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# A research on adaptive control to stabilize and synchronize a hyperchaotic system with uncertain parameters

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**Abstract.** This paper addresses the chaos control and synchronization problems of a hyperchaotic system. It is assumed that the parameters of the hyperchaotic system are unknown and the system is perturbed by the external disturbance. Based on the Lyapunov stability theory and the adaptive control theory, suitable nonlinear controllers are designed for the asymptotic stability of the closed-loop system both for stabilization of hyperchaos at the origin and complete synchronization of two identical hyperchaotic systems. Accordingly, suitable update laws are proposed to estimate the fully uncertain parameters. All simulation results are carried out to validate the effectiveness of the theoretical findings. The effect of external disturbance is under our discussion.

**Keywords:** Adaptive control; chaos stabilization; synchronization; Lyapunov stability theory; hyperchaotic system.
**AMS Classification:** 93C10, 93C15, 34D06

## 1. Introduction

Chaos theory is the study of nonlinear dynamical systems, in which apparently random events are foreseeable from simple deterministic equations. In today's research, mathematical theory of chaos has become an essential base for natural sciences [1]. A chaotic system in nature is a complex system that exhibits a complicated, random and disorganized behavior. The chaotic systems are highly sensitive to tiny changes in their initial conditions and parameters variation [2]. For the last three decades, chaos theory has become the subject of intensive research field, paying a wide range of applications in different technical and scientific disciplines such as laser physics, atomic physics, biological systems, ecological systems, Traffic control, economics, secure communications and information systems [3] etc.

Chaos control and synchronization are especially noteworthy and important research fields levelling to affect dynamics of chaotic systems in order to apply them for different kinds of applications that can be examined within many different scientific research [3-5]. At present, there are different kinds of control methods and techniques that have been proposed for carrying out chaos control and synchronization of chaotic dynamical systems. These include the active control techniques [6], adaptive control [7], modified projective synchronization [8], linear feedback control [9] and the sliding mode control [10], which are worth citing here among others. [11-13].

Recently, Jia *et al.* [8] proposed and investigated a class of new hyperchaotic systems

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that contain two cross product terms and a quadratic term. The hyperchaotic systems [8] show rich complex dynamics. The modified projective synchronization of the disturbance free hyperchaotic systems with general form which linearly depends on uncertain parameters, are investigated by presenting an adaptive control technique together with parameter update laws and a nonlinear control scheme based on the Lyapunov stability theory.

Although, in real situations, the effect of external (environmental) disturbance cannot be ignored, which may affect the synchronization performance. Therefore, it is important analytically as well as practically that the stabilization of hyperchaos and complete synchronization in the presence of external disturbance objectives are achieved.

Most of the conventional control problems assumed that the states of a system is measurable and the parameters are completely or partially known in advance [11-15]. Although, in practice, uncertainty in the parameters and the presence of environmental disturbances badly affect the control and synchronization performance. This gives rise to a high frequency resonance within the system, which may break the control and synchronization behavior completely. If the parameters of the chaotic systems are unknown, then the adaptive control techniques are used for chaos control and synchronization of hyperchaotic (chaotic) systems [16]. These characteristics of the adaptive control techniques have motivated a huge research in the area of chaos control and synchronization.

In this paper, the authors further investigated the stabilization of hyperchaos at one of its equilibrium point at the origin and complete synchronization behavior of uncertain hyperchaotic system [8] in the presence of external disturbance. To the best of the authors' knowledge, this problem has not been discussed before. Motivated by the above discussions, there are three main objectives of the present study that can be summarized as follows:

i. Based on the Lyapunov stability theory and adaptive control theory, a suitable nonlinear adaptive control functions and an appropriate update laws will be designed to guarantee the asymptotic stabilization of hyperchaotic system [8] from its current hyperchaotic state to regular state with estimation of uncertain parameters in the present of external disturbances.
ii. Based on the Lyapunov stability theory and using the adaptive control technique, a suitable nonlinear control functions and parameter update laws will be proposed for asymptotic chaos synchronization and identification of fully uncertain parameters of the hyperchaotic system [8] in the presence of external disturbance.
iii. Numerical simulations and graphs will be provided to justify the efficiency and the performance of the proposed adaptive control approach.

The rest of the paper is organized as follows. In Section 2, description of the new hyperchaotic system is given. Section 3 presents the problem statement and stabilization of hyperchaos. In Section 4, a complete synchronization scheme for identical hyperchaotic systems [8] is derived. Finally, this paper is concluded in Section 5.

## 2. System description

The mathematical model of the hyperchaotic system [8] that contains both linear and nonlinear terms that is described as follows:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ \dot{w}(t) \end{bmatrix} = \left. \begin{matrix} \begin{bmatrix} -a & a & 0 & 0 \\ b & 1 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} + \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -x(t)z(t) \\ x^2(t) \\ y(t)w(t) \end{bmatrix} \end{matrix} \right\} \tag{1}$$

where $\left[x(t),\ y(t),\ z(t),\ w(t)\right]^T \in R^4$ are the state variables and $a, b, c, d$ are the system positive parameters. To discuss the stability of the hyperchaotic system (1), let us linearize system (1) at the equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ for the parameters values $a = 20,\ b = 35,\ c = 3,\ d = 10$. The Jacobean matrix is given as under:

$$J_{(0,\ 0,\ 0,\ 0)} = \begin{bmatrix} -a & a & 0 & 0 \\ b & 1 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{bmatrix}$$

For the parameters values, $a = 20,\ b = 35,\ c = 3,$ $d = 10$ and using the mathematica 10*v*, the four eigenvalues are estimated as: $\lambda_1 = -37.8819$, $\lambda_2 = 18.5980$, $\lambda_3 = 0.2839$ and $\lambda_4 = -4$. In continuous nonlinear dynamical systems, the real part of all eigenvalues are negative and at least one positive Lyapunov exponent. We can see that the two eigenvalues $(\lambda_1,\ \lambda_4 < 0)$ are negative and two eigenvalues $(\lambda_2,\ \lambda_3 > 0)$ are non-negative and with the Lyapunov exponents: $L_1 = 1.0677, L_2 = 0.094,\ L_3 = 0$ and $L_4 = -23.1526$ [8] respectively, which confirms that the hyperchaotic system (1) is globally unstable at the equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$. On any initial condition on one of the negative eigenvector, the orbit will converge to the equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ through the eigenplane of these two negative eigenvectors. On the other side, any deviation along $\lambda_2,\ \lambda_3 > 0$ will expand and the orbit becomes unstable which shows a saddle node unstable point. Hence from the Lyapunov stability theory [1], the equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ is globally unstable. Physically, this result bears the fact that the system can oscillate chaotically and forbids the existence of stable fixed point motion in the system. Thus, the new hyperchaotic system [8] exhibits a hyperchaos for parameters values: $a = 20,\ b = 35,\ c = 3$, $d = 10$ and hence, the control problem takes place. In the following sections, a systematic procedure for hyperchaos stabilization and synchronization of hyperchaotic systems are given which ensures asymptotic stability of the closed-loop systems in the presence of external disturbance with estimation of fully uncertain parameters. The proposed schemes can be applied to a variety of chaotic and hyperchaotic systems for stabilization of chaos and complete synchronization.

