perp T_{x(0)} M_0, \tag{10}$$

and

$$p(T) - p^0 \frac{\partial g}{\partial x}(T, x(T)) \perp T_{x(T)} M_1. \tag{11}$$

We apply the Pontryagin Maximum Principle to our specific optimal control problem.
The Hamiltonian of System (4) is

$$H(x,p,p^0,u,v) = p_x(-\alpha\ x+u+v)+p_y(-b\ y\ z+\sqrt{(M-m)(m+u)})+p_z(z(cy-d)-v)$$

where $p(t) = \begin{pmatrix} p_x(t) \\ p_y(t) \\ p_z(t) \end{pmatrix}$, $t \in [0,T]$ is adjoint vector solution of the following system:

$$\begin{cases} \dot{p}_x = -\frac{\partial H}{\partial x} = \alpha p_x, \\ \dot{p}_y = -\frac{\partial H}{\partial y} = bp_y z - cp_z z = \\ bzp_y - czp_z = z(bp_y - cp_z), \\ \dot{p}_z = -\frac{\partial H}{\partial z} = byp_y - p_z(cy-d) = \\ y(bp_y - cp_z) + p_z d. \end{cases} \tag{12}$$

The final transversality condition
We know that $x(T)$ is free, the final transversality condition leads to

$$p(T) = p^0 \nabla g(x(T)).$$

In other words:

$$p_x(T) = p^0 \frac{\partial}{\partial x}(\beta x(T) - y(T)),$$

$$p_y(T) = p^0 \frac{\partial}{\partial y}(\beta x(T) - y(T)),$$

$$p_z(T) = p^0 \frac{\partial}{\partial z}(\beta x(T) - y(T)).$$

Then $p_x(T) = -\beta, \quad p_y(T) = 1, \quad p_z(T) = 0.$
It is easy to see that

$$p_x(t) = -\beta e^{\alpha(t-T)}, \quad t \in [0,T]$$

Remark 1. *Since $\beta > 0$, it follows that, for every $t \in [0,T]$, $p_x(t) < 0$.*

Lemma 1. $p^0 \neq 0$.

Proof. We argue by contradiction. If $p^0 = 0$ then $p(T) = 0$ so $(p(T), p^0) = (0,0)$; this is in contradiction with the Pontryagin Maximum Principle. □

Remark 2. *If $p^0 = 0$ then the extremal $(x(.), p(.), u(.))$ is said normal and in this case it is usual to normalize the adjoint vector so that $p^0 = -1$. If $p^0 = 0$, then the extremal $(x(.), p(.), u(.))$ is said abnormal. Note that several works have been devoted to the investigation of abnormal extremals in a generic context (see [1],[2],[3]). In our example, the abnormal case does not occur.*

Lemma 2. *$p_y(.)$ is not canceled identically on a sub interval.*

Proof. Assume that $p_y \equiv 0$ is canceled identically on a sub interval of $[0,T]$,

$$p_y \equiv 0 \Rightarrow \dot{p_y} \equiv 0$$
$$\Rightarrow z(t)(bp_y(t) - c(t)p_z(t)) = 0$$
$$\Rightarrow p_z \equiv 0$$

So by unicity of Cauchy,

$$p_y \equiv p_z \equiv 0 \ sur\ [0,T],$$

this is in contradiction with $p_y(T) = 1$. □

For the proof of the following lemma see [11].

Lemma 3. *$p_x(.) - p_z(.)$ does not vanish identically on a subset interval.*

To study the maximization condition, we search the maximum on v and u of the following function

$$p_x u(t) + p_y(t)\sqrt{(M-u(t))(m+u(t))} + (p_x(t) - p_z(t))v(t)$$

It is clear that

$$v(t) = \begin{cases} 0 & if\ p_x(t) - p_z(t) < 0 \quad , \\ V & if\ p_x(t) - p_z(t) > 0 \quad . \end{cases} \tag{13}$$

To determine the optimal control $u(.)$, we search in $[-m, M]$, the maximum of the following function

$$\phi(u) = p_x u + p_y \sqrt{(M-u)(m+u)}.$$

Function ϕ is defined on $[-m, M]$. To found its absolute maximum proceed as follows

$$\phi'(u) = p_x + p_y \frac{-u + \frac{M-m}{2}}{\sqrt{(M-u)(m+u)}}$$

$$\phi'(u) = 0 \Leftrightarrow p_y \frac{-u + \frac{M-m}{2}}{\sqrt{(M-u)(m+u)}} = -p_x$$

Note that $p_x(t) < 0$, then

$$p_y(-u + \frac{M-m}{2}) > 0$$

$$(p_x^2 + p_y^2)u^2 - (M-m)(p_x^2 + p_y^2)u + \frac{(M-m)^2}{2} p_y^2$$

$$-p_x^2 Mm = 0.$$

The absolute maximum of ϕ on $[-m, M]$ is

$$u_\phi = \frac{M-m}{2} + \frac{M+m}{2\sqrt{p_x^2(t)+p_y^2(t)}} p_x sign(p_y(t))$$

