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Mond-Weir type second order multiobjective mixed symmetric duality with square root term under generalized univex function

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Abstract. In this paper, a new class of second order (ϕ, ρ) -univex and second order (ϕ, ρ) pseudo univex function are introduced with example. A pair Mond-Weir type second order mixed symmetric duality for multiobjective nondifferentiable programming is formulated and the duality results are established under the mild assumption of second order (ϕ, ρ) univexity and second order pseudo univexity. Special cases are discussed to show that this study extends some of the known results in related domain.

Keywords: Second order (ϕ, ρ) univex, mixed symmetric duality, efficient solution, square root term, Schwartz inequality.
AMS Classification: 90C29, 90C30, 90C46

1. Introduction

Mond [16] initiated second order symmetric duality type in nonlinear programming and proved second order symmetric duality theorems under second order convexity. Mangasarian [12] discussed second order duality in nonlinear programming under inclusion condition. Mond [16] and Mangasarian [12] also indicated possible computational advantages of the second order dual over the first order dual. Later, Bector and Chandra [6] presented a pair of Mond-Weir type second order dual programs and proved weak, strong and self duality theorems under pseudo bonvexity and pseudo convexity assumption. Ahmad and Sharma [5] established second order duality for non differentiable multiobjective programming under generalized F-convexity. The concept of mixed duality is interesting and useful both from theoretical as well as from algorithmic point of view. Xu [19] formulated two mixed type duals in multiobjective programming and also proved duality theorems. Earlier, Chandra et al. [9] and Bector et al. [7,8] formulated mixed symmetric duality for a class of nonlinear programming problems. Yang et al. [20] discussed a mixed symmetric duality for a class of nondifferentiable nonlinear programming problems. Aghezzaf [2] formulated a second order multiobjective mixed type dual and obtained various duality results involving a new class of generalized second order (F,ρ)-convex function. Mishra et al. [14, 15] and Mishra [13] presented mixed symmetric first and second order duality in nondifferentiable mathematical programming problem under F-convexity. Recently, Ahmad and Husain [3] discussed a pair of multiobjective mixed symmetric dual programs over arbitrary cones and established duality results under K-preinvex /K-pseudo invexity assumption. Also, Kailey et al. [10] established mixed second order multiobjective symmetric duality with cone

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constrain under -bonvexity. Li and Gao [11] and Agarwal et al. [1] introduced a model of mixed symmetric duality for a class of non differentiable multiobjective programming problem with multiple arguments.

In this paper, a pair of second order multiobjective mixed symmetric dual programs using square root term is formulated and duality theorems are proved for these programs under generalized (ϕ,ρ) univexity. Special cases are discussed to show that this study extends some of the known results in related domain.

2. Notations and definitions

Let R^n and R^m are n-dimensional and m-dimensional Euclidean space respectively. R_+^n and R_+^m their respective nonnegative orthant. The following conventions for vectors $x,u \in R^n$ will be followed throughout this paper: $x<u \Leftrightarrow x_i<u_i$, $x \le u \Leftrightarrow x_i \le u_i$; $i=1,2,....n$. For any vector $x,u \in R^n$ we denote $x^T u = \sum_{i=1}^{n} x_i u_i$.

Let X and Y are open subset sets of R^n and R^m respectively. Let $f_i(x,y)$ be a real valued twice differentiable function defined on $X \times Y$. Let $\nabla_x f_i(x,y)$ and $\nabla_y f_i(x,y)$ denote the gradient vectors of $f_i(x,y)$ with respect to first variable x and second variable y respectively. Also $\nabla_{xx} f_i(x,y)$ and $\nabla_{yy} f_i(x,y)$ denote the Hessian matrix of $f_i(x,y)$ with respect to the first variable x and second variable y respectively. For $p \in R^m$, $\nabla_x(\nabla_{yy} f_i(x,y)p)$ and $\nabla_y(\nabla_{yy} f_i(x,y)p)$ are $n \times m$ and $m \times m$ matrix obtained by differentiating the elements of $\nabla_{yy} f_i(x,y)p$ with respect to x and y respectively.

Consider the following multiobjective programming problem (MP):

MP :(**Primal**) Minimize

$$f(x) = (f_1(x), f_2(x),....., f_r(x))$$

Subject to $h(x) \le 0, x \in X \subseteq R^n$,

where $f: X \to R^r, h: X \to R^m$.

Let X_0 be the set of all feasible solutions of problem (P); that i.e. $X_0 = \{x \in X \mid h(x) \le 0\}$.

Definition 2.1 A vector $\bar{x} \in X_0$ is said to be an efficient solution of problem (P) if there exists no $x \in X_0$ such that $f(x) \le f(\bar{x}),\ f(x) \ne f(\bar{x})$.

Definition 2.2 A vector $\bar{x} \in X_0$ is said to be a weakly efficient solution of problem (P) if there exists no $x \in X_0$ such that $f(x) < f(\bar{x})$.

Definition 2.3 A vector $\bar{x} \in X_0$ is said to be a properly efficient of problem (P) if it is efficient and there exists a positive constant M such that whenever $f_i(x) < f_i(\bar{x})$ for $x \in X_0$ and for $i \in \{1,2,...r\}$, there exist at least one $j \in \{1,2,...,r\}$ such that $f_j(\bar{x}) < f_j(x)$ and $f_i(\bar{x}) - f_i(x) \le M(f_j(x) - f_j(\bar{x}))$.

Definition 2.4 :(Schwartz Inequality) Let $x,y \in R^n$ and $A \in R^n \times R^n$ be a positive semi definite matrix, then $x^T Ay \le (x^T Ax)^{\frac{1}{2}} (y^T Ay)^{\frac{1}{2}}$, equality holds if for some $\lambda \ge 0, Ax = \lambda Ay$.

Let $r, \rho \in R$. Suppose ϕ_0^i and ϕ_1^i are a real valued function defined on $R^{|J_i|} \times R^{|J_i|} \times R^{|J_i|+1}$ and $R^{|K_i|} \times R^{|K_i|} \times R^{|K_i|+1}$, for $i=1,2;$ respectively such that $\phi_0^i(x^i,u^i,*)$ and $\phi_1^i(v^i,y^i,*)$, $i=1,2$. are convex on R^{n+1}, R^{m+1} respectively with $\phi_0^i(x^i,u^i,(0,r)) \ge 0$, and $\phi_1^i(v^i,y^i,(0,r)) \ge 0$, for $i=1,2$ and $r \in R_+$. b_0^i and b_1^i are non negative function defined on $R^{|J_i|} \times R^{|J_i|}$ and $R^{|K_i|} \times R^{|K_i|}$, $i=1,2$. respectively .