## 3. Stabilization scheme for suppressing hyperchaos

In the following sub-section, a recursive approach via the adaptive control technique will be proposed to stabilize the hyperchaos at its equilibrium $E_0 = (0,\ 0,\ 0,\ 0)$ point at the origin.

### 3.1. Problem statement

Let us consider the controlled hyperchaotic system (1), which is described by the following:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} -a & a & 0 & 0 \\ b & 1 & 0 & 1 \\ 0 & 0 & -c & 0 \\ 0 & -d & 0 & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{bmatrix} + \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -x(t)z(t) \\ x^2(t) \\ y(t)w(t) \end{bmatrix} + \\ \begin{bmatrix} \psi_1(t) \\ \psi_2(t) \\ \psi_3(t) \\ \psi_4(t) \end{bmatrix} + D_i(t) \end{cases} \qquad (2)$$

where $X(t) = \left[x(t),\ y(t),\ z(t),\ w(t)\right]^T \in R^4$ is the state vector and $a,\ b,\ c,\ d$ are the uncertain system parameters, $D_i(t)$ is the external disturbance present in the hyperchaotic system and $\boldsymbol{\psi}(t) = \left[\psi_1(t),\ \psi_2(t),\ \psi_3(t),\ \psi_4(t)\right]^T \in R^4$ is the control input. In this situation, it is desired to design a nonlinear feedback control law which describes the synthesis of a bounded control input $\boldsymbol{\psi}(t) \in R^4$. This control input accomplishes the asymptotic stabilization of the hyperchaotic system (2) to its equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ at the origin.

### 3.2. Controller design

In the absence of a proper feedback controller $(\boldsymbol{\psi}(t) \in R^4)$; the trajectories of the chaotic system will quickly bifurcate from each other in all future states and the system will become unstable. Hence, the role of a proper feedback controller for the chaos stabilization problem is to restrict the system converges to the equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ for all initial conditions. Thus, the main focus of this part is to design an adaptive control function and parameters updated law that will stabilize the hyperchaotic system (2) to its equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ at the origin asymptotically. The asymptotic stabilization of the hyperchaotic system [8] from its current chaotic state to a stable regular state will be ensured with the estimation of uncertain parameters, where a desired dynamic of the hyperchaotic system is obtained. In order to ensure that the controlled system (2) globally converges to the origin asymptotically, let us state the following theorem.

***Theorem 1.*** *The hyperchaotic systems (1) can be stabilized asymptotically with the following adaptive control law:*

$$\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \psi_3(t)\\ \psi_4(t)\end{bmatrix} = \left\{ \begin{array}{l} -\begin{bmatrix} -\hat{a}+1 & a & 0 & 0 \\ \hat{b} & 2 & 0 & 1 \\ 0 & 0 & -\hat{c}+1 & 0 \\ 0 & -\hat{d} & 0 & 0 \end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\\ z(t)\\ w(t)\end{bmatrix} - \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ -x(t)z(t) \\ x^2(t) \\ y(t)w(t) \end{bmatrix} - D_i(t) \end{array}\right. \tag{3}$$

where $\tilde{a} = a - \hat{a},\ \tilde{b} = b - \hat{b},\ \tilde{c} = c - \hat{c}\ ,\ \tilde{d} = d - \hat{d}$ and $\hat{a},\ \hat{b},\ \hat{c},\ \hat{d}$ are the estimation of uncertain parameters $a,\ b,\ c$ and $d$, alternatively with $\dot{\tilde{a}} = -\dot{\hat{a}},\ \dot{\tilde{b}} = -\dot{\hat{b}},\ \dot{\tilde{c}} = -\dot{\hat{c}}$ and $\dot{\tilde{d}} = -\dot{\hat{d}}$ .

***Proof of theorem 1.*** *Using systems of Eq. (2) and (3), the closed-loop system is given by the following:*

$$\Rightarrow \quad \begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix} \tilde{a}\left(y(t) - x(t)\right) - x(t) \\ \tilde{b}x(t) - y(t) \\ -\tilde{c}z(t) - z(t) \\ -\tilde{d}y(t) - w(t) \end{bmatrix} \tag{4}$$

***Corollary 1.*** *The asymptotic stabilization of the hyperchaotic system (1) at its equilibrium point* $E_0 = (0,\ 0,\ 0,\ 0)$ *is accomplished, if the closed-loop dynamic (4) is such that:*

$$\lim_{t\to\infty}\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\\ \dot{z}(t)\\ \dot{w}(t)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

***Corollary 2.*** *The uncertain parameters are estimated from the system parameters in the following sense:*

$$\lim_{t\to\infty}\tilde{P} = \left. \begin{array}{l} \lim\limits_{t\to\infty}\left(\tilde{a} = a - \hat{a}\right),\ \lim\limits_{t\to\infty}\left(\tilde{b} = b - \hat{b}\right) \\ \lim\limits_{t\to\infty}\left(\tilde{c} = c - \hat{c}\right),\ \lim\limits_{t\to\infty}\left(\tilde{d} = d - \hat{d}\right) \end{array}\right\} \to 0$$

