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A new method to find fuzzy nth order derivation and applications to fuzzy nth order arithmetic based on generalized h-derivation

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Abstract. In this paper, fuzzy nth-order derivative for $n \in N$ is introduced. To do this, nth-order derivation under generalized Hukuhara derivative here in discussed. Calculations on the fuzzy nth-order derivative on fuzzy functions and their relationships, in general, are introduced. Then, the fuzzy nth-order differential equations is solved, for $n \in N$.

Keywords: General nth-order derivative; fuzzy nth-order differential equations; H-derivative; H-difference.

AMS Classification: 34A07.

1. Introduction

The H-derivative of fuzzy number valued function was introduced by Siekalla in [22]. This derivation amplifies the fuzziness when time goes by [8], thus strongly general differentiability was introduced in this paper and have been studied by many researchers, this concept allows us to solve the problems of H-derivative. The fuzzy derivations are very important for solving fuzzy equations for instance, fuzzy differential equations and fuzzy integro-differential equations.

The first order equations under H-derivation studied by Bede initially at [3, 8]. He explained four cases of derivatives for fuzzy first order derivative. Two cases of them are always very important and the two others are important to acquire switching point. He used these four cases of derivatives for solving fuzzy differential equations. Chalco, used first two cases of derivations, because the two others cases are constant, [11].

General H-derivative has been used to study the second order derivation by Allahviranloo, [5] and Zhang, [26]. Their studies were used to get the existence of the fuzzy second order equations under general H-derivative. Allahviranloo et.al obtained the solutions of nth-order fuzzy linear differential equations by approximating method in [1, 2]. Allahviranloo and hooshangian introduced fuzzy generalized H-differential and used it to solve fuzzy differential equations of second-order, [4].

In this paper we use general H-derivative to find high order derivation. We acquire cases of derivations and we use them to invent relations

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between derivatives and their cases, then we apply them to investigate summation and minus of fuzzy derivatives and relationships between them. Indeed, with general differentiability, we can find more relationships for a larger classes of them rather than using H-derivative.

In section 2, we review briefly some needed concepts. In section 3 we introduce nth-order derivation for all $n \in N$, the minus and summation of two the fuzzy functions under H-derivative which are approved for nth-order derivation. Indeed, an algorithm is introduced to find fuzzy nth-order derivation and its cases for all, in general. In section 4, the fuzzy H-difference between two nth-order derivative of fuzzy functions is demonstrated and the examples to illustrate more are presented. In the final section, fuzzy differential equations in general form are solved. Our solution have been based on the generalized H-derivation. Finally, conclusion will be drawn in Section 5.

2. Basic Concepts

The basic definitions of a fuzzy number are given as follows:

Definition 1. [14] *A fuzzy number is a fuzzy set like* $u : R \to [0,1]$ *which satisfies:*

1. *u is an upper semi-continuous function,*
2. *$u(x) = 0$ outside some interval [a,d],*
3. *There are real numbers b, c such as $a \leq b \leq c \leq d$ and*
   - 3.1 *$u(x)$ is a monotonic increasing function on [a, b],*
   - 3.2 *$u(x)$ is a monotonic decreasing function on [c, d],*
   - 3.3 *$u(x) = 1$ for all $x \in [b, c]$.*

Definition 2. [14] *The metric structure is given by Hausdorff distance satisfying the following properties, that R_F is denoted the class of fuzzy subsets of real axis:*
$D : R_F \times R_F \longrightarrow R_+ \cup 0$
$D(u(r), v(r)) = Max\{sup|\underline{u} - \underline{v}|, sup|\overline{u} - \overline{v}|\}$
(R_F, D) is a complete metric space and following properties are well known:
$D(u + w, v + w) = D(u, v), \quad \forall u, v, w \in R_F$
$D(ku, kv) = |k|D(u, v), \quad \forall u, v \in R_F, \quad \forall k \in R$
$D(u+v, w+e) \leq D(u, w)+D(v, e), \quad \forall u, v, w, e \in R_F$

Definition 3. [16] *Let $x, y \in R_F$. If there exists $z \in R_F$ such that $x = y + z$ then z is called the H-differential of x, y and it is denoted $x \ominus y$.*

Definition 4. [7] *Let $F : I \to R_F$ and $t_0 \in I$. We say that F is differentiable at t_0 if there is $F'(t_0) \in R_F$ such that either*
(I) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0+h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0-h)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F(t_0+h)\ominus F(t_0)}{h} = lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0-h)}{h} = F'(t_0) \ or$$

(II) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0)\ominus F(t_0+h)$ and $F(t_0-h)\ominus F(t_0)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0+h)}{-h} = lim_{h\searrow 0}\frac{F(t_0-h)\ominus F(t_0)}{-h} = F'(t_0) \ or$$

(III) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0+h)\ominus F(t_0)$ and $F(t_0-h)\ominus F(t_0)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F(t_0+h)\ominus F(t_0)}{h} = lim_{h\searrow 0}\frac{F(t_0-h)\ominus F(t_0)}{-h} = F'(t_0) \ or$$

(IV) For $h > 0$ sufficiently close to 0, the H-differences $F(t_0)\ominus F(t_0+h)$ and $F(t_0)\ominus F(t_0-h)$ exist and the following limits

$$lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0+h)}{-h} = lim_{h\searrow 0}\frac{F(t_0)\ominus F(t_0-h)}{h} = F'(t_0)$$

Theorem 1. [7] *Let $F : [a, b] \to I$ be a function and denote $[F(t)]_\alpha = [f_\alpha(t), g_\alpha(t)]$ for each $\alpha \in [0, 1]$. Then:*
(i) If F is differentiable in the first form (I), then f_α and g_α are differentiable functions and $[F'(t)]_\alpha = [f'_\alpha(t), g'_\alpha(t)]$.
(ii) If F is differentiable in the second form (II), then f_α and g_α are differentiable functions and $[F'(t)]_\alpha = [g'_\alpha(t), f'_\alpha(t)]$.

Definition 5. [24] *Let $F : I \to R_F$ be a set-valued function. A point $t_0 \in I$ is said to be a switching point for the differentiability of F, if in any neighborhood T of t_0 there exist points $t_1 < t_0 < t_2$ such that:*
Type 1: F is differentiable at t_1 in the sense (I) of Definition 4 while it is not differentiable in the sense (II) of Definition 4 and F is differentiable at t_2 in the sense (II) of Definition 4 while it is not differentiable in the sense (I) of Definition 4.
or
Type 2: F is differentiable at t_1 in the sense (II) of Definition 4 while it is not differentiable in the sense (I) of Definition 4 and F is differentiable at t_2 in the sense (I) of Definition 4 while it is not differentiable in the sense (II) of Definition 4.