Throughout this chapter, we assume that $\psi_0^i, \psi_1^i : R \to R$ satisfying $\psi_0^i(u) \ge 0 \Rightarrow u \ge 0$, $\psi_1^i(u) \le 0 \Rightarrow u \le 0$ and $\psi_0^i(-a) = -\psi_0^i(a)$, $\psi_1^i(-a) = -\psi_1^i(a)$ for $i=1,2$.

Now we define a new class of second order (ϕ,ρ) -univex, second order (ϕ,ρ) -pseudo univex and second order (ϕ,ρ) -quasi-univex as follows.

Definition 2.5 A real-valued twice differentiable function $f_i(*,y): X \times X \to R$ is said to be second order (ϕ,ρ)-univex at $u \in X$ with respect to $q \in R^n$ for $b: X\times X \to R_+$, $\psi: R \to R$, if there exist $\phi: X\times X\times R^{n+1} \to R$, $\rho \in R$ such that

$$b(x,u)\psi\begin{pmatrix} f_i(x,y)-f_i(u,y) \\ +\frac{1}{2}q^T\nabla_{uu}f_i(u,y)q \end{pmatrix} \ge \phi\left(x,u;\begin{pmatrix}\nabla_u f_i(u,y) \\ +\nabla_{uu}f_i(u,y)q,\rho\end{pmatrix}\right)$$

Definition2.6 A real-valued twice differentiable function $f_i(*,y): X\times X \to R$ is said to be second order (ϕ,ρ)-pseudo univex at $u \in X$ with respect to $q \in R^n$ for $b: X\times X \to R_+$ and $\psi: R \to R$, if there exist $\phi: X\times X\times R^{n+1} \to R$, $\rho \in R$ such that

$$\phi(x,u;(\nabla_u f_i(u,y)+\nabla_{uu}f_i(u,y)q,\rho)) \ge 0$$

$$\Rightarrow b(x,u)\psi\begin{pmatrix} f_i(x,y)-f_i(u,y) \\ +\frac{1}{2}q^T\nabla_{uu}f_i(u,y)q \end{pmatrix} \ge 0.$$

Definition 2.7 A real-valued twice differentiable function $f_i(*,y): X\times X \to R$ is said to be second order (ϕ,ρ)-quasiunivex at $u \in X$ with respect to $q \in R^n$ for $b: X\times X \to R_+$ and $\psi: R \to R$, if there exist $\phi: X\times X\times R^{n+1} \to R$, $\rho \in R$ such that

$$b(x,u)\psi\begin{pmatrix} f_i(x,y)-f_i(u,y) \\ +\frac{1}{2}q^T\nabla_{uu}f_i(u,y)q \end{pmatrix} \ge 0$$

$$\Rightarrow \phi(x,u;(\nabla_u f_i(u,y)+\nabla_{uu}f_i(u,y)q,\rho)) \ge 0.$$

Definition 2.8 A real valued twice differentiable function f is second order (ϕ,ρ)- unicave and second order (ϕ,ρ)-pseudounicave if $-f$ is second order (ϕ,ρ)-univex and second order (ϕ,ρ) pseudounivex respectively.

Remark 2.1 If we consider the case $b \equiv 1$,

$$\phi(x,u;(\nabla_u f(u)+\nabla_{uu}f(u)q,\rho))$$
$$= F(x,u;\nabla_u f(u)+\nabla_{uu}f(u)q)+\rho d^2(x,u)$$

with F is sub linear in third argument and ρ=0, then the above definitions reduce to second order F-convexity and second order F-pseudo convexity as introduced by Mishra [13].

Example 2.1 Let $f: R\times R \to R$ defined as

$$f(x,y) = e^{-x^2} - e^{-y^2}.$$

So we have $\nabla_x f(x,y) = -2xe^{-x^2}$,

$$\nabla_{xx} f(x,y) = (4x^2-2)e^{-x^2},$$

Now $f(x,y)$ is not convex at u=0, as

$$f(x,y)-f(u,y)-(x-u)^T\nabla_x f(u,y)$$
$$= e^{-x^2}-e^{-u^2}-(x-u)(-2ue^{-u^2})$$
$$= e^{-x^2}-1 \ngeq 0 \text{ for } u=0, x\neq 0.$$

Let $q = \begin{bmatrix}1\\0\end{bmatrix}$. Then

$$f(x,y)-f(u,y)+\tfrac{1}{2}q^T\nabla_{xx}f(u,y)q$$
$$-(x-u)^T[\nabla_x f(u,y)+\nabla_{xx}f(u,y)q]$$
$$= e^{-x^2}-e^{-u^2}+\tfrac{1}{2}(4u^2-2)e^{-u^2}$$
$$-(x-u)[-2ue^{-u^2}+(4u^2-2)e^{-u^2}]$$
$$= e^{-x^2}+2x-2 \ngeq 0, \text{ for } u=0, x=\tfrac{1}{2}.$$

So $f(x,y)$ is not second order convex/bonvex at u=0,

Let $\phi: R\times R\times R^2 \to R$ defined as

$$\phi(x,u;(a,\rho)) = (1-2^{\rho})e^{-xu}+u^Ta;$$

$\psi: R \to R$ defined as $\psi(x) = x$;

$b: R\times R \to R_+$ defined as $b(x,u) = 1$.

Now

$$b(x,u)\psi[f(x,y)-f(u,y)+\tfrac{1}{2}q^T\nabla_{xx}f(u,y)q]$$
$$-\phi(x,u,(\nabla_x f(u,y)+\nabla_{xx}f(u,y)q,\rho))$$
$$= e^{-x^2}-e^{-u^2}+\tfrac{1}{2}(4u^2-2)e^{-u^2}-(1-2^{\rho})e^{-xu}$$
$$-[-2ue^{-u^2}+(4u^2-2)e^{-u^2}]$$
$$= e^{-x^2}-1-1-(1-2^{\rho})+2 \quad \text{at } u=0,$$
$$= e^{-x^2}+2^{\rho}-1>0 \quad \text{for } \rho>0, \forall x\in R.$$

So $f(x,y)$ is second order (ϕ,ρ)-univex function at u=0 .

Hence from the above example it is clear that second order (ϕ,ρ)-univex function is more generalized than convex and second order convex function.