***Proof of theorem 1.*** Let us construct a quadratic Lyapunov error function candidate as follows:

$$V\left(X(t),\ \tilde{P}\right) = \frac{1}{2}\left(\begin{bmatrix}x(t)\\ y(t)\\ z(t)\\ w(t)\\ \tilde{a}\\ \tilde{b}\\ \tilde{c}\\ \tilde{d}\end{bmatrix}^T A \begin{bmatrix}x(t)\\ y(t)\\ z(t)\\ w(t)\\ \tilde{a}\\ \tilde{b}\\ \tilde{c}\\ \tilde{d}\end{bmatrix}\right) \geq 0 \tag{5}$$

where $X(t) = \left[x(t),\ y(t),\ z(t),\ w(t)\right]^T \in R^4$ and $A = \left[a_{ii} = 0.5\right],\ i = 1,\ 2,\ \ldots\ 8$. It is easy to see that $A$ is a positive definite matrix (PDM) and henceforth, $V\left(X(t),\ \tilde{P}(t)\right)$ is a positive definite function. According to the Lyapunov stability theory [1], the closed-loop system (4) is asymptotically stable if the time derivative of (5) becomes negative. The time derivative of the Lyapunov error function (5) is given by the following:

$$\dot{V}\left(X(t),\ \tilde{P}\right) = \begin{bmatrix} x(t)\dot{x}(t) + y(t)\dot{y}(t) + z(t)\dot{z}(t) + \\ w(t)\dot{w}(t) + (\tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}}) \end{bmatrix}$$

$$\dot{V}\left(X(t),\ \tilde{P}\right) = \begin{bmatrix} x(t)\left(\tilde{a}\left(y(t) - x(t)\right) - x(t)\right) + \\ y(t)\left(\tilde{b}x(t) - y(t)\right) + z(t)\left(-\tilde{c}z(t) - z(t)\right) + \\ w(t)\left(-\tilde{d}y(t) - w(t)\right) + \left(\tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}}\right) \end{bmatrix}$$

$$\dot{V}\left(X(t),\ \tilde{P}\right) = \begin{bmatrix} -\left(x^2(t) + y^2(t) + z^2(t) + w^2(t)\right) \\ +\tilde{a}\left(\left(y(t) - x(t)\right)x(t) - \dot{\hat{a}}\right) \\ +\tilde{b}\left(x(t)y(t) - \dot{\hat{b}}\right) - \tilde{c}\left(z^2(t) + \dot{\hat{c}}\right) \\ -\tilde{d}\left(y(t)w(t) + \dot{\hat{d}}\right) \end{bmatrix} \tag{6}$$

Now based on Eq. (6), the update law $\hat{a},\ \hat{b},\ \hat{c}$ and $\hat{d}$ for the parameter estimation is developed as under:

$$\dot{\hat{P}} = \begin{bmatrix}\dot{\hat{a}}\\ \dot{\hat{b}}\\ \dot{\hat{c}}\\ \dot{\hat{d}}\end{bmatrix} = \begin{bmatrix} \left(y(t) - x(t)\right)x(t) \\ x(t)y(t) \\ -z^2(t) \\ -y(t)w(t) \end{bmatrix} \tag{7}$$

With this choice of controller (3) and the parameters estimation updated law (7) that yields the following:

$$\dot{V}\left(X(t),\, \tilde{P}\right) = -\begin{pmatrix} x^2(t) + y^2(t) + \\ z^2(t) + w^2(t) \end{pmatrix} \le 0 \qquad (8)$$

Thus, $\dot{V}\left(X(t),\, \tilde{P}\right)$ is negative semi-definite function. This proves that the closed-loop (4) is locally stable. Since $\dot{V}\left(X(t),\, \tilde{P}\right)$ is negative semi-definite function and we cannot immediately conclude that the origin is asymptotically stable. To achieve asymptotic stability, we proceed as follows.

The states trajectories of the hyperchaotic (chaotic) systems and vectors of the uncertain positive parameters are always bounded [18]. Thus, from Eq. (3) and Eq. (4), $\psi(t) \in R^4$ and $\hat{a},\, \hat{b},\, \hat{c}$ and $\hat{d}$ respectively, are also bounded. Therefore, if, $\lambda_{\min}(A)$ is the minimum eigenvalue of PDM $A$, then, by the Barbalat's lemma [19]:

$$\int_0^t \lambda_{\min}(A)\|X(t)\|^2 dt \le \int_0^t X(t)^T AX(t)dt$$
$$= \int_0^t -\dot{V}\left(X(t),\, \tilde{P}\right)dt$$
$$= V(0) - V(t) \le 0$$

This confirms that: $\lim_{t\to\infty}\|X(t)\| = 0$, for $i = 1,\ 2,\ 3,\ 4$.

Thus, the hyperchaotic system (1) is stabilized asymptotically.

### 3.3. Numerical simulations

In this sub-section of the paper, the Mathematica 10*v* software is used to provide the numerical simulations to verify the efficiency and performances of the proposed adaptive control approach. The parameters of the hyperchaotic system [8] are set as $a = 20,\ b = 35,\ c = 3$ and $d = 10$, with initial condition being taken as $\left[x(0),\, y(0),\, z(0),\, w(0)\right]^T = \begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^T$, so that the hyperchaotic system (1) can exhibit chaotic behavior. The following external disturbances are applied to the hyperchaotic system (1).

$$D_1(t) = -0.01\sin(20t),\ D_2(t) = 0.02\cos(15t),$$
$$D_3(t) = -0.05\sin(10t),\ D_4(t) = 0.05\sin(30t).$$

The time histories of the controlled (red lines) and uncontrolled states trajectories (blue lines) are depicted in Figures 1 to 4. These figures illustrate that the state trajectories of the controlled system converged to its equilibrium point $E_0 = (0,\ 0,\ 0,\ 0)$ at the origin under the synthesized control action (3), while the uncontrolled state trajectories are completely different than the controlled state trajectories.