Theorem 2. [20] *Let $F : I \to R_F$ be differentiable on each $t \in I$ in the sense (III) or (IV) in Definition 4. Then $F'(t) \in R$ for all $t \in I$.*

Theorem 3. [20] *If $f : [a, b] \to R_F$ be integrable and $c \in [a, b]$, $\lambda \in R$. Then:*
(i) $\int_{t_0}^{t_0+a} F(t)dt = \int_{t_0}^{c} F(t)dt + \int_{c}^{t_0+a} F(t)dt$,
(ii) $\int_I (F(t) + G(t))dt = \int_I F(t)dt + \int_I G(t)dt$,
(iii) $\int_I \lambda F(t)dt = \lambda \int_I F(t)dt$,
(iv) $D(F, G)$ is integrable,
(V) $D(\int_I F(t)dt, \int_I G(t)dt) \le \int_I D(F, G)$

Definition 6. [20] *Let $f(x)$ be a fuzzy valued function on $[a, b]$. Suppose that $f(x, r)$ and $f(x, r)$ are improper Riemman-integrable on $[a, b]$ then we say that $f(x)$ is improper on $[a, b]$, furthermore,*

$$\underline{(\int_a^b f(t,r)dt)} = (\int_a^b \underline{f(t,r)}dt)$$

$$\overline{(\int_a^b f(t,r)dt)} = (\int_a^b \overline{f(t,r)}dt)$$

3. Generalized Fuzzy Nth-order Derivative

In this article is necessary to introduce the E and E^j items in the following terms:

$$E(F(t)) = \begin{cases} F(t) & F(t) \text{ is } (I) - differentiable, \\ \ominus F(t) & F(t) \text{ is } (II) - differentiable \end{cases}$$

and

$$E^j(F(t)) = \begin{cases} E^{j-1}(F(t)) & F^{(j-1)}(t) \text{ is } (I) - differentiable, \\ E^{j-1}(\ominus F(t)) & F^{(j-1)}(t) \text{ is } (II) - differentiable \end{cases}$$

which $E^j(F(t)) = E(E^{j-1}(F(t))$, for all j that $j = 2, 3, ..., n$.

Theorem 4. *For all $F, G \in R_F$ and $c \in R$, for all $j = 1, 2, ..., n$, is approved the following items:*
a) $E^j(c \odot F(t)) = c \odot E^j(F(t))$.
b) $E^j(F + G)(t) = E^j(F(t)) + E^j(G(t))$.
c) $E^j(F \ominus G)(t) = E^j(F(t)) \ominus E^j(G(t)) = E^j(F(t)) + E^{j+1}(G(t))$.
d) Let $j = 2k$, then $E^j(F(t)) = F(t)$ and let $j = 2k - 1$, then $E^j(F(t)) = E(F(t))$, for all $k = 1, 2, ...$

Proof. The proof is clear. □

At first we approve a theorem on the Hakuhara difference that are needed here under:

Theorem 5. *For all $x, y, z \in R_F$ and $a \in R$*
a) $0 \ominus x = \ominus x$
b) $\ominus x = \ominus y \Rightarrow x = y$
c) $\ominus(\ominus x) = x$
d) $x \ominus y = z \Rightarrow x \ominus z = y$
e) $x \ominus (y + z) = x \ominus y \ominus z$
f) $x \ominus (y \ominus z) = x \ominus y + z$
g) $\ominus ax = a(\ominus x)$
h) $\ominus(x \ominus y) = \ominus x + y$

Proof. a) The proof is trivial.
b) $\ominus x = \ominus y \Rightarrow 0 \ominus x = 0 \ominus y$, thus by Definition 3, there exists $u \in R_F$ that $0 \ominus x = 0 \ominus y = u$, thus $0 = x + u$ and also $0 = y + u$, thus $x + u = y + u$ and it is mean $x = y$
c) If $\ominus(\ominus x)$ exists, then there is $u \in R_F$ that $0 \ominus (\ominus x) = u$. In following we prove that $u = x$: $0 \ominus (\ominus x) = u$, thus $0 = u + (\ominus x)$, then $0 \ominus u = \ominus x$ and by using (a) we have $0 \ominus u = \ominus u = \ominus x$, using (b) we can result $x = u$.
d) If $x \ominus y = z$, thus we have $x = z + y$, then we can write $x = y + z \Rightarrow x \ominus z = y$
e) If $x \ominus (y + z)$ exists, then there exists $u \in R_F$ that $x \ominus (y + z) = u$ now by Definition 3 $x = u + y + z$, thus $x \ominus y = u + z$, now we can gain $x \ominus y \ominus z = u$
f) If there exists $x \ominus (y \ominus z)$, then $u \in R_F$ which $x \ominus (y \ominus z) = u$, by using Definition 3 we can write

$x = u + (y \ominus z)$ and $x = u + y \ominus z$, now we can write $x + z = u + y$ and therefore $x + z \ominus y = u$.
g) By using (a) we have $\ominus ax = 0 \ominus ax$, thus there exists $u \in R_F$ which $\ominus ax = 0 \ominus ax = u$ now by Definition 3 we have $0 = u + ax$, thus $0 = \frac{u}{a} + x$ and by using (a) we write $0 \ominus x = \frac{u}{a}$, thus $0 + (\ominus x) = \frac{u}{a}$ and $0 + a(\ominus x) = u$ therefore $a(\ominus x) = u$.
h) If there exists $\ominus(x \ominus y)$ then there is a $u \in R_F$ which $\ominus(x \ominus y) = u$, thus $0 \ominus (x \ominus y) = u$ and by Definition 3 we have $0 = u + x \ominus y$ and $0 + y = u + x$, thus $0 + y \ominus x = u$ therefore $y \ominus x = u$. □

Using H-derivative definition, Definition 4 and Theorem 4, we will have the following definition:

Definition 7. *Let $F : I \to R_F$ and $t_0 \in I$. We can say that F is differentiable of $n-$ordered at t_0 if there is $F^{(n-1)}(t_0) \in R_F$ such that either:*
(I) For $h > 0$ sufficiently close to 0, for all $n \in N$, the H-differences $F^{(n-1)}(t_0 + h) \ominus F^{(n-1)}(t_0)$ and $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0 - h)$ exist , and the following limits:

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h}$$
$$= lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h} = F^{(n)}(t_0)$$
or

(II) For $h > 0$ sufficiently close to 0, for all $n \in N$, the H-differences $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0 + h)$ and $F^{(n-1)}(t_0 - h) \ominus F^{(n-1)}(t_0)$ exist and the following limits:

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0+h)}{-h}$$
$$= lim_{h\searrow 0}\frac{F^{(n-1)}(t_0-h)\ominus F^{(n-1)}(t_0)}{-h} = F^{(n)}(t_0)$$
or

(III) For $h > 0$ sufficiently close to 0, for all $n \in N$, the H-differences $F^{(n-1)}(t_0 + h) \ominus F^{(n-1)}(t_0)$ and $F^{(n-1)}(t_0 - h) \ominus F^{(n-1)}(t_0)$ exist and the following limits:

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h}$$
$$= lim_{h\searrow 0}\frac{F^{(n-1)}(t_0-h)\ominus F^{(n-1)}(t_0)}{-h} = F^{(n)}(t_0)$$
or

(IV) For $h > 0$ sufficiently close to 0, for all $n \in N$, the H-differences $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0 + h)$ and $F^{(n-1)}(t_0) \ominus F^{(n-1)}(t_0 - h)$ exist and the following limits:

$$lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0+h)}{-h}$$
$$= lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h} = F^{(n)}(t_0)$$

Remark 1. *In Definition 7, by placing $n = 1$, the Definition 4 can be obtained.*

In Definition 7, it is clear that the nth-ordered derivative is depend on the $(n-1)$th-ordered derivative, $(n-1)$th-ordered derivative depend on the $(n-2)$th-ordered derivative and so on. Using this dependance and by using Theorem 5, for $F : I \longrightarrow R_F$, we have four cases of derivatives that can be proved as follows:

Remark 2. *In following theorem for all $n \in N$ and $k \in \{N \cup \{0\}\}, n \geq k$ we have $\binom{n}{k} = \frac{n!}{k!(n-k)!}$*

Theorem 6. *For all integer n -even and odd- we have four cases for H-derivative:*
(A): If $n = 2k, k = 1, 2, ...$ we have four cases:

(1): If even quantity of $F^{(i)}(t_0), i = 1, 2, ..., n$ are differentiable in case (I) and the rest in case (II) of Definition 7:

$$F^{(n)}(t_0) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0 + (n-j)h)))}{h^n}$$
$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0 + (n-j-1)h)))}{h^n}$$
$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0 + (n-j-2)h)))}{h^n}$$
$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0 - jh)))}{h^n} \qquad (1)$$

(2): If odd quantity of $F^{(i)}(t_0), i = 1, 2, ..., n-1$ are in case (I) and the rest in case (II) of Definition 7:

$$F^{(n)}(t_0) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0 + jh))}{-h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-1)h)))}{-h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-2)h)))}{-h^n}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-n+1)h)))}{-h^n} \tag{2}$$

(3): If even quantity of $F^{(i)}(t_0), i = 1,2,...,n$ are differentiable in case (III) and the rest in case (IV) of Definition 7:

$$F^{(n)}(t_0)$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j)h)))}{h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-1)h)))}{-h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j-2)h)))}{h^n}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-n+1)h)))}{-h^n} \tag{3}$$

(4): If odd quantity of $F^{(i)}(t_0), i = 1,2,...,n-1$ be in case (III) and the rest in case (IV) of Definition 7:

$$F^{(n)}(t_0) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+jh)))}{-h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j-1)h)))}{h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-2)h)))}{-h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0-jh)))}{h^n} \tag{4}$$

B): If $n = 2k-1, k = 1,2,...,$ we have four cases.

(1): If odd quantity of $F^{(i)}(t), i = 1,2,...,n$ are differentiable in case (I) and the rest in case (II) of Definition 7:

$$F^{(n)}(t_0)$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j)h)))}{h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j-1)h)))}{h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j-2)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0-jh)))}{h^n} \tag{5}$$

(2): If even quantity of $F^{(i)}(t_0), i = 1,2,...,n-1$ are in case (I) and the rest be in case (II) of Definition 7:

$$F^{(n)}(t_0) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+jh))}{-h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-1)h)))}{-h^n}$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-2)h)))}{-h^n}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-n+1)h)))}{-h^n} \tag{6}$$

(3): If odd quantity of $F^{(i)}(t_0), i = 1, 2, ..., n$ are differentiable in case (III) and the rest in case (IV) of Definition 7:

$$\begin{aligned} &F^{(n)}(t_0) \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j)h)))}{-h^n} \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-1)h)))}{h^n} \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j-2)h)))}{-h^n} \\ &= ... \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-n+1)h)))}{h^n} \end{aligned} \tag{7}$$

(4): If even quantity of $F^{(i)(t_0)}, i = 1, 2, ..., n-1$ are in case (III) and the rest be in case (IV) of Definition 7:

$$\begin{aligned} F^{(n)}(t_0) &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+jh)))}{-h^n} \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j-1)h)))}{h^n} \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(j-2)h)))}{-h^n} \\ &= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0-jh)))}{h^n} \end{aligned} \tag{8}$$

Proof. By induction, we consider the method for nth-order fuzzy derivative as accurate, the method should be approved for $(n+1)$th-order fuzzy derivation.

The theorem is proved for case (1) of (A), in the other cases are proved similarly. In the case (I) of Definition 7, the nth-order derivative is in following:

$$\begin{aligned} F^{(n)}(t) &= lim_{h\searrow 0}\frac{F^{(n-1)}(t_0+h)\ominus F^{(n-1)}(t_0)}{h} \\ &= lim_{h\searrow 0}\frac{F^{(n-1)}(t_0)\ominus F^{(n-1)}(t_0-h)}{h} \end{aligned} \tag{9}$$

and $(n+1)$th-order derivation in case (I) is:

$$\begin{aligned} &F^{(n+1)}(t) \\ &= lim_{h\searrow 0}\frac{F^{(n)}(t_0+h)\ominus F^{(n)}(t_0)}{h} \\ &= lim_{h\searrow 0}\frac{F^{(n)}(t_0)\ominus F^{(n)}(t_0-h)}{h} \end{aligned} \tag{10}$$

in the other hand by Theorem 6 for even n, we have:

$$\begin{aligned} &F^{(n)}(t_0) \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j)h)))}{h^n} \\ &= ... \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0-jh)))}{h^n}. \end{aligned} \tag{11}$$

By replacing elements of Eq. (11) by (10) we have

$$\begin{aligned} &F^{(n+1)}(t_0) \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n-j+1)h)))\ominus\binom{n}{j}(E^j(F(t_0+(n-j)h)))}{h^{n+1}} \\ &= ... \\ &= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(F(t_0+(n+(j+2))h))+(E^j((\binom{n}{j}(E^j(F(t_0+(n+(j+1))h)))}{h^{n+1}}. \end{aligned} \tag{12}$$

By expanding limits and by the following formulate:

$$\binom{n}{j}+\binom{n}{j+1}=\binom{n+1}{j+1}$$

we can reach the followings:

$$\binom{n}{j}(E^j(F(t_0+(n-j+1)h))$$

$$\ominus((\binom{n}{j+1}(E^j(F(t_0+(n-j+1)h)))$$

$$= \binom{n+1}{j}(E^j(F(t_0+(n-j+1)h))$$

$$\vdots$$

$$\binom{n}{j}(E^j(F(t_0 + (n + (j + 2))h)) \ominus \left(\binom{n}{j+1}(E^j(F(t_0 + (n + (j + 2))h))\right) = \binom{n+1}{j}(E^j(F(t_0+(n+(j+2))h))$$

thus

$$F^{(n+1)}(t_0)$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n+1}{j}(E^j(F(t_0+(n-j+1)h))}{h^{n+1}}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n+1}{j}(E^j(F(t_0+(n+(j+2))h))}{h^{n+1}} \qquad \square$$

Remark 3. *Now by replacing $n = 1$ in Eqs. (5), (6), (7) and (8), the Definition 4 and the other definitions in Ref [3] can be got.*

Theorem 7. *Let $F : I \to R_f$ is nth-ordered differentiable on each $t \in I$ in the case (III) or (IV) in Definition 7. Then $F^{(n)} \in R$ for all $t \in I$.*

Proof. Suppose that, (I) and (III) are coincided simultaneously. Then there are $A, B, C \in R_F$, which for only two first limits in cases (1) and (3) in Theorem 6 we have here:

$$\begin{aligned} A &= \sum_{i=0}^{\frac{n}{2}} E^i(F(t_0+(n-2i)h)) \\ &\ominus \sum_{i=0}^{\frac{n}{2}} E^i(F(t_0+(n-(2i+1))h)) \end{aligned}$$

and

$$\begin{aligned} B &= \sum_{i=0}^{\frac{n}{2}} E^i(F(t_0+(n-(2i+1))h)) \\ &\ominus \sum_{i=0}^{\frac{n}{2}-1} E^i(F(t_0+(n-(2i+2))h)) \end{aligned}$$

and

$$\begin{aligned} C &= \sum_{i=0}^{\frac{n}{2}-1} E^i(F(t_0+(n-(2i+2))h)) \\ &\ominus \sum_{i=0}^{\frac{n}{2}} E^i(F(t_0+(n-(2i+1))h)). \end{aligned}$$

Thus we get

$$\sum_{i=0}^{\frac{n}{2}-1} E^i(F(t_0+(n-(2i+2))h))$$

$$= \sum_{i=0}^{\frac{n}{2}-1} E^i(F(t_0+(n-(2i+2))h)) + (B+C)$$

i.e. $B + C = \tilde{0}$ which implies $B = C = \tilde{0}$, in case where $F^{(n)}(t_0) = \tilde{0}$ or $B, C \in R$, $B = -C$, then $F^{(n)}(t_0) \in R$ is resulted. $\square$

4. Arithmetics on the Fuzzy Nth-ordered Derivations

In this section calculations of the fuzzy nth-ordered derivation and their relationships are researched. These calculations are concluded summation and minus of two fuzzy functions and scalar multipliers of one fuzzy function.