Example 2.2

Let $g : R_+ \times R_+ \to R$ defined as

$g(x,y) = x\ln x - y\ln y,\ \nabla_x g(x,y) = 1+\ln x,$

$\nabla_{xx} g(x,y) = \frac{1}{x}.$

Now $g(x,y)$ is not convex at u=1, since

$g(x,y) - g(u,y) - (x-u)^T \nabla_x g(u,y)$

$= x\ln x - u\ln u - x - x\ln u + u + u\ln u$

$= x\ln x - x - x\ln u = x\ln x - x \ngeq 0$ for $x \in (0,e).$

Again $g(x,y) - g(u,y) + \frac{1}{2} q^T \nabla_{xx} g(u,y) q$

$-(x-u)^T [\nabla_x g(u,y) + \nabla_{xx} g(u,y) q]$

$= x\ln x + u + \frac{1}{2u} + 1 - x - x\ln u - \frac{x}{u}$

$= x\ln x + \frac{5}{2} - 2x$ for $u = 1.$

$= x\ln x + \frac{5}{2} - 2x \ngeq 0,$ for $x \in [2,\infty)$

So $g(x,y)$ is not second order convex/bonvex at u=1,

Let $\phi : R \times R \times R^2 \to R$ defined as

$\phi(x,u;(a,\rho)) = (1-2^{\rho})e^{-xu} + u^T a;$

$\psi : R \to R$ defined as $\psi(x) = x;$

$b : R \times R \to R_+$ defined as $b(x,u) = 1.$

$b(x,u)\psi[g(x,y) - g(u,y) + \frac{1}{2} q^T \nabla_{xx} g(u,y) q]$

$= x\ln x - u\ln u + \frac{1}{2u}$

Let $\phi(x,u;(\nabla_u g + \nabla_{uu} g q),\rho)) \geq 0$

$\Rightarrow (1-2^{\rho})e^{-xu} + (\nabla_u g + \nabla_{uu} g q)$

$= (1-2^{\rho})e^{-xu} + (1 + \ln u + \frac{1}{u}) \geq 0$

$\Rightarrow (1-2^{\rho})e^{-x} + 2 \geq 0,$ at $u = 1,$

$\Rightarrow (1-2^{\rho})e^{-x} + 2 \geq 0 \Rightarrow x > 0,$ for $\rho = 1.$

Let $q = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$

Now

$b(x,y)\psi[g(x,y) - g(u,y) + \frac{1}{2} q^T \nabla_{xx} g(u,y) q]$

$= x\ln x - u\ln u + \frac{1}{2u} \geq 0$, for $x > 0.$

So $g(x,y)$ is second order (ϕ,ρ) -pseudo-univex function at u=1.

Hence from the above example it is clear that second order (ϕ,ρ) -pseudo-univex function is more generalize than convex and second order convex function.

3. Mond-Weir type second order mixed symmetric dual program

For N={1,2,…,n} and M={1,2,…,m}, let $J_1 \subseteq N$ and $J_2 = N \setminus J_1$. Similarly $K_1 \subseteq M$ and $K_2 = M \setminus K_1$. Let $|J_1|$ denote the number of elements in J_1. The numbers $|J_2|, |K_1|, |K_2|$ are defined similarly. Notice that if $|J_1| = 0$, then $|J_2| = n$. It is clear that any $x \in X \subseteq R^n$ can be written as $x = (x^1, x^2),\ x^1 \in R^{|J_1|}$ and $x^2 \in R^{|J_2|}$. Similarly, any $y \in Y \subseteq R^m$, can be written as $y = (y^1, y^2)$, $y^1 \in R^{|K_1|}, y^2 \in R^{|K_2|}$.

Let $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ and $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ are thrice continuously differentiable functions.

Let $\lambda_i \in R,\ p_i^1 \in R^{|K_1|}, w_i^1 \in R^{|K_1|}, p_i^2 \in R^{|K_2|}, w_i^2 \in R^{|K_2|},$ $q_i^1 \in R^{|J_1|}, q_i^2 \in R^{|J_2|},\ a_i^1 \in R^{|J_1|},\ a_i^2 \in R^{|J_2|},$

$p^1 = (p_1^1, p_2^1, \ldots, p_r^1),\ p^2 = (p_1^2, p_2^2, \ldots, p_r^2),$

$q^1 = (q_1^1, q_2^1, \ldots, q_r^1),\ q^2 = (q_1^2, q_2^2, \ldots, q_r^2),$

$a^1 = (a_1^1, a_2^1, \ldots, a_r^1),\ a^2 = (a_1^2, a_2^2, \ldots, a_r^2),$

$w^1 = (w_1^1, w_2^1, \ldots, w_r^1),\ w^2 = (w_1^2, w_2^2, \ldots, w_r^2),$ and $E_i^1,\ E_i^2,\ C_i^1, C_i^2$ are positive semi definite matrices of order $|J_1|, |J_2|, |K_1|$ and $|K_2|$ respectively for $i = 1, 2, \ldots, r.$

Now we formulate the following pair of multiobjective mixed symmetric dual programs and prove duality theorems:

(SMSP): Second order mixed symmetric primal:

Minimize $H(x,y,w,p) = \begin{pmatrix} H_1(x,y,w_1,p_1), \\ H_2(x,y,w_2,p_2), \\ \ldots, H_r(x,y,w_r,p_r) \end{pmatrix}$

Subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_{y^1} f_i(x^1,y^1) - C_i^1 w_i^1 + \nabla_{y^1 y^1} f_i(x^1,y^1) p_i^1] \leq 0, \quad (3.1)$$

$$\sum_{i=1}^{r} \lambda_i [\nabla_{y^2} g_i(x^2,y^2) - C_i^2 w_i^2 + \nabla_{y^2 y^2} g_i(x^2,y^2) p_i^2] \leq 0, \quad (3.2)$$

$$(y^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_{y^1} f_i(x^1,y^1) - C_i^1 w_i^1 + \nabla_{y^1 y^1} f_i(x^1,y^1) p_i^1] \geq 0, \quad (3.3)$$

$$(y^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_{y^2} g_i(x^2, y^2) - C_i^2 w_i^2 + \nabla_{y^2 y^2} f_i(x^2, y^1) p_i^2] \geq 0, \quad (3.4)$$

$$(x^1, x^2) \geq 0, \quad (3.5)$$

$$(w_i^1)^T C_i^1 w_i^1 \leq 1, i = 1, 2, \ldots r. \quad (3.6)$$

$$(w_i^2)^T C_i^2 w_i^2 \leq 1, i = 1, 2, \ldots r. \quad (3.7)$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1 \quad (3.8)$$

(SMSD): Second order Mixed Symmetric Dual:

Maximize $G(u, v, a, q) = \begin{pmatrix} G_1(u, v, a_1, q_1), \\ G_2(u, v, a_2, q_2), \\ \ldots, G_r(u, v, a_r, q_r) \end{pmatrix}$