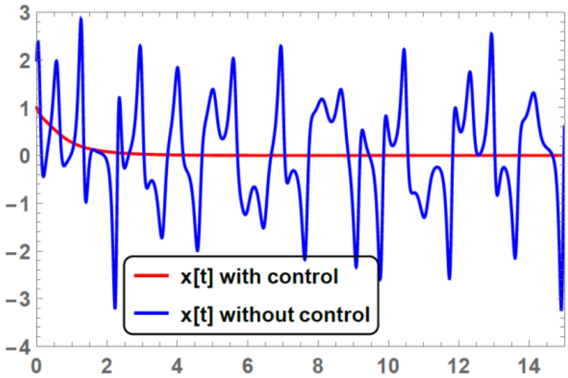


**Figure 1.** Time series of the state variable x[t] (time in seconds)

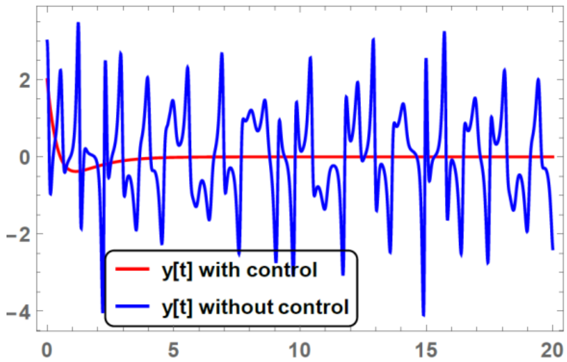


**Figure 2.** Time series of the state variable y[t] (time in seconds)

The initial values of the estimated parameters are chosen as: $\hat{a}(0) = 25,\ \hat{b}(0) = 40,\ \hat{c}(0) = -2$ and $\hat{d}(0) = 5$. The simulation results are demonstrated in Figures 5-6. It is clear that the estimations $\hat{a},\, \hat{b},\, \hat{c}$ and $\hat{d}$ of the unknown parameters, $a = 20 + 0.1\, sin(20t)$, $b = 35 + 0.1\cos(30t),\ c = 3 - 0.05\ \sin(50t)$ and $d = 10 + 0.05\, cos(50t)$ converged to the true values of $a = 20,\ b = 35,\ c = 3$ and $d = 10$ alternatively, as time goes to infinity under the updated law (7).

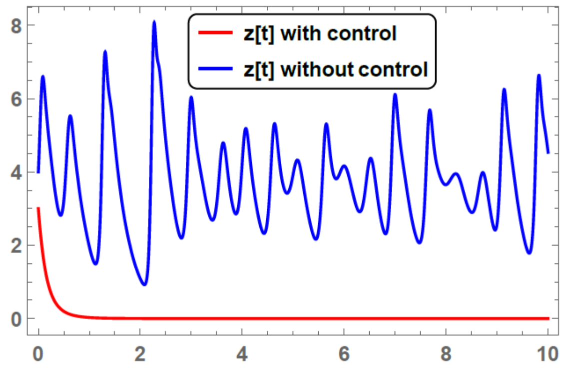


**Figure 3.** Time series of the state variable z[t] (time in seconds)

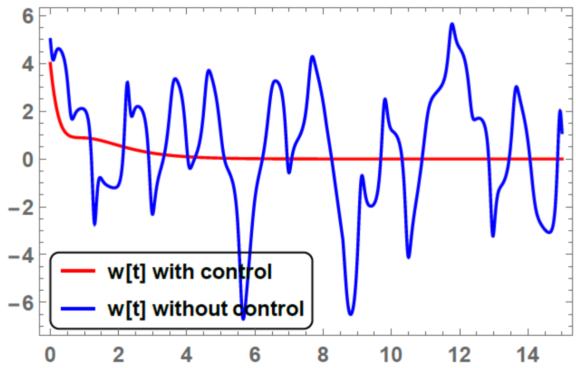


**Figure 4.** Time series of the state variable w[t] (time in seconds)

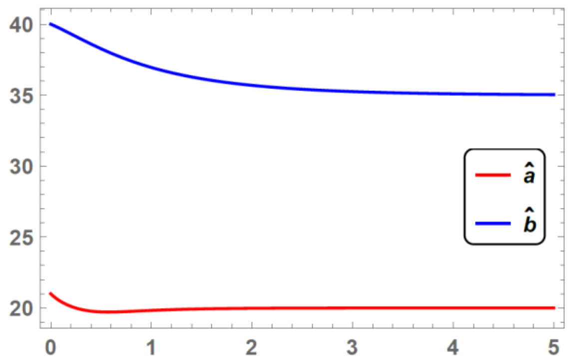


**Figure 5.** Time series of the updated vector (a & b) (time in seconds)

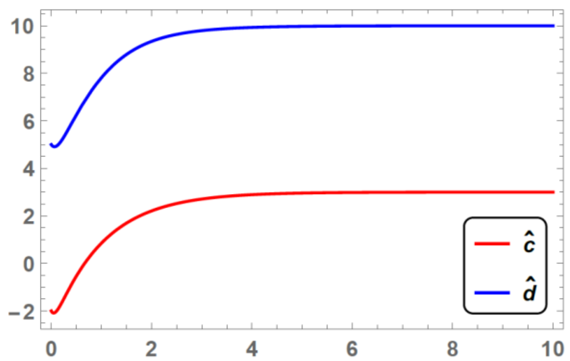


**Figure 6.** Time series of the updated vector (c & d) (time in seconds)

In order to discuss the asymptotic stability of the closed-loop system at the origin, the time series of the derivatives of the Lyapunov error function has been plotted in Figure 6. One can notice that the investigated controllers are robust against the external disturbance, which is helpful in certain physical and engineering applications.

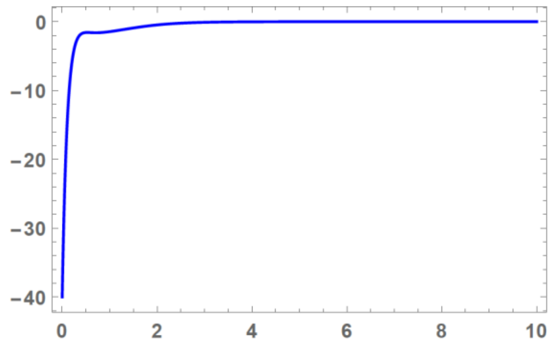

**Figure 7.** Time series of $\dot{V}\left(E(t),\tilde{P}\right)$ (time in seconds)

## 4. Complete synchronization scheme for two identical hyperchaotic systems

### 4.1. Problem statement

In order to observe the complete synchronization behavior of hyperchaotic system (1), let us assume that the hyperchaotic system (1) with subscript 1 as the master system and is described as below:

$$\begin{aligned}&(\text{Master system})\\&\begin{bmatrix}\dot{x}_1(t)\\ \dot{y}_1(t)\\ \dot{z}_1(t)\\ \dot{w}_1(t)\end{bmatrix}=\begin{bmatrix}-a & a & 0 & 0\\ b & 1 & 0 & 1\\ 0 & 0 & -c & 0\\ 0 & -d & 0 & 0\end{bmatrix}\begin{bmatrix}x_1(t)\\ y_1(t)\\ z_1(t)\\ w_1(t)\end{bmatrix}+\\&\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ -x_1(t)z_1(t)\\ x_1^2(t)\\ y_1(t)w_1(t)\end{bmatrix}+D_i(t)\end{aligned} \quad (9)$$

where, $\left[x_1(t),\, y_1(t),\, z_1(t),\, w_1(t)\right]^T \in R^4$ are the state vectors, $a$, $b$, $c$ and $d$ are the uncertain parameters of the master hyperchaotic system (9), and $D_i(t)$ is the external disturbance presents in the master system.