Theorem 8. *If $g : I \longrightarrow R_F$, $c \in R_F$ and $f : I \longrightarrow R_F$ by $f(t) = c \odot g(t)$, for all $t \in I$. If g is differentiable on I of nth-order in $t_0 \in I$, then f is differentiable on I of nth-order in $t_0 \in I$ with $f^{(n)}(t_0) = c \odot g^{(n)}(t_0)$.*

Proof. Without loosing generality for even n, if even quantity of $f^{(i)}, i = 1, 2, ..., n$ are differentiable in case (I) and $f(t) = c \odot g(t)$, for all $t \in I$ are considered. Using Theorem 6 we will have:

$$f^{(n)}(t) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(n-j)h))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(j+1)h))}{h^n}$$

by putting $f(t) = c \odot g(t)$, the above equations will be written as below:

$(c \odot g(t))^{(n)}$

$$= lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} c \odot (E^j(g(t+(n-j)h))}{h^n}$$

$$= ... = lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} c \odot (E^j(g(t+(j+1)h))}{h^n}$$

By Theorem 6 the equations can be written in the following case:

$$f^{(n)}(t) = lim_{h \searrow 0} c \odot \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t+(n-j)h))}{h^n}$$

$$= ... = lim_{h \searrow 0} c \odot \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t+(j+1)h))}{h^n}$$

then we have

$$f^{(n)}(t) = c \odot lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t+(n-j)h))}{h^n}$$

$$= ... = c \odot lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t+(j+1)h))}{h^n}$$

$$= c \odot g^{(n)}(t)$$

proof for the other cases is similar and omitted. □

Theorem 9. *For odd n, we have:*
(a) If j quantity of $f^{(k)}(t), k = 1, 2, ..., n$, for all odd j, are (I)-differentiable and the rest (II)-differentiable on (a, b), and element i of $g^{(l)}(t), l = 1, 2, ..., n-1$, for all odd i, are (I)-differentiable on (a, b). Thus $f(t) + g(t)$ is nth-order differentiable and for all $t \in (a, b)$,
$(f + g)^{(n)}(t) = f^{(n)}(t) \ominus (-1)g^{(n)}(t)$.
(b) If j quantity of $f^{(j)}(t), j = 1, 2, ..., n-1$, for all even j, are (I)-differentiable on (a, b) and for i quantity of $g^{(l)}(t), l = 1, 2, ..., n$, for all odd i, are (I)-differentiable on (a, b). Then $f+g$ is nth-order differentiable and for all $t \in (a, b)$,
$(f + g)^{(n)}(t) = (-1)g^{(n)}(t) + (-1)f^{(n)}(t)$.
(c) If j quantity of $f^{(k)}(t), k = 1, 2, ..., n$, for all even j, are (I)-differentiable and the rest (II)-differentiable on (a, b), and element i of $g^{(l)}(t), l = 1, 2, ..., n-1$, for all even i, are (I)-differentiable on (a, b). Then $f + g$ is nth-order differentiable and for all $t \in (a, b)$,
$(f + g)^{(n)}(t) = g^{(n)}(t) \ominus (-1)f^{(n)}(t)$.
(d) If j quantity of $f^{(k)}(t), k = 1, 2, ..., n$, for all odd j, are (I)-differentiable and the rest (II)-differentiable on (a, b), and element i of $g^{(l)}(t), l = 1, 2, ..., n-1$, for all odd i, are (I)-differentiable on (a, b). Then $f + g$ is nth-order differentiable and for all $t \in (a, b)$,
$(f + g)^{(n)}(t) = f^{(n)}(t) + g^{(n)}(t)$.

Proof. (a) Without loosing generality, it is considered that j quantity of $f^{(k)}(t), k = 1, 2, ..., n$, for all odd j, are (I)-differentiable and i quantity of $g^{(l)}(t), l = 1, 2, ..., n$, for all even j, are (II)-differentiable on (a, b). By applying Theorem 6 we have:

$$f^{(n)}(t) = lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t+(j+1)h)))}{h^n}$$

and

$$g^{(n)}(t) = lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t+jh)))}{-h^n}$$
$$= ...$$
$$= lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t-(n-(j+1)h))))}{-h^n}$$

then

$(f + g)^{(n)}(t)$

$$= lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f+g)(t+(n-j)h))}{h^n}$$

$$= ... = lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f+g)(t+(j+1)h)))}{h^n}$$

$\Rightarrow (f+g)^{(n)}(t)$

$$= lim_{h \searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t+(n-j)h)))}{h^n}$$

$$+\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(j+1)h)))}{h^n}$$

$$+\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(j+1)h)))}{h^n}$$

$\Rightarrow (f{+}g)^{(n)}(t)$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(n-j)h)))}{h^n}$$

$$+\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(j+1)h)))}{h^n}$$

$$\ominus\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(j+1)h)))}{h^n}$$

$= f^{(n)} + (\ominus(-1)g^{(n)}) = f^{(n)}(t) \ominus (-1)g^{(n)}(t)$ □

Theorem 10. *Let n, be an odd number:*
(a) If j quantity of $f^{(k)}(t), k = 1,2,...,n$, for all odd j, are (I)-differentiable and the rest (II)-differentiable on (a,b), and element i of $g^{(l)}(t), l = 1,2,...,n-1$, for all odd i, are (I)-differentiable on (a,b). If H-differences $f^{(k)}(t) \ominus g^{(k)}(t), k = 1,2,...,n-1$, exist for $t \in (a,b)$ then $f(t) \ominus g(t)$ is n-order differentiable and for all $t \in (a,b)$, $(f \ominus g)^{(n)}(t) = f^{(n)}(t) + (-1)g^{(n)}(t)$.
(b) If j quantity of $f^{(j)}(t), j = 1,2,...,n-1$, for all even j, are (I)-differentiable on (a,b) and for i quantity of $g^{(l)}(t), l = 1,2,...,n$, for all odd i, are (I)-differentiable on (a,b). If H-differences $(f(t) \ominus g^{(k)}(t), k = 1,2,...,n-1$, exist for $t \in (a,b)$ then $f(t) \ominus g(t)$ is n-order differentiable and for all $t \in (a,b)$, $(f \ominus g)^{(n)}(t) = (-1)f^{(n)}(t) + g^{(n)}(t)$.
(c) If j quantity of $f^{(j)}(t), j = 1,2,...,n$ and i quantity of $g^{(l)}(t), l = 1,2,...,n$, for all even j and i, are (I)-differentiable and the rest are (II)-differentiable on (a,b). If H-differences $(f(t) \ominus g(t))^{(k)}, k = 1,2,...,n-1$, exist for $t \in (a,b)$ then $f(t) \ominus g(t)$ is n-order differentiable and for all $t \in (a,b)$, $(f \ominus g)^{(n)}(t) = g^{(n)}(t) \ominus (-1)f^{(n)}(t)$.
(d) If j quantity of $f^{(j)}(t), j = 1,2,...,n$ and i quantity of $g^{(l)}(t), l = 1,2,...,n$, for all odd j and i, are (I)-differentiable and the rest (II)-differentiable on (a,b). If $(f(t) \ominus g(t))^{(k)}, k = 1,2,...,n-1$, exist for all $t \in (a,b)$ then $f(t) \ominus g(t)$ is n-order differentiable and for all $t \in (a,b)$, $(f \ominus g)^{(n)}(t) = f^{(n)}(t) \ominus g^{(n)}(t)$.