Subject to

$$\sum_{i=1}^{r} \lambda_i [\nabla_{u^1} f_i(u^1, v^1) + E_i^1 a_i^1 + \nabla_{u^1 u^1} f_i(u^1, v^1) q_i^1] \geq 0, \quad (3.9)$$

$$\sum_{i=1}^{r} \lambda_i [\nabla_{u^2} g_i(u^2, v^2) + E_i^2 a_i^2 + \nabla_{u^2 u^2} g(u^2, v^2)] \geq 0, \quad (3.10)$$

$$(u^1)^T \sum_{i=1}^{r} \lambda_i [\nabla_{u^1} f_i(u^1, v^1) + E_i^1 a_i^1 + \nabla_{u^1 u^1} f(u^1, v^1) q_i^1] \leq 0, \quad (3.11)$$

$$(u^2)^T \sum_{i=1}^{r} \lambda_i [\nabla_{u^2} g_i(u^2, v^2) + E_i^2 a_i^2 + \nabla_{u^2 u^2} g_i(u^2, v^2) q_i^2] \leq 0, \quad (3.12)$$

$$(v^1, v^2) \geq 0, \quad (3.13)$$

$$(a_i^1)^T E_i^1 a_i^1 \leq 1; \quad i = 1, 2, \ldots.., r, \quad (3.14)$$

$$(a_i^2)^T E_i^2 a_i^2 \leq 1; , i = 1, 2, \ldots.., r, \quad (3.15)$$

$$\lambda > 0, \sum_{i=1}^{r} \lambda_i = 1. \quad (3.16)$$

where

$$H_i(x, y, w, p) = \left\{ \begin{array}{c} f_i(x^1, y^1) + g_i(x^2, y^2) + ((x^1)^T E_i^1 x^1)^{\frac{1}{2}} \\ +((x^2)^T E_i^2 x^2)^{\frac{1}{2}} - (y^1)^T C_i^1 w_i^1 \\ -(y^2)^T C_i^2 w_i^2 - \frac{1}{2}(p_i^1)^T \nabla_{y^1 y^1} f(x^1, y^1) p_i^1 \\ -\frac{1}{2}(p_i^2)^T \nabla_{y^2 y^2} g(x^2, y^2) p_i^2 \end{array} \right\},$$

$$G_i(u, v, a, q) = \left\{ \begin{array}{c} f_i(u^1, v^1) + g_i(u^2, v^2) + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} \\ +((v^2)^T C_i^2 v^2)^{\frac{1}{2}} - (u^1)^T E_i^1 a_i^1 \\ -(u^2)^T E_i^2 a_i^2 - \frac{1}{2}(q_i^1)^T \nabla_{u^1 u^1} f(u^1, v^1) q_i^1 \\ -\frac{1}{2}(q_i^2)^T \nabla_{u^2 u^2} g(u^2, u^2) q_i^2 \end{array} \right\},$$

for $i = 1, 2, \ldots, r$.

Remark 3.1

Since the objective function of (MSP) and (MSD) contain the quadratic term like ($x^T Ax$) these problems are nondifferentiable multiobjective programming problems.

Theorem 3.1**(Weak duality)** *Let* $(x^1, x^2, y^1, y^2, \lambda, w^1, w^2, p^1, p^2)$ *be feasible solution of (SMSP) and* $(u^1, u^2, v^1, v^2, \lambda, a^1, a^2, z^1, z^2)$ *be feasible solution (MSD) and*

1. $\sum_{i=1}^{r} \lambda_i [f_i(*, v^1) + (*)^T E_i^1 a_i^1]$ *is second order* (ϕ_0^1, ρ) *-univex at* u^1 *for fixed* v^1.
2. $\sum_{i=1}^{r} \lambda_i [f_i(x^1, *) - (*)^T C_i^1 w_i^1]$ *is second order* (ϕ_1^1, ρ) *-unicave at* y^1 *for fixed* x^1.
3. $\sum_{i=1}^{r} \lambda_i [g_i(*, v^2) + (*)^T E_i^2 a_i^2]$ *is second order* (ϕ_o^2, ρ) *-pseudo univex at* u^2 *for fixed* v^2
4. $\sum_{i=1}^{r} \lambda_i [g_i(x^2, *) - (*)^T C_i^2 w_i^2]$ *is second order* (ϕ_1^2, ρ) *-pseudo unicave at* y^2 *for fixed* x^2.
5. $\phi_0^1(x^1, u^1; (\xi^1, \rho)) + (u^1)^T \xi^1 \geq 0;$ *where*
$$\xi^1 = \sum_{i=1}^{r} \lambda_i [\nabla_{u^1} f(u^1, v^1) + E_i^1 a_i^1 + \nabla_{u^1 u^1} f(u^1, v^1) q_i^1].$$
6. $\phi_0^2(x^2, u^2; (\xi^2, \rho)) + (u^2)^T \xi^2 \geq 0;$ *where*
$$\xi^2 = \sum_{i=1}^{r} \lambda_i [\nabla_{u^2} g(u^2, v^2) + E_i^2 a_i^2 + \nabla_{u^2 u^2} g(u^2, v^2) q_i^2].$$
7. $\phi_1^1(v^1, y^1; (\varsigma^1, \rho)) + (y^1)^T \varsigma^1 \leq 0;$ *where*
$$\varsigma^1 = \sum_{i=1}^{r} \lambda_i [\nabla_{y^1} f(x^1, y^1) - C_i^1 w_i^1 + \nabla_{y^1 y^1} f(x^1, y^1) p_i^1].$$
8. $\phi_1^2(v^2, y^2; (\varsigma^2, \rho)) + (y^2)^T \varsigma^2 \leq 0,$ *where*
$$\varsigma^2 = \sum_{i=1}^{r} \lambda_i [\nabla_{y^2} g(x^2, y^2) - C_i^2 w_i^2 + \nabla_{y^2 y^2} g(x^2, y^2) p_i^2].$$

Then $Inf(SMSP) \geq Sup(SMSD)$.

Proof: Since $\sum_{i=1}^{r} \lambda_i [f_i(*, v^1) + (*)^T E_i^1 a_i^1]$ is second order (ϕ_0^1, ρ) -univex at u^1 for fixed v^1 and $\lambda > 0$, with respect to $b_0^1 : R^{|J_1|} \times R^{|J_1|} \to R_+$, $\phi_0^1 : R^{|J_1|} \times R^{|J_1|} \times R^{|J_1|+1} \to R$, we have

$$b_0^1(x^1,u^1)\psi_0^1\left\{\sum_{i=1}^r \lambda_i \left\{\begin{array}{l}[f_i(x^1,v^1)+(x^1)^T E_i^1 a_i^1]\\ -[f_i(u^1,v^1)+(u^1)^T E_i^1 a_i^1]\\ +\frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f(u^1,v^1)q_i^1\end{array}\right\}\right\}$$

$$\geq \phi_0^1(x^1,u^1;\sum_{i=1}^r(\lambda_i[\nabla_{u^1} f_i(u^1,v^1)+E_i^1 a_i^1+\nabla_{u^1u^1} f_i(u^1,v^1)q_i^1],\rho)). \quad (3.17)$$