We define the slave system with subscript 2 to be the identical hyperchaotic system with the same parameters values as hyperchaotic system (1) and is described as follows:

(Slave system)

$$\begin{bmatrix}\dot{x}_2(t)\\ \dot{y}_2(t)\\ \dot{z}_2(t)\\ \dot{w}_2(t)\end{bmatrix}=\begin{bmatrix}-a & a & 0 & 0\\ b & 1 & 0 & 1\\ 0 & 0 & -c & 0\\ 0 & -d & 0 & 0\end{bmatrix}\begin{bmatrix}x_2(t)\\ y_2(t)\\ z_2(t)\\ w_2(t)\end{bmatrix}+ \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ -x_2(t)z_2(t)\\ x_2^2(t)\\ y_2(t)w_2(t)\end{bmatrix}+d_i(t)+\boldsymbol{\psi}(t) \quad (10)$$

where $\left[x_2(t),\ y_2(t),\ z_2(t),\ w_2(t)\right]^T \in R^4$ are the state vectors, $a, b, c$ and $d$ are the uncertain parameters of the slave hyperchaotic system (10), and $d_i(t),\ i=1,\ 2,\ 3,\ 4,$ is the external disturbance presents in the slave system. The control input is described as; $\boldsymbol{\psi}(t)=\left[\psi_1(t),\ \psi_2(t),\ \psi_3(t),\ \psi_4(t)\right]^T \in R^4$.

The main objective of this section is to design the controllers $\boldsymbol{\psi}(t)\in R^4$ and update laws such that the controllers $\boldsymbol{\psi}(t)\in R^4$ force the slave hyperchaotic system to synchronize with the master hyperchaotic system for different initial conditions in the presence of the external disturbances and despite the fact that the parameters of both systems are unknown.

### 4.2. Adaptive controller design

***Theorem 2.*** *The hyperchaotic systems (9) and (10) can be synchronized asymptotically for different initial conditions with the following adaptive control law:*

$$\begin{bmatrix}\psi_1(t)\\ \psi_2(t)\\ \psi_3(t)\\ \psi_4(t)\end{bmatrix}=\begin{bmatrix}-\hat{a}+1 & \hat{a} & 0 & 0\\ -\hat{b} & -2 & 0 & -1\\ 0 & 0 & \hat{c}-1 & 0\\ 0 & \hat{d} & 0 & -1\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\end{bmatrix}+ \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ -x_1(t)z_1(t)+x_2(t)z_2(t)\\ x_1^2(t)-x_2^2(t)\\ -y_1(t)w_1(t)+y_2(t)w_2(t)\end{bmatrix} +D_i(t)-d_i(t) \quad (11)$$

where $\hat{a},\ \hat{b},\ \hat{c},\ \hat{d}$ are the estimated values of the uncertain parameters $a,\ b,\ c$ and $d$, alternatively with $\tilde{a}=a-\hat{a},\ \ \tilde{b}=b-\hat{b},$ $\tilde{c}=c-\hat{c}$ and $\tilde{d}=d-\hat{d}$ with $\dot{\tilde{a}}=-\dot{\hat{a}},\ \dot{\tilde{b}}=-\dot{\hat{b}},$ $\dot{\tilde{c}}=-\dot{\hat{c}}$, $\dot{\tilde{d}}=-\dot{\hat{d}}$ and $e_1(t)=x_2(t)-x_1(t)$, $e_2(t)=y_2(t)-y_1(t)$, $e_3(t)=z_2(t)-z_1(t)$ and $e_4(t)=w_2(t)-w_1(t)$.

***Definition 2.*** *The error dynamic $E$ of the master and slave systems synchronization scheme is given as follows:*

$$\dot{E}=\begin{bmatrix}\dot{e}_1(t)\\ \dot{e}_2(t)\\ \dot{e}_3(t)\\ \dot{e}_4(t)\end{bmatrix}=\begin{bmatrix}-a & a & 0 & 0\\ b & 1 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -d & 0 & 0\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\end{bmatrix}+ \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 10 & 0 & 0\\ 0 & 0 & 10 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ x_1(t)z_1(t)-x_2(t)z_2(t)\\ -x_1^{\ 2}(t)+x_2^{\ 2}(t)\\ y_1(t)w_1(t)-y_2(t)w_2(t)\end{bmatrix} +\boldsymbol{\psi}(t) \quad (12)$$

***Proof of theorem 2.*** *Using systems of Eq. (11) and (12), the closed-loop system is given by the following:*

$$\dot{E}=\begin{bmatrix}-\tilde{a}-1 & \tilde{a} & 0 & 0\\ \tilde{b} & -1 & 0 & 0\\ 0 & 0 & -\tilde{c}-1 & 0\\ 0 & -\tilde{d} & 0 & -1\end{bmatrix}\begin{bmatrix}e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\end{bmatrix} \quad (13)$$

***Corollary 3.*** *The asymptotic synchronization is accomplished if:*

$$\lim_{t\to\infty} E=\lim_{t\to\infty}\left\|\begin{matrix}\dot{e}_1(t)\\ \dot{e}_2(t)\\ \dot{e}_3(t)\\ \dot{e}_4(t)\end{matrix}\right\|_2=\lim_{t\to\infty}\begin{bmatrix}x_2(t)-x_1(t)\\ y_2(t)-y_1(t)\\ z_2(t)-z_1(t)\\ w_2(t)-w_1(t)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

***Corollary 4.*** *The uncertain parameters are estimated from the system parameters in the sense of:*

$$\lim_{t\to\infty}\tilde{P}=\left.\begin{matrix}\lim\limits_{t\to\infty}\left(\tilde{a}=a-\hat{a}\right),\ \lim\limits_{t\to\infty}\left(\tilde{b}=b-\hat{b}\right)\\ \lim\limits_{t\to\infty}\left(\tilde{c}=c-\hat{c}\right),\ \lim\limits_{t\to\infty}\left(\tilde{d}=d-\hat{d}\right)\end{matrix}\right\}\to 0$$

Let us construct a quadratic Lyapunov error function candidate as:

$$V\left(E,\tilde{P}\right)=\frac{1}{2}\left(\begin{bmatrix}e_1(t)\\e_2(t)\\e_3(t)\\e_4(t)\\\tilde{a}\\\tilde{b}\\\tilde{c}\\\tilde{d}\end{bmatrix}^T A\begin{bmatrix}e_1(t)\\e_2(t)\\e_3(t)\\e_4(t)\\\tilde{a}\\\tilde{b}\\\tilde{c}\\\tilde{d}\end{bmatrix}\right)\geq 0 \qquad (14)$$

where $A=\left[a_{ii}=0.5\right],\ i=1,\ 2,\ \cdots\ 8$.