Proof. (a) Let us consider j element of $f^{(k)}(t)$, $k = 1,2,...,n$, for all odd j, are (I)-differentiable and i element of, $g^{(l)}(t)$, $l = 1,2,...,n$, for all even j, (II)-differentiable on (a,b).
For odd n in Theorem 6 we have:

$$f^{(n)}(t) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(j+1)h)))}{h^n}$$

and

$$g^{(n)}(t) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+jh)))}{-h^n}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t-(n-(j+1)h)))}{-h^n}$$

then

$(f \ominus g)^{(n)}(t)$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j((f\ominus g)(t+(n-j)h)))}{h^n}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j((f\ominus g)(t+(j+1)h)))}{h^n}$$

$\Rightarrow (f{\ominus}g)^{(n)}(t)$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(n-j)h)))}{h^n}$$

$$\ominus\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^{j}(f(t+(j+1)h)))}{h^{n}}$$

$$\ominus\frac{\sum_{j=0}^{n}\binom{n}{j}(E^{j}(g(t+(j+1)h)))}{h^{n}}$$

$$\Rightarrow (f\ominus g)^{(n)}(t)$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^{j}(f(t+(n-j)h)))}{h^{n}}$$

$$+\frac{\sum_{j=0}^{n}\binom{n}{j}(E^{j}(g(t+(n-j)h)))}{h^{n}}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^{j}(f(t+(j+1)h)))}{h^{n}}$$

$$\ominus\frac{\sum_{j=0}^{n}\binom{n}{j}(E^{j}(g(t+(j+1)h)))}{h^{n}}$$

$$= f^{(n)}\ominus(\ominus(-1)g^{(n)}) = f^{(n)}(t)+(-1)g^{(n)}(t) \quad \square$$

Example 1. *Let* $f, g : [0, \pi/2] \to R_F$, $f = [5r-4, 3-2r]sint$ *and* $g = [-3+r, -1-r]t^4$*:*

(a) If f, f', f'' *be (I)-differentiable on* $(0, \pi/2)$ *and* g, g', g'' *are (II)-differentiable on* $(0, \pi/2)$ *or if one of* f, f', f'' *is (I)-differentiable and two of them are (II)-differentiable on* $(0, \pi/2)$ *and one of* g, g', g'' *is (II)-differentiable and two of them are (I)-differentiable on* $(0, \pi/2)$*. If H-differences* $f(t)\ominus g(t)$, $f'(t)\ominus g'(t)$ *and* $f''(t)\ominus g''(t)$ *exist for* $t \in (0, \pi/2)$ *then* $f(t)\ominus g(t)$ *is third order differentiable and for all* $t \in (0, \pi/2)$, $(f \ominus g)'''(t) = [5r-4, 3-2r](-cost) + (-1)[-72+24r, -24-24r]t$.

(b) If f, f', f'' *are (II)-differentiable on* $(0, \pi/2)$ *and* g, g', g'' *are (I)-differentiable on* $(0, \pi/2)$ *or if one of* f, f', f'' *is (II)-differentiable and two of them are (I)-differentiable on* $(0, \pi/2)$ *and one of* g, g', g'' *is (I)-differentiable and two of them are (II)-differentiable on* $(0, \pi/2)$*. If H-differences* $f(t) \ominus g(t)$ *and* $f'(t) \ominus g'(t)$ *and* $f''(t) \ominus g''(t)$ *exist for* $t \in (0, \pi/2)$ *then* $f(t) \ominus g(t)$ *is third order differentiable and for all* $t \in (0, \pi/2)$, $(f \ominus g)'''(t) = (-1)[5r-4, 3-2r](-cost)+[-72+24r, -24-24r]t$.

(c) If one of f, f', f'' *is (II)-differentiable and two of them are (I)-differentiable on* $(0, \pi/2)$ *and all of* g, g', g'' *are (II)-differentiable on* $(0, \pi/2)$*. If H-differences* $f(t) \ominus g(t)$ *and* $f'(t) \ominus g'(t)$ *and* $f''(t) \ominus g''(t)$ *exist for* $t \in (0, \pi/2)$ *then* $f(t) \ominus g(t)$ *is third order differentiable and for all* $t \in (0, \pi/2)$, $(f \ominus g)'''(t) = [-72+24r, -24-24r]t \ominus (-1)[5r-4, 3-2r](-cost)$*. (d) If one of* f, f', f'' *is (I)-differentiable and two of them are (II)-differentiable on* $(0, \pi/2)$ *and all of* g, g', g'' *are (I)-differentiable on* $(0, \pi/2)$*. If H-differences* $f(t)\ominus g(t)$ *and* $f'(t)\ominus g'(t)$ *and* $f''(t)\ominus g''(t)$ *exist for* $t \in (0, \pi/2)$ *then* $f(t)\ominus g(t)$ *is third order differentiable and for all* $t \in (0, \pi/2)$, $(f \ominus g)'''(t) = [5r-4, 3-2r](-cost) \ominus [-72+24r, -24-24r]t$*. We show that (a) is correct. The other results are provable similar.*

$$(f \ominus g)'''(t_0) =$$

$$lim_{h\searrow 0}\frac{(f\ominus g)(t_0+3h)+(f\ominus g)(t_0+h)\ominus(3(f\ominus g)(t_0+2h)+(f\ominus g)(t_0))}{h^3} =$$

$$\frac{\begin{aligned}&[5r-4,3-2r]sin(t_0+3h)\ominus[-3+r,-1-r](t_0+3h)^4+[5r-4,3-2r]sin(t_0+h)\\&\ominus[-3+r,-1-r](t_0+h)^4\ominus(3[5r-4,3-2r]sin(t_0+2h))\ominus 3[-3+r,-1-r](t_0+2h)^4\\&+[5r-4,3-2r]sin(t_0)\ominus[-3+r,-1-r](t_0)^4)=[5r-4,3-2r]sin(t_0+3h)\\&+[5r-4,3-2r]sin(t_0+h)\ominus 3[5r-4,3-2r]sin(t_0+2h)+[5r-4,3-2r]sin(t_0)\\&\ominus[-3+r,-1-r](t_0+3h)^4\ominus[-3+r,-1-r](t_0+h)^4\ominus(3[-3+r,-1-r](t_0+2h)^4)\\&\ominus[-3+r,-1-r](t_0)^4\end{aligned}}{h^3} = f'''(t_0)+(-1)g'''(t_0)$$

Theorem 11. *Let* n*, be an even:*
(a) If j *quantity of* $f^{(k)}(t), k = 1, 2, ..., n,$ *for all odd* j*, are (I)-differentiable and the rest (II)-differentiable on* (a, b)*, and element* i *of* $g^{(l)}(t), l = 1, 2, ..., n-1,$ *for all odd* i*, are (I)-differentiable on* (a, b)*. Then* $f(t) \ominus g(t)$ *is n-order differentiable and for all* $t \in (a, b)$, $(f + g)^{(n)}(t) = f^{(n)}(t) + g^{(n)}(t)$.