We deduce two cases:

$p_y(t) > 0$, in this case the maximum of ϕ on $[0, M]$ is

$$0 \;\; if \;\; u_\phi < 0,$$

$$u_\phi \;\; if \;\; u_\phi \geq 0,$$

$p_y(t) < 0$

$$\phi(0) = p_y(t)\sqrt{Mm} < 0$$

and

$$\phi(M) = p_x(t)M < 0.$$

The maximum of ϕ on $[0, M]$ is

$$0 \;\; if \;\; p_y(t)\sqrt{Mm} > p_x(t)M$$

$$M \;\; if \;\; p_x(t)M > p_y(t)\sqrt{Mm}, \; t \in [0,T]$$

Conclusion

The optimal control of system (4) is

$$u(t) = \begin{cases} 0 \; if \; p_y(t) > 0 \\ and \; \frac{M-m}{2} + \frac{M+m}{2\sqrt{p_x^2(t)+p_y^2(t)}} p_x(t) \leq 0, \\ \\ \frac{M-m}{2} + \frac{(M+m)p_x(t)}{2\sqrt{p_x^2(t)+p_y^2(t)}} \; if \;\; p_y(t) > 0 \\ and \; \frac{M-m}{2} + \frac{M+m}{2\sqrt{p_x^2(t)+p_y^2(t)}} p_x(t) > 0, \\ \\ 0 \;\; if \;\; p_y(t) < 0 \\ and \;\; p_y(t)\sqrt{m} > p_x(t)\sqrt{M} \\ \\ M \;\; if \;\; p_y(t) < 0 \\ and \;\; p_y(t)\sqrt{m} < p_x(t)\sqrt{M}. \end{cases} \tag{14}$$

We proved the following theorem.

Theorem 2. *If* $p_y(t) > 0$ *and* $\frac{M-m}{2} + \frac{M+m}{2\sqrt{p_x^2(t)+p_y^2(t)}} p_x(t) < 0$, *then*

$$u(t) = 0, \quad t \in [0,T].$$

If $p_y(t) > 0$ *et* $\frac{M-m}{2} + \frac{M+m}{2\sqrt{p_x^2(t)+p_y^2(t)}} p_x(t) > 0$, *then*

$$u(t) = \frac{M-m}{2} + \frac{(M+m)p_x(t)}{2\sqrt{p_x^2(t)+p_y^2(t)}}, t \in [0,T].$$

If $p_y(t) < 0$ *and* $p_y(t)\sqrt{m} > p_x(t)\sqrt{M}$, *then*

$$u(t) = 0, \quad t \in [0,T].$$

If $p_y(t) < 0$ *and* $p_y(t)\sqrt{m} < p_x(t)\sqrt{M}$, *then*

$$u(t) = M, \; t \in [0,T].$$

4. Numerical simulations

We give here a brief overview of the indirect method, this method is based on the Pontryagin Maximum Principle, which gives necessary condition for optimality, and states that every optimal trajectory is the projection of an extremal. If one is able from the condition of maximization to express the extremal control in function of $(x(t), p(t))$, then the extremal system is a differential system of the form $\dot{z}(t) = F(t, z(t))$, where $z(t) = (x(t), p(t))$, and the values of initial, final and transversality conditions are put in the form $R(z(0), z(T)) = 0$.

Finally we obtain a problem of the form

$$\begin{cases} \dot{z}(t) = F(t, z(t)), \\ R(z(0), z(T)) = 0 \end{cases}. \tag{15}$$

Let $z(t, z_0)$ the solution of Cauchy's problem

$$\dot{z}(t) = F(t, z(t)), \quad z(0) = z_0.$$

Put $G(z_0) = R(z_0, z(T, z_0))$. The problem (16) it equivalent to

$$G(z_0) = 0$$

which is solved using the Newton's method. For more details on the shooting method, the reader can refer to [12].

Let us consider now that the ground is fertilized from September to July, and the insecticides is put in a continuous way, from May to July. To solve the problem, the indirect method based on the Pontryagin Maximum Principle is used. We provide in Table 3 numerical results of $x(T)$, $y(T)$ and $z(T)$ for several values of the weight parameter β. The numerical simulations were led using *Matlab* on a desktop computer.

Table 3. Pollution, cereal output and locusts final as time function of β.

0	909	5.94	248
10	36.40	16.90	13.39
50	35.97	20.76	2.50
70	35.91	18.47	1.92
100	35.90	15.65	1.39
150	35.87	12.18	1
200	35.84	9.85	1

We note that pollution decreases slowly, yield is decreased significantly, this is due to the fact that the wheat been ravaged by locusts.