It is easy to see that $A$ is a PDM and henceforth $V\left(E,\ \tilde{P}\right)$ is a positive definite function. According to the Lyapunov stability theory, the closed-loop system (13) will be asymptotically stable if the time derivative of (14) becomes negative. Thus, the time derivative of the Lyapunov error function (14) is given as follows:

$$\dot{V}\left(E,\ \tilde{P}\right)=\begin{bmatrix}e_1(t)\left(\tilde{a}\left(e_2(t)-e_1(t)\right)\right)\\+e_2(t)\left(\tilde{b}e_2(t)-e_2(t)\right)\\+e_3(t)\left(\tilde{c}e_3(t)-e_3(t)\right)\\+e_4(t)\left(\tilde{d}e_4(t)-e_4(t)\right)\end{bmatrix}$$

$$\dot{V}\left(E,\tilde{P}\right)=\begin{pmatrix}-\left(e_1^2(t)+e_2^2(t)+e_3^2(t)+e_4^2(t)\right)+\\\tilde{a}\left(\left(e_2(t)-e_1(t)\right)e_1(t)-\dot{\hat{a}}\right)+\\\tilde{b}\left(e_1(t)e_2(t)-\dot{\hat{b}}\right)-\tilde{c}\left(e_3^{\ 2}(t)+\dot{\hat{c}}\right)\\-\tilde{d}\left(e_4(t)+\dot{\hat{d}}\right)\end{pmatrix} \qquad (15)$$

Now based on (15), the updated law $\hat{a},\ \hat{b},\ \hat{c}$ and $\hat{d}$ for the parameters estimation is developed as under:

$$\dot{\hat{P}}=\begin{bmatrix}\dot{\hat{a}}\\\dot{\hat{b}}\\\dot{\hat{c}}\\\dot{\hat{d}}\end{bmatrix}=\begin{bmatrix}\left(e_2(t)-e_1(t)\right)e_1(t)\\e_1(t)e_2(t)\\-e_3^{\ 2}(t)\\-e_2(t)e_4(t)\end{bmatrix} \qquad (16)$$

with this choice of controller (11) and the parameters estimation updated law (16) that yields:

$$\dot{V}\left(E,\ \tilde{P}\right)=-\begin{pmatrix}e_1^2(t)+e_2^2(t)\\+e_3^2(t)+e_4^2(t)\end{pmatrix}\leq 0 \qquad (17)$$

Thus, $\dot{V}\left(E,\ \tilde{P}\right)$ is negative semi-definite function. This proves that the closed-loop (13) is locally stable. Since,

$$\left[e_1(t),\ e_2(t),\ e_3(t),\ e_4(t)\right]\in L^2\cap L^\infty<\infty.$$

Hence, from (13), $\left[e_1(t),\ e_2(t),\ e_3(t),\ e_4(t)\right]$ is bounded [17]. The states trajectories of hyperchaotic (chaotic) systems and vectors of the uncertain parameters are always bounded [18]. Thus, from (11) and (15), $\psi(t)\in R^4$ and $\hat{a},\ \hat{b},\ \hat{c}$ and $\hat{d}$, respectively are also bounded. Therefore, if, $\lambda_{\min}(A)$ is the smallest eigenvalue of a PDM $A$, then, by the Barbalat's lemma [19]:

$$\int_0^t \lambda_{\min}(A)\|e(t)\|^2dt\leq\int_0^t\left(e(t)^T Ae(t)\right)dt$$
$$=\int_0^t -\dot{V}\left(E,\ \tilde{P}\right)dt$$
$$=V(0)-V(0)\leq 0$$

This confirms that:

$$\lim_{t\to\infty}\left\|e_i(t)\right\|=0,\ \text{for}\ i=1,\ 2,\ 3,\ 4.$$

Thus, the two identical hyperchaotic systems (9) and (10) with uncertain parameters in the presence of the external disturbances are asymptotically synchronized.

## 4.3. Numerical simulations and discussions

In this sub-section of the paper, numerical simulations are presented to verify the efficiency of the proposed adaptive complete synchronization approach. The parameters of the hyperchaotic system [8] are chosen to be: $a=20,\ b=35,\ c=3$ and $d=10$, with initial conditions being taken as: $\left[x_1(0),\ y_1(0),\ z_1(0),\ w_1(0)\right]^T=\left[-1\ \ -2\ \ -3\ \ -2\right]^T$ and $\left[x_2(0),\ y_2(0),\ z_2(0),\ w_2(0)\right]^T=\left[1\ \ 2\ \ 3\ \ 4\right]^T$ respectively. The following external disturbances are applied to the master and slave systems respectively.

$D_1(t)=0.02\sin(10t),\ D_2(t)=0.02\cos(15t),$
$D_3(t)=-0.02\sin(20t),\ D_4(t)=0.01\sin(20t)$

$d_1(t) = -0.01\cos(15t),\ d_2(t) = 0.01\sin(10t),$
$d_3(t) = -0.02\cos(10t),\ d_4(t) = -0.01\sin(20t)$

Figures 8-11, elaborate the time history of state vectors of the two identical hyperchaotic systems [8] to observe the asymptotic synchronization. These simulations results depict that the states of the slave hyperchaotic system converged to that of the master hyperchaotic system under the synthesized control action (11), while the unsynchronized states trajectories of the slave hyperchaotic system (red line) are completely different than the master states trajectories (blue line). The simulation results of asymptotic complete synchronization using adaptive control technique have good performances and confirm that the slave hyperchaotic system shows similar behavior (red and blue lines) to the identical master hyperchaotic system showing that the developed approach guarantees high security.