(b) If j *quantity of* $f^{(k)}(t), k = 1, 2, ..., n,$ *for all even* j*, are (I)-differentiable and the rest (II)-differentiable on* (a, b)*, and element* i *of* $g^{(l)}(t), l = 1, 2, ..., n-1,$ *for all odd* i*, are (I)-differentiable on* (a, b)*. Then* $f(t) + g(t)$ *is n-order differentiable and for all* $t \in (a, b)$, $(f + g)^{(n)}(t) = (-1)f^{(n)}(t) + (-1)g^{(n)}(t)$.

(c) If j *quantity of* $f^{(k)}(t), k = 1, 2, ..., n,$ *for all odd* j*, are (I)-differentiable and the rest*

(II)-differentiable on (a,b), and element i of $g^{(l)}(t), l = 1,2,...,n-1$, for all odd i, are (I)-differentiable on (a,b). Then $f(t)+g(t)$ is n-order differentiable and for all $t \in (a,b)$, $(f+g)^{(n)}(t) = (-1)f^{(n)}(t) \ominus g^{(n)}(t)$.

(d) If j quantity of $f^{(k)}(t), k = 1,2,...,n$, for all even j, are (I)-differentiable and the rest (II)-differentiable on (a,b), and element i of $g^{(l)}(t), l = 1,2,...,n-1$, for all even i, are (I)-differentiable on (a,b). Then $f(t)+g(t)$ is n-order differentiable and for all $t \in (a,b)$, $(f+g)^{(n)}(t) = f^{(n)}(t) \ominus (-1)g^{(n)}(t)$.

Proof. (a)If n be a even number and $f^{(k)}(t), k = 1,2,...,n$ be (I)-differentiable or (II)-differentiable:

$$f^{(n)}(t) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t+(j+1)h)))}{h^n}$$

and $g^{(k)}, k = 1,2,...,n$ be (II)-differentiable we have

$$g^{(n)}(t_0) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(g(t+(j+1)h)))}{h^n}$$

Now we can write

$$(f+g)^{(n)}(t) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f+g)(t_0+(n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f+g)(t_0+(j+1)h)))}{h^n}.$$

Then

$$(f+g)^{(n)}(t_0) = lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t_0+(n-j)h)))+(E^j(g(t_0+(n-j)h))))}{h^n}$$

$$= ...$$

$$= lim_{h\searrow 0}\frac{\sum_{j=0}^{n}\binom{n}{j}(E^j(f(t_0+(n-j)h))))+(E^j(g(t_0+(n-j)h)))}{h^n}$$

Proof (b), (c) and (d) are similar and omitted. □

Theorem 12. *For all even n we have four cases for $(f \ominus g)^{(n)}$, in respect to (I)-differentiability or (II)-differentiability:*

(a) If j elements of $f^{(j)}(t), j = 1,2,...,n$, for all odd j, are (I)-differentiable and the rest (II)-differentiable on (a,b), and i elements of $g^{(l)}(t), l = 1,2,...,n-1$, are (I)-differentiable on (a,b) and the rest are (II)-differentiable. If H-differences $f^{(k)}(t) \ominus g^{(k)}(t), k = 1,2,...,n-1$ exist for $t \in (a,b)$ then $f \ominus g$ is n-order differentiable at t on (a,b) and $(f \ominus g)^{(n)}(t) = f^{(n)}(t) \ominus g^{(n)}(t)$.

(b) If j elements of $f^{(j)}(t), j = 1,2,...,n$, are (I)-differentiable and the rest (II)-differentiable on (a,b),and i elements of $g^{(l)}(t), l = 1,2,...,n-1$, for all odd j and i, are (I)-differentiable on (a,b) and the rest (II)-differentiable. If H-differences $f^{(k)}(t) \ominus g^{(k)}(t), k = 1,2,...,n-1$ exist for $t \in (a,b)$ then $f(t) \ominus g(t)$ is n-order differentiable at t and $(f \ominus g)^{(n)}(t) = (-1)(f^{(n)}(t) \ominus g^{(n)}(t)$.

(c) If for odd number j, $f^{(j)}(t), j = 1,2,...,n$ be (I)-differentiable on (a,b) and for even i, $g^{(i)}(t), i = 1,2,...,n$, be (I)-differentiable on (a,b). If H-differences $f^{(k)}(t) \ominus g^{(k)}(t), k = 1,2,...,n-1$ exist for $t \in (a,b)$ then $f(t) \ominus g(t)$ is n-order differentiable at t and $(f \ominus g)^{(n)}(t) = g^{(n)}(t) \ominus (-1)f^{(n)}(t)$.

(d) If for even number j , $f^{(j)}(t), j = 1,2,...,n$ be (I)-differentiable on (a,b) and for all odd i, $g^{(k)}(t), k = 1,2,...,n-1$, be (I)-differentiable on (a,b). If H-differences $f^{(k)}(t) \ominus g(t)^{(k)}, k = 1,2,...,n-1$ exist for $t \in (a,b)$ then $f(t) \ominus g(t)$ is fourth order differentiable at t and $(f+g)^{(n)} =$

$f^{(n)} + (-1)g^{(n)}$.

Proof. (a) If n be a even number and $f^{(k)}(t), k = 1, 2, ..., n$ be (I)-differentiable or (II)-differentiable:

$$f^{(n)}(t) = lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t + (n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t + (j+1)h)))}{h^n}$$

and $g^{(k)}, k = 1, 2, ..., n$ be (II)-differentiable we have

$$g^{(n)}(t_0) = lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t + (n-j)h)))}{h^n}$$

$$= ... = lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(g(t + (j+1)h)))}{h^n}$$

Now we can write

$$(f + g)^{(n)}(t) =$$
$$lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f \ominus g)(t_0 + (n-j)h)))}{h^n}$$
$$= ... = lim_{h\searrow 0} \frac{\sum_{j=0}^{n} (-1)^{j+1} \ominus \binom{n}{j} (E^j(f \ominus g)(t_0 + (j+1)h))}{h^n}.$$

Then

$$(f + g)^{(n)}(t_0) =$$
$$lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t_0 + (n-j)h))) \ominus (E^j(g(t_0 + (n-j)h)))}{h^n}$$
$$= ...$$
$$= lim_{h\searrow 0} \frac{\sum_{j=0}^{n} \binom{n}{j} (E^j(f(t_0 + (n-j)h))) \ominus (E^j(g(t_0 + (n-j)h))))}{h^n}.$$

Proof (b), (c) and (d) are similar and omitted. □

Example 2. *Let $f, g : [0, \pi/2] \longrightarrow R_F$, $f = [5r - 4, 3 - 2r]sint$ and $g = [-3 + r, -1 - r]t^4$.*

(a) If $f, f', f'', f''', g, g', g'', g'''$ are differentiable in same case ((I) or (II)) on (a, b) or f, f', f'', f''' are (I)-differentiable and g, g', g'', g''' are (II)-differentiable on $(0, \pi/2)$ or inverse and if H-differences $f(t) \ominus g(t)$, $f'(t) \ominus g'(t), f''(t) \ominus g''(t)$ and $f'''(t) \ominus g'''(t)$ exist for $t \in (0, \pi/2)$ then $f \ominus g$ is fourth order differentiable at t on (a, b) and $(f \ominus g)^{(4)} = [5r - 4, 3 - 2r]sint \ominus [-72 + 24r, -24 - 24r]$.