From the hypothesis (5), we get

$$\phi_0^1(x^1,u^1;(\xi^1,\rho))+(u^1)^T\xi^1 \geq 0$$

$$\Rightarrow \phi_0^1(x^1,u^1;(\xi^1,\rho)) \geq -(u^1)^T\xi^1.$$

Using (3.11) in the above inequality, we get $\phi_0^1(x^1,u^1;(\xi^1,\rho)) \geq -(u^1)^T\xi^1 \geq 0$ and with the property of b_0^1 and ψ_0^1, (3.17) becomes

$$\sum_{i=1}^r \lambda_i[f_i(x^1,v^1)+(x^1)^T E_i^1 a_i^1]$$

$$\geq \sum_{i=1}^r \lambda_i[f_i(u^1,v^1)+(u^1)^T E_i^1 a_i^1 - \frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f_i(u^1,v^1)q_i^1]. \quad (3.18)$$

Now $\sum_{i=1}^r \lambda_i[f_i(x^1,*)-(*)^T C_i^1 w_i^1]$ is second order (ϕ_1^1,ρ)-unicave at y^1 for fixed x^1, for $\lambda > 0$ with respect to $b_1^1 : R^{|K_1|}\times R^{|K_1|} \to R_+, \psi_1^1 : R \to R$, $\phi_1^1 : R^{|K_1|}\times R^{|K_1|}\times R^{|K_1|+1} \to R$, we have

$$b_1^1(v^1,y^1)\psi_1^1\left\{\sum_{i=1}^r \lambda_i \left\{\begin{array}{l}[f_i(x^1,v^1)-(v^1)^T C_i^1 w_i^1]\\ -[f_i(x^1,y^1)-(y^1)^T C_i^1 w_i^1]\\ +\frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f(x^1,y^1)p_i^1\end{array}\right\}\right\}$$

$$\leq \phi_1^1(v^1,y^1;(\sum_{i=1}^r \lambda_i[\nabla_{y^1} f_i(x^1,y^1)-C_i^1 w_i^1+\nabla_{y^1y^1} f_i(x^1,y^1)p_i^1],\rho)). \quad (3.19)$$

Hypothesis (7) in light of (3.3) implies

$$\phi_1^1(v^1,y^1;(\sum_{i=1}^r \lambda_i[\nabla_{y^1} f_i(x^1,y^1)-C_i^1 w_i^1+\nabla_{y^1y^1} f_i(x^1,y^1)p_i^1],\rho)) \leq 0. \quad (3.20)$$

So from (3.19), (3.20) and with the property of b_1^1 and ψ_1^1, we get

$$\sum_{i=1}^r \lambda_i[f_i(x^1,v^1)-(v^1)^T C_i^1 w_i^1]$$

$$\leq \sum_{i=1}^r \left\{[f_i(x^1,y^1)-(y^1)^T C_i^1 w_i^1]+\frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f(x^1,y^1)p_i^1\right\} \quad (3.21)$$

Subtracting (3.21) from (3.18), we get

$$\sum_{i=1}^r \lambda_i[(x^1)^T E_i^1 a_i^1+(v^1)^T C_i^1 w_i^1] \geq$$

$$\sum_{i=1}^r \lambda_i \left\{\begin{array}{l}[f_i(u^1,v^1)+(u^1)^T E_i^1 a_i^1 - \frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f_i(u^1,v^1)q_i^1]\\ -[f_i(x^1,y^1)+(y^1)^T C_i^1 w_i^1 - \frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f_i(x^1,y^1)p_i^1]\end{array}\right\} \quad (3.22)$$

Now from (6), we get

$\phi_0^2(x^2,u^2;(\xi^2,\rho))+(u^2)^T\xi^2 \geq 0$, which implies in light of (3.12) as

$$\phi_0^2(x^2,u^2;(\xi^2,\rho)) \geq -(u^2)^T\xi^2 \geq 0. \quad (3.23)$$

Since $\sum_{i=1}^r \lambda_i[g_i(*,v^2)+(*)^T E_i^2 a_i^2]$ is second order (ϕ_o^2,ρ)-pseudo univex at u^2 for fixed v^2 and using (3.23) with the properties of b_0^2 and ψ_0^2, we get $\sum_{i=1}^r \lambda_i[g_i(x^2,v^2)+(x^2)^T E_i^2 a_i^2]$

$$\geq \sum_{i=1}^r \lambda_i[g_i(u^2,v^2)+(u^2)^T E_i^2 a_i^2 - \tfrac{1}{2}(q_i^1)^T \nabla_{u^2u^2} g_i(u^2,v^2)q_i^2] \quad (3.24)$$

Similarly from the hypothesis (6), (8) and (3.12) with the property of b_1^2, ψ_1^2 we get

$$\sum_{i=1}^r \lambda_i[g_i(x^2,v^2)-(v^2)^T C_i^2 w_i^2]$$

$$\leq \sum_{i=1}^r \lambda_i[g_i(x^2,y^2)-(y^2)^T C_i^2 w_i^2 - \tfrac{1}{2}(p_i^1)^T \nabla_{y^2y^2} g_i(x^2,y^2)p_i^2] \quad (3.25)$$

Subtracting (3.25) from (3.24), we get

$$\sum_{i=1}^r \lambda_i[x^{2T}E_i^2 a_i^2 + v^{2T}C_i^2 w_i^2]$$

$$\geq \sum_{i=1}^r \lambda_i \left[\begin{array}{l} g_i(u^2,v^2)+u^{2T}E_i^2 a_i^2 - \frac{1}{2}q_i^{1T}\nabla_{u^2u^2} g_i(u^2,v^2)q_i^2 + \\ g_i(x^2,y^2)+y^{2T}C_i^2 w_i^2 - \frac{1}{2}p_i^{1T}\nabla_{y^2y^2} g_i(x^2,y^2)p_i^2\end{array}\right] \quad (3.26)$$

Adding (3.22) and (3.26), we obtain

$$\sum_{i=1}^r \left[(x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1 + (x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2\right]$$

$$\geq \sum_{i=1}^{r} \lambda_i \begin{Bmatrix} [f_i(u^1,v^1) + (u^1)^T E_i^1 a_i^1 - \frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f_i(u^1,v^1) q_i^1] \\ -[f_i(x^1,y^1) + (y^1)^T C_i^1 w_i^1 - \frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1] \\ +[g_i(u^2,v^2) + (u^2)^T E_i^2 a_i^2 - \frac{1}{2}(q_i^2)^T \nabla_{u^2u^2} g_i(u^2,v^2) q_i^2] \\ -[g_i(x^2,y^2) + (y^2)^T C_i^2 w_i^2 - \frac{1}{2}(p_i^2)^T \nabla_{y^2y^2} g_i(x^2,y^2) p_i^2] \end{Bmatrix}$$

$$\Rightarrow \sum_{i}^{r} \lambda_i \begin{bmatrix} (x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1 \\ +(x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2 \end{bmatrix}$$

$$\geq \sum_{i=1}^{r} \lambda_i \begin{Bmatrix} [f_i(u^1,v^1) + (u^1)^T E_i^1 a_i^1 - \frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f_i(u^1,v^1) q_i^1] \\ -[f_i(x^1,y^1) + (y^1)^T C_i^1 w_i^1 - \frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1] \end{Bmatrix}$$

$$+\sum_{i=1}^{r} \lambda_i \begin{Bmatrix} [g_i(u^2,v^2) + (u^2)^T E_i^2 a_i^2 - \frac{1}{2}(q_i^2)^T \nabla_{u^2u^2} g_i(u^2,v^2) q_i^2] \\ -[g_i(x^2,y^2) + (y^2)^T C_i^2 w_i^2 - \frac{1}{2}(p_i^2)^T \nabla_{y^2y^2} g_i(x^2,y^2) p_i^2] \end{Bmatrix}. \qquad (3.27)$$