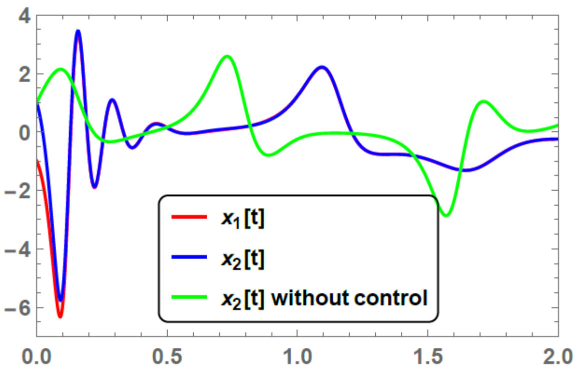


**Figure 8.** Time series of the synchronized states $x_1$[t] & $x_2$[t] (time in seconds)

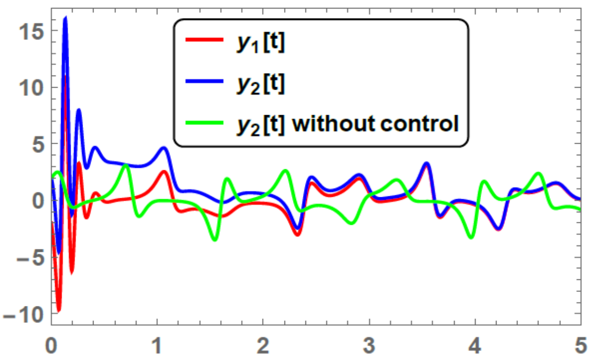


**Figure 9.** Time series of the synchronized states $y_1$[t] & $y_2$[t] (time in seconds)

The initial values of the estimated parameters are chosen as: $\hat{a}(0) = 25,\ \hat{b}(0) = 20,\ \hat{c}(0) = 1,\ \hat{d}(0) = 5$. The convergence of the estimated parameters to the true parameters is illustrated in Figs. 13-14. It is clear that the estimations $\hat{a},\ \hat{b},\ \hat{c}$ and $\hat{d}$ of the uncertain parameters, $a = 20 - 0.01\cos(15t)$, $b = 35 + 0.05\sin(10t)$, $c = 3 - 0.01cos(30t)$ and $d = 10 - 0.05\cos(45t)$ converged to the true values of $a,\ b,\ c$ and $d$ alternatively, as time goes to infinity under the updated law (16).

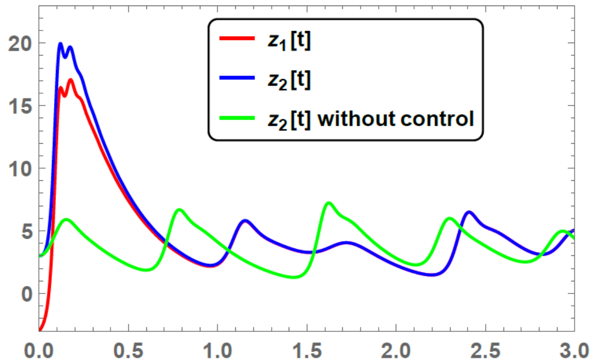


**Figure 10.** Time series of the synchronized states $z_1$[t] & $z_2$[t] (time in seconds)

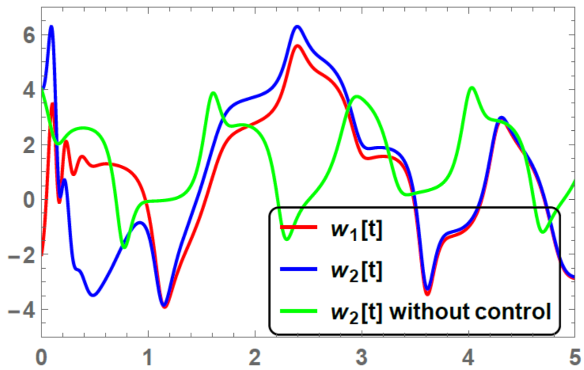


**Figure 11.** Time series of the synchronized states $w_1$[t] & $w_2$[t] (time in seconds)

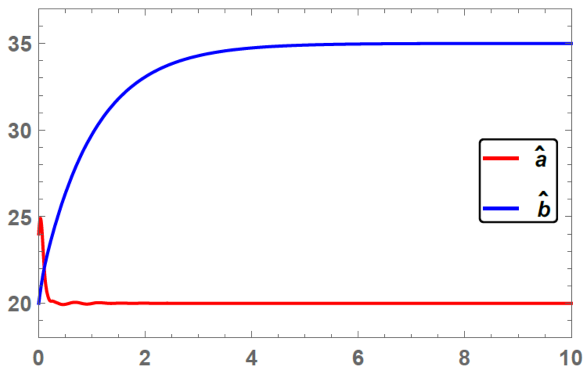


**Figure 12.** Time series of the updated vectors (a & b) (time in seconds)

The convergence of the error states for complete synchronization is depicted in Figure 13. It is observed that the error states converged to the equilibrium point in the presence of external disturbances when the controller are switched on at 6 seconds. This indicates the smooth and fast convergence rates to the equilibrium point. Thus, it is being confirmed that the proposed control approach is robust against the external disturbances and parameters mismatches.

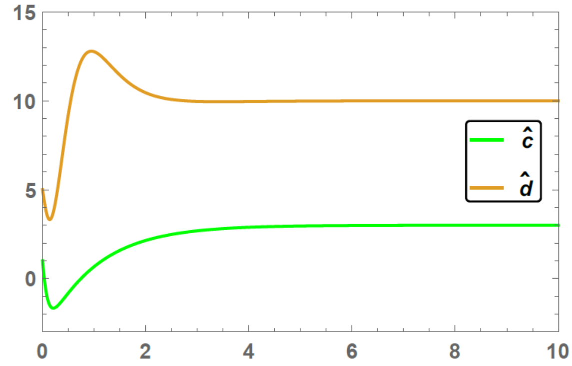


**Figure 13.** Time series of the updated vectors (c & d) (time in seconds)

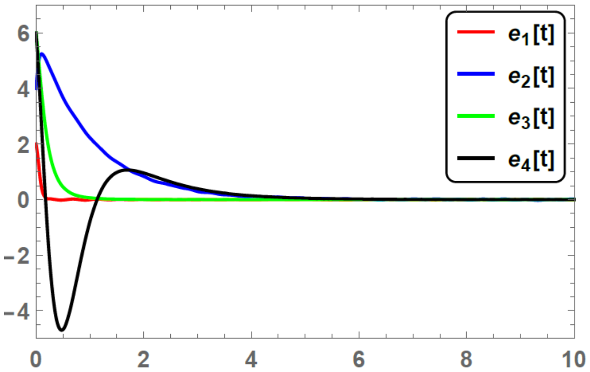


**Figure 14.** Time series of the synchronized error states (time in seconds)

To confirm the asymptotic stability, the time series of the derivative of the Lyapunov function (16) is illustrated in Figure 15.