Variations of controls $u(.)$ and $v(.)$ depending on t for $\beta = 0$ and $\beta = 50$ respectively are shown in the Figure 4 and Figure 5 :

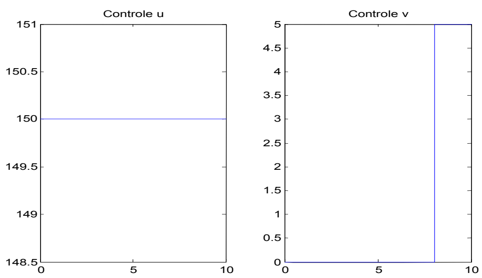

Figure 4. Optimal controls for $\beta = 0$.

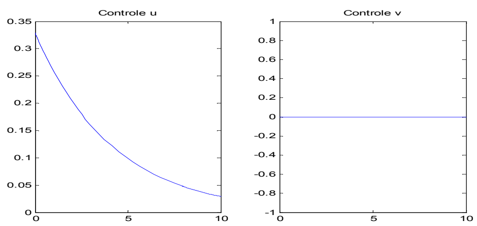

Figure 5. Optimal controls for $\beta = 50$.

It is visible from Figure 4 that when the farmer is only interested to increase the output, without taking into account the pollution, the optimal weight to be considered is of course "$\beta = 0$". Then $u(t) = 150qx/ha$, for all $t \in [0, 10]$ and quantities of insecticides are zero before the arrival of locusts i.e before May, $v(t) = 0$, but from May $v(t) = 5l/ha$.

Whereas, if he does not want to pollute the ground, he should use smaller quantities of fertilizers, for example, $\beta = 50$, the optimal control $u(.)$ decrease from $u = 0.34\ qx/ha$ at time $t = 0$ to $u = 0.05\ qx/ha$ at time $t = 10$. The quantities of insecticides in this case are zero (Figure 5).

Figure 6 shows the variations of pollution, cereal output and the number of locusts function of time t, for $\beta = 0$.

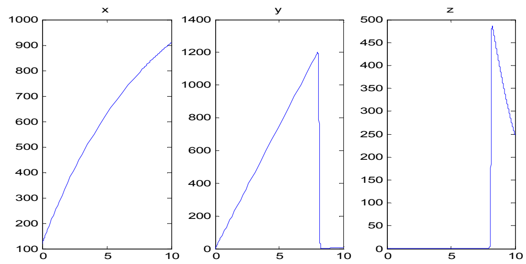

Figure 6. Optimal trajectories versus to t for $\beta = 0$.

We note that when $\beta = 0$, pollution increases, cereal output also increases until time $t = 8$ i.e until the coming of the locusts. At the same time, the number of locusts is maximum, then the curve decreases because of the maximum amount of insecticides $v(t) = 5l/ha$ which is applied.

If $\beta = 50$, in other words, taking into account the pollution, this last decreases according to t, cereal output increases until time $t = 8$ months, in May, then curve y decreases. But the curve representing the number of locusts does not decrease because no amount of insecticides is applied (see Figure 7).

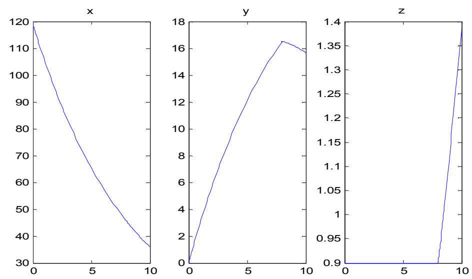

Figure 7. Optimal trajectories versus to t for $\beta = 50$.

5. Conclusion

In this work, we have modeled a practical problem in agriculture which is the optimization problem of a cereal production by introducing the constraint of the presence of locusts that are a real nuisance in Algeria. Controls resulting from the model are nonlinear. Different parameters of the model are identified using real-life data from the

National Institute of Plant Protection (INPV) located in the capital of Algeria Algiers. The theoretical resolution is done using the Pontryagin maximum principle. For the numerical resolution, we used the shooting method based on the Pontryagin maximum principle.

Our simulations show that the strategy of spreading fertilizers and insecticides can be improved in Algeria compared to what is done at present, so as to increase the rate of production and however minimizing the pollution effect.

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Nacima MOUSSOUNI *is currently a professor at the University Mouloud Mammeri of Tizi Ouzou in Algeria, which she joined in November 2002. She began her research in the laboratory, L2CSP at the University of Tizi Ouzou with Professor Mohamed AIDENE in the same laboratory, subsequently, this allowed us to publish an article entitled An Algorithm for Optimization of Cereal Output. Our works are based on real data of the Ministry of Agriculture in order to improve cereal production in Algeria.*

Aidene MOHAMED *is currently a professor at the University Mouloud Mammeri of Tizi Ouzou in Algeria. He is also director of laboratory L2CSP and Graduate School of Operations Research and Optimization.*