(b) If one of f, f', f'', f''' are (II)-differentiable and the others are (I)-differentiable and one of g, g', g'', g''' be (I)-differentiable and the others be (II)-differentiable on $(0, \pi/2)$ or if one f, f', f'', f''' be (II)-differentiable and the others be (I)-differentiable on $(0, \pi/2)$ and similar for g, g', g'', g''', or f one f, f', f'', f''' are (I)-differentiable and the others be (II)-differentiable on $(0, \pi/2)$ and similar for g, g', g'', g'''. If H-differences $f(t) \ominus g(t)$, $f'(t) \ominus g'(t), f''(t) \ominus g''(t)$ and $f'''(t) \ominus g'''(t)$ exist for $t \in (0, \pi/2)$ then $f \ominus g$ is fourth order differentiable at t and $(f \ominus g)^{(4)} = (-1)([5r - 4, 3 - 2r]sint \ominus [-72 + 24r, -24 - 24r])$.

(c) If one of f, f', f'', f''' be (I)-differentiable and the others be (II)-differentiable and all of g, g', g'', g''' be (II)-differentiable or (I)-differentiable on $(0, \pi/2)$. If H-differences $f(t) \ominus g(t)$, $f'(t) \ominus g'(t), f''(t) \ominus g''(t)$ and $f'''(t) \ominus g'''(t)$ exist for $t \in (0, \pi/2)$ then $f + g$ is fourth order differentiable at t and $(f \ominus g)^{(4)} = (-1)[5r - 4, 3 - 2r]sint + [-72 + 24r, -24 - 24r]$.

(d) If f, f', f'', f''' be (II)-differentiable and one of g, g', g'', g''' be (I)-differentiable and the others be (II)-differentiable on $(0, \pi/2)$ or if f, f', f'', f''' be (I)-differentiable on $(0, \pi/2)$ and one of g, g', g'', g''' be (II)-differentiable and the others be (I)-differentiable on $(0, \pi/2)$. If H-differences $f(t) \ominus g(t)$, $f'(t) \ominus g'(t), f''(t) \ominus g''(t)$ and $f'''(t) \ominus g'''(t)$ exist for $t \in (0, \pi/2)$ then $f \ominus g$ is fourth

order differentiable at t and $(f \ominus g)^{(4)} = [5r-4, 3-2r]sint + (-1)[-72+24r, -24-24r]$.

We show that (a) is correct, the other results are provable similar.

$$(f \ominus g)^{(4)}(t_0) =$$
$$lim_{h \searrow 0} \frac{(f\ominus g)(t_0+4h)+6(f\ominus g)(t_0+2h)+(f\ominus g)(t_0)\ominus(4(f\ominus g)(t_0+3h)+(f\ominus g)(t_0+h))}{h^4}$$
$$= lim_{h \searrow 0} \frac{[5r-4,3-2r]sin(t_0+4h)\ominus[-3+r,-1-r](t_0+4h)^4+6[5r-4,3-2r]sin(t_0+2h)\ominus 6[-3+r,-1-r](t_0+2h)^4+[5r-4,3-2r]sin(t_0) \ominus[-3+r,-1-r](t_0)^4\ominus(4[5r-4,3-2r]sin(t_0+3h)\ominus 4[-3+r,-1-r](t_0+3h)^4)\ominus[5r-4,3-2r]sin(t_0+3h)\ominus[-3+r,-1-r](t_0+3h)^4}{h^4}$$
$$= lim_{h \searrow 0} \frac{[5r-4,3-2r]sin(t_0+4h)+6[5r-4,3-2r]sin(t_0+2h)\ominus[5r-4,3-2r]sin(t_0)+(4[5r-4,3-2r]sin(t_0+3h)\ominus[5r-4,3-2r]sin(t_0+3h)}{h^4}$$
$$+lim_{h \searrow 0} \frac{\ominus[-3+r,-1-r](t_0+4h)^4+6[-3+r,-1-r](t_0+2h)^4\ominus[-3+r,-1-r](t_0)^4\ominus 4[-3+r,-1-r](t_0+3h)^4)+[-3+r,-1-r](t_0+3h)^4}{h^4}$$
$$= f^{(4)}(t_0) \ominus g^{(4)}(t_0)$$

5. Solving Fuzzy Nth-order Differential Equations

We define an nth-order fuzzy differentiable equation by:

$$x^{(n)}(t) = f(t, x(t), x'(t), x''(t), ..., x^{(n-1)}(t)),$$

where $x(t)$ is a fuzzy function of t, $f(t, x(t), x'(t), x''(t), ..., x^{(n-1)}(t))$ is a fuzzy-valued function and the fuzzy variables $x'(t), x''(t), ..., x^{(n)}(t)$ are the defined derivatives of $x(t), x'(t), ..., x^{(n-1)}(t)$ respectively. Given the initial values $x(t_0) = k_0, x'(t_0) = k_1, ..., x^{(n-1)}(t_0) = k_{n-1}$, we obtain a fuzzy cauchy problem of the n-order

$$\begin{cases} x^{(n)}(t) = f(t, x(t), x'(t), x''(t), ..., x^{(n-1)}(t)), \\ x(t_0) = k_0, \\ x'(t_0) = k_1, \\ \vdots \\ x^{(n-1)}(t_0) = k_{n-1} \end{cases} \tag{13}$$

Theorem 13. *Let* $f : [a, b] \times E \times E \times ... \times E \to E$ *be continuous, and suppose that there exist* $M_1, M_2, ..., M_n > 0$ *such that*

$$D(f(t, x_1, x_2, ..., x_n); f(t, y_1, y_2, ..., y_n)) \leq \sum M_i D(x_i, y_i)$$

for all $t \in [a, b]$, $x_i, y_i \in E, i = 1, 2, ..., n$. *Then the initial value problem (13) has a unique solution on* $[a, b]$ *in each sense of differentiability.*

Proof. See Theorem 3.3 in [21]. □

Theorem 14. *For even number n, if $f : [a,b] \longrightarrow R_F$ and let $a = b_0 < b_1 < ... < b_n = b$ be a division of the interval $[a,b]$ such that f is n-order differentiable of (I) or (II) differentiable in the sense of Definition 7 on each of the intervals $[b_{i-1}, b_i], i = 1, 2, ..., n$, with the same case of $(n-1)$-order differentiable on each subinterval. Then:*
$\int_a^b(\int_a^b ...(\int_a^b f^{(n)}(t)dt...)dt)dt = f(b_i) - a_1 f(b_{i-1}) + a_2 f(b_{i-2}) - ... - a_{n-1} f(b_{i-n+1}) + f(b_{i-n}) + (-1) \odot (f(b_n) - a_1 f(b_{i-n+1}) + a_2 f(b_{i-n+2}) - ... - a_{n-1} f(b_{i-1}) + f(b_i))$.
here $a_i = \binom{n}{i}$ *and*
$I = \{i \in \{1, ..., n\}$ *such that for even number* k, $f^{(k)}, k = 1, 2, ..., n$, *be (I)-differentiable*$\}$.
$J = \{j \in \{1, ..., n\}$ *such that for odd number* k, $f^{(k)}, k = 1, 2, ..., n-1$, *be (I)-differentiable*$\}$

Proof.