From Schwartz inequality, (3.6), (3.7), (3.14) and (3.15), we have

$$(x^1)^T E_i^1 a_i^1 + (v^1)^T C_i^1 w_i^1 + (x^2)^T E_i^2 a_i^2 + (v^2)^T C_i^2 w_i^2$$

$$\leq ((x^1)^T E_i^1 x^1)^{\frac{1}{2}} ((a_i^1)^T E_i^1 a_i^1)^{\frac{1}{2}} + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} ((w_i^1)^T) C_i^1 w_i^1)^{\frac{1}{2}}$$

$$+((x^2)^T E_i^2 x^2)^{\frac{1}{2}} ((a_i^2)^T E_i^2 a_i^2)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}} ((w_i^2)^T C_i^2 w_i^2$$

$$\leq ((x^1)^T E_i^1 x^1)^{\frac{1}{2}} + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} + ((x^2)^T E_i^2 x^2)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}} \qquad (3.28)$$

Using (3.28) in (3.27), we get

$$\sum_{i=1}^{r} \lambda_i [((x^1)^T E_i^1 x^1)^{\frac{1}{2}} + ((v^1)^T C_i^1 v^1)^{\frac{1}{2}} + ((x^2)^T E_i^2 x^2)^{\frac{1}{2}} + ((v^2)^T C_i^2 v^2)^{\frac{1}{2}}]$$

$$\geq \sum_{i=1}^{r} \lambda_i \begin{Bmatrix} [f_i(u^1,v^1) + (u^1)^T E_i^1 a_i^1 - \frac{1}{2}(q_i^1)^T \nabla_{u^1u^1} f_i(u^1,v^1) q_i^1] \\ -[f_i(x^1,y^1) + (y^1)^T C_i^1 w_i^1 - \frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1] \\ +[g_i(u^2,v^2) + (u^2)^T E_i^2 a_i^2 - \frac{1}{2}(q_i^2)^T \nabla_{u^2u^2} g_i(u^2,v^2) q_i^2] \\ -[g_i(x^2,y^2) + (y^2)^T C_i^2 w_i^2 - \frac{1}{2}(p_i^2)^T \nabla_{y^2y^2} g_i(x^2,y^2) p_i^2] \end{Bmatrix}$$

$$\Rightarrow \sum_{i=1}^{r} \lambda_i \begin{Bmatrix} f_i(x^1,y^1) + g_i(x^2,y^2) + ((x^1)^T E_i^1 x^1)^{\frac{1}{2}} \\ +((x^2)^T E_i^2 x^2)^{\frac{1}{2}} - (y^1)^T C_i^1 w_i^1 - (y^2)^T C_i^2 w_i^2 \\ -\frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 - \frac{1}{2}(p_i^2)^T \nabla_{y^2y^2} g_i(x^2,y^2) p_i^2 \end{Bmatrix}$$

$$\geq \sum_{i=1}^{r} \lambda_i \begin{Bmatrix} f_i(u^1,v^1) + g_i(u^2,v^2) + ((u^1)^T E_i^1 u^1)^{\frac{1}{2}} \\ +((v^2)^T E_i^2 v^2)^{\frac{1}{2}} - (y^1)^T C_i^1 w_i^1 - (y^2)^T C_i^2 w_i^2 \\ -\frac{1}{2}(p_i^1)^T \nabla_{y^1y^1} f_i(x^1,y^1) p_i^1 - \frac{1}{2}(p_i^2)^T \nabla_{y^2y^2} g_i(x^2,y^2) p_i^2 \end{Bmatrix}$$

$$\Rightarrow Inf(MSP) \geq Sup(MSD).$$ □

Theorem 3.2 (Strong Duality) *Suppose* $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ *and* $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ *be thrice differentiable function and let* $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{w}_i^1, \hat{w}_i^2, \hat{p}_i^1, \hat{p}_i^2)$ *be a weak efficient solution for (SMSP). Let* $\bar{\lambda} = \lambda$ *be fixed in (MSD). Assume that*

(i) the Hessian matrices $\nabla_{y^1y^1} f_i(\hat{x}^1, \hat{y}^1)$ *and* $\nabla_{y^2y^2} g_i(\hat{x}^2, \hat{y}^2)$ *are nonsingular for all* $i = 1, 2, ..., r$;

(ii) the matrix $\sum_{i=1}^{r} \lambda_i \nabla_{y^1}(\nabla_{y^1y^1} f_i \hat{p}_i^1)$ *and* $\sum_{i=1}^{r} \lambda_i \nabla_{y^2}(\nabla_{y^2y^2} g_i \hat{p}_i^2)$ *are positive definite or negative definite and*

(iii) the set

$\{\nabla_{y^1} f_1 - C_1^1 \hat{w}_1^1 + \nabla_{y^1y^1} f_1 \hat{p}_1^1, ..., \nabla_{y^1} f_r - C_r^1 \hat{w}_r^1 + \nabla_{y^1y^1} f_r \hat{p}_r^1\}$

and

$\{\nabla_{y^2} g_1 - C_1^2 \hat{w}_1^2 + \nabla_{y^2y^2} g_1 \hat{p}_1^2, ..., \nabla_{y^2} g_r - C_r^2 \hat{w}_r^2 + \nabla_{y^2y^2} g_r \hat{p}_r^2\}$

are linearly independent.