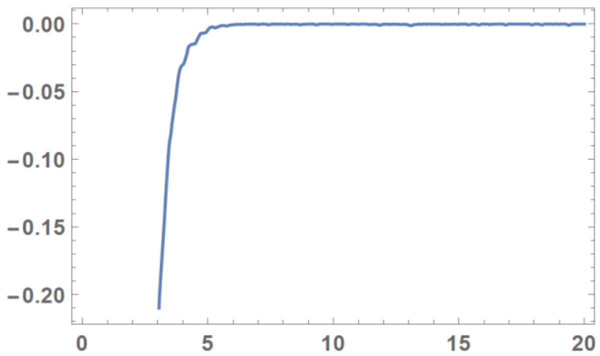

**Figure 15.** Time series of $\dot{V}\left(E(t),\tilde{P}\right)$ (time in seconds)

## 5. Conclusion

In this paper, the stabilization and synchronization schemes for a hyperchaotic system have been investigated based on a technique derived from adaptive control theory. There are two main objectives that the authors have achieved in this paper. Firstly, based on the Lyapunov stability theory and using the adaptive control technique, a class of adaptive control functions were designed that guaranteed the asymptotic stability of the closed-loop system for chaos stabilization with the estimation of fully uncertain parameters. Secondly, based on the Lyapunov stability theory, adaptive control functions were designed to synchronize two identical hyperchaotic systems in the presence of external disturbances. Accordingly, suitable updated laws were designed to estimate the fully uncertain parameters. The simulations results show that the synchronization error states converged to zero and the parameter estimates converged to the true values. The proposed approach relaxes the calculation of the Lyapunov exponents for asymptotic stability of the closed-loop system both for chaos stabilization and synchronization. These characteristics give advantages to the proposed approach.

We believe that the results of this research work should be beneficial and could be employed in the field of hybrid image encryption, secure communications and genetic networks.


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## References


[1] H. K. Khalil, Nonlinear Systems. 3rd ed. Prentice Hall, Englewood Cliffs, NJ (2002).

[2] Lorenz, E., Deterministic nonperiodic flow. Journal of the Atmospheric Sciences, 20(2), 130–141 (1963).

[3] Andrievskii, B. R., Fradkov, A. l., Control of Chaos: Methods and Applications. Automation and Remote Control, 65(4), 505-533 (2004).

[4] Aghababa, M. P., Fractional modeling and control of a complex nonlinear energy supply-demand system. Complexity, doi: 10.1002/cplx.21533 (2014).

[5] Vaidyanathan, S., Rajagopal, K., Anti-Synchronization of Tigan and Li Systems with Unknown Parameters via Adaptive Control. An International Journal of Optimization and Control: Theories & Applications, 2(1), 17-28 (2012).

[6] Ahmad, I., Saaban, A., Ibrahim, A., Shahzad, M., Global Chaos Synchronization of New Chaotic System using Linear Active Control. Complexity, doi: 10.1002/cplx.21573 (2014).

[7] Aghababa, M. P., Aghababa, H. P., Synchronization of nonlinear chaotic

electromechanical gyrostat systems with uncertainties. Nonlinear Dynamics 67, 2689-2701 (2012).

[8] Jia, N., Wang, T., Generation and Modified projective Synchronization for a class of New Hyperchaotic Systems. Abstract and Applied Analysis, Article ID 804964, 11 pages (2013).

[9] Shi, X., Wang, Z., Hybrid synchronization phenomenon in two coupled delay hyperchaotic Lorenz systems with unknown parameters via a single linear controller. Journal of Vibrational and Control, 19(7), 1051-1060 (2013).

[10] Pai, N. S., Yau, H. T., Hybrid synchronization phenomenon in two coupled delay hyperchaotic Lorenz systems with unknown parameters via a single linear controller. Journal of Vibrational and Control, 17(1) 11-17 (2010).

[11] Al-Hadhrami, S., Saaban, A., Ibrahim, A., Shazad, M., Ahmad, I., Linear Active Control Algorithm to Synchronize a Nonlinear HIV/AIDS Dynamical System. Asian Journal of Applied Sciences and Engineering, 3(2), 96-115 (2014).

[12] Ojo, K. S., Njah, A. N., Ogunjo, O., Comparison of backsteeping and modified active control in projective synchronization of chaos in an extended Bonhoffer-van der Pol Oscillator. Pranam Journal of Physics, 80(5), 825-835 (2013).

[13] Njah, A. N., Synchronization and Ant-synchronization of double hump Duffing-Van der Pol Oscillator via Active Control. Journal of Information and Computer Science, 4(4), 243-250 (2009).

[14] Idowu, B. A., Vincent, U. E., Synchronization and stabilization of Chaotic Dynamics in Quasi-1D Bose-einstein Condensate. Journal of Chaos, (2013).

[15] Khan, A., Shahzad, M., Synchronization of circular restricted three body problem with Lorenz hyper chaotic system using a robust adaptive sliding mode controller. Complexity, 18, 58-64 (2013).

[16] Zhao, J., Zhou, D., Li, Y., A new impulsive synchronization of Chen hyper-chaotic system and Lü hyper-chaotic system. Journal of Vibration and Control, 19(12), 1773-1778 (2013).

[17] Park, J. H., Adaptive control for modified projective synchronization of a four-dimensional chaotic system with uncertain parameters. Journal of Computational and Applied mathematics, 213, 288-293 (2008).

[18] Ahmad, I., Saaban, A., Ibrahim, A., Shahzad, M., Chaos Control and Synchronization of Chaotic System Based upon Adaptive Control Algorithm, Int. Journal of Advances in Tel. comm. Electronics, Signals & Systems, 3(2), 53-62 (2014).

[19] Ying-ying, M., Yun-gang, L., Barbalat's Lemma and its application in analysis of system stability. Journal of Shandong University of Technology, 37(1), 51-55 (2007).

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