$$f^{(n)}(t) = lim_{h\to 0} \frac{\sum_{j=0}^{n}(-1)^{j+1} \ominus \binom{n}{j} f(t + (n-j)h)}{h^n}$$

$$= ... = lim_{h\to 0} \frac{\sum_{j=0}^{n}(-1)^{j+1} \ominus \binom{n}{j} f(t)(t + (j+1)h)}{h^n}$$

then

$$\int_a^b(\int_a^b ...(\int_a^b f^{(n)}(t)dt...)dt)dt$$

$$= lim_{h\to 0} \int_a^b(\int_a^b ...(\int_a^b$$

$$\frac{\sum_{j=0}^{n}(-1)^{j+1} \ominus \binom{n}{j} f(t + (n-j)h)}{h^n} dt...)dt)dt$$

$$= ... = lim_{h\to 0} \int_a^b(\int_a^b ...$$

$$(\int_a^b \frac{\sum_{j=0}^{n}(-1)^{j+1} \ominus \binom{n}{j} f(t)(t + (j+1)h)}{h^n} dt$$

$$...)dt)dt$$

□

Theorem 15. *For odd number n, if $f : [a,b] \longrightarrow R_F$ and let $a = b_0 < b_1 < ... < b_n = b$ be a division of the interval $[a,b]$ such that f is n-order differentiable of (I) or (II) differentiable in the sense of Definition 7 on each of the intervals $[b_{i-1}, b_i], i = 1, 2, ..., n$, with the same case of $(n-1)$-order differentiable on each subinterval. Then:*
$\int_a^b(\int_a^b f^{(n)}(t)dt)...)dt = f(b_i) - a_1 f(b_{i-1}) + ... - a_{n-1} f(b_{i-n+1}) + f(b_{i-n}) + (-1) \odot (f(b_n) - a_1 f(b_{i-n+1}) + ... - a_{n-1} f(b_{i-1}) + f(b_i))$.
where $a_i = \binom{n}{i}$ *and*
$I = \{i \in \{1, ..., n\}$ *such that for odd number* k, $f^{(k)}, k = 1, 2, ..., n$ *be (I)-differentiable*$\}$.
$J = \{j \in \{1, ..., n\}$ *such that for even number* k, $f^{(k)}, k = 1, 2, ..., n-1$ *be (I)-differentiable*$\}$.

Theorem 16. *Let $t_0 \in [a,b]$, and assume that $f : [a,b] \times R_F \times R_F \times ... \times R_F \to R_F$ is continuous. A mapping $x : [a,b] \to R_F$ is a solution of the initial value problem (13) if and only if $x \in \mathcal{C}^n(I, R_F)$, and satisfies the following integral equations for all $t \in [a,b]$:*

$$\begin{aligned} x(t) = k_0 + c_1(k_1(t-t_0) + c_2(\frac{(t-t_0)^2}{2!} + ... \\ + c_{n-1}(\frac{k_{n-1}}{(n-1)!}(t-t_0)^{n-1} \\ + c_n \underbrace{\int_{t_0}^t \int_{t_0}^t ... \int_{t_0}^t}_{n} f(s, x(s), x'(s), ... \\ , x^{(n-1)}(s))ds...dsds)...)) \end{aligned} \tag{14}$$

where

$$c_i = \begin{cases} 1, & x^{(i)}(t) \;\; is \;\; (I) - differentiable, \\ \ominus(-1), & x^{(i)}(t) \;\; is \;\; (II) - differentiable. \end{cases}$$

for all $i = 1, 2, ..., n$.

Proof. Since f is continuous, it must be integrable. Is considered that (14) is solution of initial value problem (13). It is obvious that the solution for the following problem:

$$\begin{cases} x^{(n+1)}(t) = f(t, x(t), x'(t), x''(t), ..., x^{(n)}(t)), \\ x(t_0) = k_0, \\ x'(t_0) = k_1, \\ \vdots \\ x^{(n)}(t_0) = k_n \end{cases}$$

should be resulted as under:

$$x'(t) = k_1 + c_2(\frac{k_2}{2!}(t - t_0)^2 + c_3(\frac{k_3}{3!}(t - t_0)^3 + ... + c_{n-2}(\frac{k_{n-2}}{(n-2)!}(t - t_0)^{(n-2)} + c_{n-1}\underbrace{\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n} f(s, x(s), x'(s), ..., x^{(n)}(s))ds...ds))).$$

By exercising integral over $[t_0, t]$, we can equivalently have:

$$x(t) = k_0 + c_1(k_1(t - t_0) + c_2(\frac{k_2}{2!}(t - t_0)^2 + ... + c_{n-1}(\frac{k_{n-1}}{(n-1)!}(t - t_0)^{n-1} + c_n\underbrace{\int_{t_0}^{t}\int_{t_0}^{t} ... \int_{t_0}^{t}}_{n+1} f(s, x(s), x'(s), ..., x^{(n)}(s))ds...dsds)))).$$

□

Example 3. *Let following fuzzy differential equation with initial values:*

$$\begin{cases} x''(t) = x(t) \\ x(0) = [\alpha - 1, 1 - \alpha], \\ x'(0) = [\alpha, 2 - \alpha] \end{cases}$$

Then the solution of this fuzzy differential equation for all $t \in [0, \infty]$ is

$$x(t) = [\alpha - 1, 1 - \alpha] + c_2([\alpha, 2 - \alpha]t + c_1(\int_0^t \int_0^t x(s)dsds)).$$

Let $x(t)$ and $x'(t)$ are (I)-differentiable, the solution by Theorem 16 is obtained in the following:

$$x(t) = [\alpha - 1, 1 - \alpha] + [\alpha, 2 - \alpha]t + \int_0^t \int_0^t x(s)dsds$$

Now we can solve this interval-value integral equation, it means two crisp integral equation should be solve. The solution is gained by the Modified Adomian method in the following:

$$x(t) = [\alpha - 1 + \alpha t + \frac{t^2}{2}(\alpha - 1), 1 - \alpha + (2 - \alpha)t + \frac{t^2}{2}(1 - \alpha)].$$

Let $x(t)$ and $x'(t)$ be (II)-differentiable, the solution by Theorem 16 is gained in the following interval equation:

$$x(t) = [\alpha - 1, 1 - \alpha] \ominus (-1)[\alpha, 2 - \alpha]t + \int_0^t \int_0^t x(t)dsds$$

It means, the solution by solving two crisp integral equations by Modified Adomian method will be obtained in the following term:

$$x(t) = [\alpha - 1 + (2 - \alpha)t + \frac{t^2}{2}(\alpha - 1), 1 - \alpha + \alpha t + \frac{t^2}{2}(1 - \alpha)].$$

Let $x(t)$ be (I)-differentiable and $x'(t)$ be (II)-differentiable, the solution by Theorem 16 is in the bottom interval equation:

$$x(t) = [\alpha - 1, 1 - \alpha] \ominus (-1)[\alpha, 2 - \alpha]t \ominus (-1)\int_0^t \int_0^t x(t)dsds$$

It means, the solution by solving a crisp integral equation system by Modified Adomian method will be obtained in the bottom term:

$$x(t) = [\alpha - 1 + (2 - \alpha)t + \frac{t^2}{2}(1 - \alpha), 1 - \alpha + \alpha t + \frac{t^2}{2}(\alpha - 1)].$$

Let $x(t)$ be (II)-differentiable and $x'(t)$ be (I)-differentiable, the solution by Theorem 16 is obtained in the following:

$$x(t) = [\alpha - 1, 1 - \alpha] + [\alpha, 2 - \alpha]t \ominus (-1)\int_0^t \int_0^t x(t)dsds$$

It means, the solution by solving a crisp integral equation system by Modified Adomian method will be gained in the sequence:

$$x(t) = [\alpha - 1 + \alpha t + \frac{t^2}{2}(1 - \alpha), 1 - \alpha + (2 - \alpha)t + \frac{t^2}{2}(\alpha - 1)].$$

6. Conclusion

In this work, we introduced a new method for finding generalized fuzzy nth order derivative and we proved some theorems in the relationships between fuzzy derivatives of nth order and we presented the solution of fuzzy differential equations of nth order. For future research one can use generalized fuzzy nth order derivative for obtaining the switching point of fuzzy differential equations that is introduced by Bede [8].

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