Then there exist $a_i^1 \in R^{|J_1|}, a_i^2 \in R^{|J_2|}$ *such that* $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{a}_i^1, \hat{a}_i^2, \hat{q}_i^1 = 0, \hat{q}_i^2 = 0)$ *is feasible solution of (SMSD) and the two objective values are equal. Also if the hypotheses of theorem 3.1 are satisfied for all feasible solution of (SMSP) and (SMSD), then* $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{a}_i^1, \hat{a}_i^2, \hat{q}_i^1 = 0, \hat{q}_i^2 = 0)$ *is efficient solution of (SMSD).*

Proof. Since $(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{w}_i^1, \hat{w}_i^2, \hat{p}_i^1, \hat{p}_i^2)$ is a weak efficient solution of (SMSP) by Fritz-John condition [12] there exist $\alpha \in R^r, \tau \in R^r, \delta \in R^r$, $\gamma \in R, \beta^1 \in R^{|K_1|}, \beta \in R^{|K_2|}, \xi^1 \in R^{|J_1|}, \xi^2 \in R^{|J_2|}$ such that

$$\sum_{i=1}^{r} \alpha_i [\nabla_{x^1} f_i + E_i^1 \hat{a}_i^1 - \tfrac{1}{2} \nabla_{x^1}(\nabla_{y^1y^1} f_i \hat{p}_i^1)]$$

$$+\sum_{i=1}^{r} \lambda_i [\nabla_{y^1x^1} f_i + \nabla_{x^1}(\nabla_{y^1y^1} f_i \hat{p}_i^1)](\beta^1 - \gamma \hat{y}^1) - \xi^1 = 0, \qquad (3.29)$$

$$\sum_{i=1}^{r} \alpha_i [\nabla_{x^2} g_i + E_i^2 \hat{a}_i^2 - \tfrac{1}{2} \nabla_{x^2}(\nabla_{y^2y^2} g_i \hat{p}_i^2)]$$

$$+\sum_{i=1}^{r} \lambda_i [\nabla_{y^2x^2} g_i + \nabla_{x^2}(\nabla_{y^2y^2} g_i \hat{p}_i^2)](\beta^2 - \gamma \hat{y}^2) - \xi^2 = 0, \qquad (3.30)$$

$$\sum_{i=1}^{r}(\alpha_i-\gamma\hat{\lambda}_i)[\nabla_{y^1}f_i-C_i^1\hat{w}_i^1]+\sum_{i=1}^{r}\hat{\lambda}_i(\nabla_{y^1y^1}f)(\beta^1-\gamma\hat{y}^1-\gamma\hat{p}_i^1)$$
$$+\sum_{i=1}^{r}[\nabla_{y^1}(\nabla_{y^1y^1}f_i\hat{p}_i^1)][(\beta^1-\gamma\hat{y}^1)\hat{\lambda}_i-\tfrac{1}{2}\alpha_i\hat{p}_i^1]=0, \quad (3.31)$$

$$\sum_{i=1}^{r}(\alpha_i-\gamma\hat{\lambda}_i)[\nabla_{y^2}g_i-C_i^2\hat{w}_i^2]$$
$$+\sum_{i=1}^{r}\hat{\lambda}_i(\nabla_{y^2y^2}g_i)(\beta^2-\gamma\hat{y}^2-\gamma\hat{p}_i^2)$$
$$+\sum_{i=1}^{r}[\nabla_{y^2}(\nabla_{y^2y^2}g_i\hat{p}_i^2)][(\beta^2-\gamma\hat{y}^2)\hat{\lambda}_i-\tfrac{1}{2}\alpha_i\hat{p}_i^2]=0, \quad (3.32)$$

$$(\beta^1-\gamma\hat{y}^1)^T[\nabla_{y^1}f_i-C_i^1\hat{w}_i^1+\nabla_{y^1y^1}f_i\hat{p}_i^1]-\delta_i^1=0,$$
$$i=1,2,\ldots,r, \quad (3.33)$$

$$(\beta^2-\gamma\hat{y}^2)^T[\nabla_{y^2}g_i-C_i^2\hat{w}_i^2+\nabla_{y^2y^2}g_i\hat{p}_i^2]-\delta_i^2=0,$$
$$i=1,2,\ldots,r, \quad (3.34)$$

$$[(\beta^1-\gamma\hat{y}^1)\hat{\lambda}_i-\alpha_i\hat{p}_i^1]^T\nabla_{y^1y^1}f_i=0, i=1,2,\ldots r; \quad (3.35)$$

$$[(\beta^2-\gamma\hat{y}^2)\hat{\lambda}_i-\alpha_i\hat{p}_i^2]^T\nabla_{y^2y^2}g_i=0, i=1,2,\ldots,r; \quad (3.36)$$

$$\alpha_iC_i^1\hat{y}^1+(\beta^1-\gamma\hat{y}^1)^T\hat{\lambda}_iC_i^1=2\tau_iC_i^1\hat{w}_i^1,\ i=1,2,\ldots,r, \quad (3.37)$$

$$\alpha_iC_i^2\hat{y}^2+(\beta^2-\gamma\hat{y}^2)^T\hat{\lambda}_iC_i^2=2\tau_iC_i^2\hat{w}_i^2,\ i=1,2,\ldots,r, \quad (3.38)$$

$$(\hat{x}^1)^TE_i^1\hat{a}_i^1=((\hat{x}^1)^TE_i^1\hat{x}^1)^{\frac{1}{2}},\ i=1,2,\ldots,r, \quad (3.39)$$

$$(\hat{x}^2)^TE_i^2\hat{a}_i^2=((\hat{x}^2)^TE_i^2\hat{x}^2)^{\frac{1}{2}},\ i=1,2,\ldots,r, \quad (3.40)$$

$$(\beta^1)^T\sum_{i=1}^{r}\lambda_i[\nabla_{y^1}f_i-C_i^1\hat{w}_i^1+\nabla_{y^1y^1}f_i\hat{p}_i^1]=0,\ i=1,2,\ldots,r, \quad (3.41)$$

$$(\beta^2)^T\sum_{i=1}^{r}\lambda_i[\nabla_{y^2}g_i-C_i^2w_i^2+\nabla_{y^2y^2}g_i\hat{p}_i^2]=0,\ i=1,2,\ldots,r, \quad (3.42)$$

$$(\gamma\hat{y}^1)^T\sum_{i=1}^{r}\lambda_i[\nabla_{y^1}f_i-C_i^1\hat{w}_i^1+\nabla_{y^1y^1}f_i\hat{p}_i^1]=0, \quad (3.43)$$

$$(\gamma\hat{y}^2)^T\sum_{i=1}^{r}\lambda_i[\nabla_{y^2}g_i-C_i^2\hat{w}_i^2+\nabla_{y^2y^2}g_i\hat{p}_i^2]=0, \quad (3.44)$$

$$\tau_i[(\hat{w}_i^1)^TC_i^1w_i^1-1]=0, i=1,2,\ldots,r, \quad (3.45)$$

$$\tau_i[(\hat{w}_i^2)^TC_i^2w_i^2-1]=0, i=1,2,\ldots r, \quad (3.46)$$

$$\delta^{1T}\hat{\lambda}=0, \quad (3.47)$$

$$\delta^{2T}\hat{\lambda}=0, \quad (3.48)$$

$$\hat{x}^1\xi^1=0, \quad (3.49)$$

$$\hat{x}^2\xi^2=0, \quad (3.50)$$

$$\hat{a}_i^{1T}E_i^1\hat{a}_i^1\le 1,\ \ i=1,2,\ldots,r, \quad (3.51)$$

$$\hat{a}_i^{2T}E_i^2\hat{a}_i^2\le 1,\ i=1,2,\ldots,r, \quad (3.52)$$

$$(\alpha,\beta,\gamma,\tau,\delta,\xi)\ge 0, \quad (3.53)$$

$$(\alpha,\beta,\gamma,\tau,\delta,\xi)\ne 0. \quad (3.54)$$

Since $\hat{\lambda}>0$, inequalities (3.47) and (3.48) implies that $\delta^1=0,\delta^2=0$.
Consequently (3.34) and (3.35) implies that

$$(\beta^1-\gamma\hat{y}^1)^T[\nabla_{y^1}f_i-C_i^1\hat{w}_i^1+\nabla_{y^1y^1}f_i\hat{p}_i^1]=0. \quad (3.55)$$

and $(\beta^2-\gamma\hat{y}^2)^T[\nabla_{y^2}g_i-C_i^2\hat{w}_i^2+\nabla_{y^2y^2}g_i\hat{p}_i^2]=0.$ (3.56)

Since $\nabla_{y^1y^1}f_i$ and $\nabla_{y^2y^2}g_i$ are nonsingular matrix for $i=1,2,.....r$. (3.35) and (3.36) implies

$$(\beta^1-\gamma\hat{y}^1)\hat{\lambda}_i=\alpha_i\hat{p}_i^1 \quad (3.57)$$

and $(\beta^2-\gamma\hat{y}^2)\hat{\lambda}_i=\alpha_i\hat{p}_i^2.$ (3.58)

Using (3.57) in (3.31) we get

$$\sum_{i=1}^{r}(\alpha_i-\gamma\hat{\lambda}_i)[\nabla_{y^1}f_i-C_i^1\hat{w}_i^1+(\nabla_{y^1y^1}f_i\hat{p}_i^1)]$$
$$+\tfrac{1}{2}\sum_{i=1}^{r}\hat{\lambda}_i[\nabla_{y^1}(\nabla_{y^1y^1}f_i\hat{p}_i^1)](\beta^1-\gamma\hat{y}^1)=0. \quad (3.59)$$

Similarly by using (3.58) in (3.32), we get

$$\sum_{i=1}^{r}(\alpha_i-\gamma\hat{\lambda}_i)[\nabla_{y^2}g_i-C_i^1\hat{w}_i^1+\nabla_{y^2y^2}g_i\hat{p}_i^2)]$$
$$+\tfrac{1}{2}\sum_{i=1}^{r}\hat{\lambda}_i[\nabla_{y^2}(\nabla_{y^2y^2}g_i\hat{p}_i^2)](\beta^2-\gamma\hat{y}^2)=0. \quad (3.60)$$

Multiplying (3.59) by $(\beta^1-\gamma\hat{y}^1)^T$ and using (3.55) the result reduces to

$$(\beta^1-\gamma\hat{y}^1)^T\sum_{i=1}^{r}\lambda_i\nabla_{y^1}(\nabla_{y^1y^1}f_i\hat{p}_i^1)(\beta^1-\gamma\hat{y}^1)=0. \quad (3.61)$$

And multiplying (3.60) by $(\beta^2 - \gamma\hat{y}^2)^T$ and using (3.56), we get

$$(\beta^2 - \gamma\hat{y}^2)^T \sum_{i=1}^{r} \lambda_i \nabla_{y^2}(\nabla_{y^2y^2} g_i \hat{p}_i^2)(\beta^2 - \gamma\hat{y}^2) = 0. \quad (3.62)$$

Using the hypothesis (2) in (3.61) and (3.62), we get $\beta^1 = \gamma\hat{y}^1$ (3.63)

and $\beta^2 = \gamma\hat{y}^2$. (3.64)

Therefore (3.59) and (3.60) reduced to

$$\sum_{i=1}^{r} (\alpha_i - \gamma\hat{\lambda}_i)[\nabla_{y^1} f_i - C_i^1 \hat{w}_i^1 + (\nabla_{y^1y^1} f_i \hat{p}_i^1)] = 0. \quad (3.65)$$

$$\sum_{i=1}^{r} (\alpha_i - \gamma\hat{\lambda}_i)[\nabla_{y^2} g_i - C_i^2 \hat{w}_i^2 + \nabla_{y^2y^2} g_i \hat{p}_i^2)] = 0. \quad (3.66)$$

Using hypothesis (iii) in (3.65) and (3.66) we get

$\alpha_i = \gamma\hat{\lambda}_i,\ i = 1,2,.....r.$ (3.67)

If $\gamma = 0$, then $\alpha_i = 0;\ i = 1,2,.....r.$ and (3.63) and (3.64) implies $\beta^1 = \beta^2 = 0.$

Therefore (3.29) and (3.30) implies $\xi^1 = \xi^2 = 0$ and (3.37), (3.38) implies $\tau_i = 0,\ i = 1,2,.....r.$

Thus $(\alpha, \beta, \gamma, \tau, \delta, \xi) = 0$.

This is a contradiction to (3.54).

Hence $\gamma > 0.$ (3.68)

Since $\lambda_i > 0,\ i = 1,2,....r$, (3.67) implies $\alpha_i > 0,$

$i = 1, 2, ...r.$ (3.69)

Using (3.63) in (3.57) and (3.64) in (3.58), we get

$\alpha_i \hat{p}_i^j = 0$, for $i = 1,2,....r$ and $j = 1,2.$ (3.70)

Using (3.69) in (3.70), we obtain

$\hat{p}_i^j = 0$, for $i = 1,2,....r$ and $j = 1,2.$ (3.71)

Again using (3.63) and (3.71) in (3.29) and (3.64) and (3.71) in (3.30), it gives

$\sum_{i=1}^{r} \alpha_i[\nabla_{x^1} f_i + E_i^1 \hat{a}_i^1] = \xi^1$ and

$\sum_{i=1}^{r} \alpha_i[\nabla_{x^2} g_i + E_i^2 \hat{a}_i^2] = \xi^2$, which by (3.67) gives

$$\sum_{i=1}^{r} \hat{\lambda}_i[\nabla_{x^1} f_i + E_i^1 \hat{a}_i^1] = \frac{\xi^1}{\gamma} \geq 0. \quad (3.72)$$

and $\sum_{i=1}^{r} \hat{\lambda}_i[\nabla_{x^2} g_i + E_i^2 \hat{a}_i^2] = \dfrac{\xi^2}{\gamma} \geq 0.$ (3.73)

Now $(\hat{x}^1)^T \sum_{i=1}^{r} \lambda_i[\nabla_{x^1} f_i + E_i^1 \hat{a}_i^1] = \dfrac{\hat{x}^{1T}\xi^1}{\gamma} = 0,$ (3.74)

(by using (3.49))

and $(\hat{x}^2)^T \sum_{i=1}^{r} \lambda_i[\nabla_{x^2} g_i + E_i^2 \hat{a}_i^2] = \dfrac{\hat{x}^{2T}\xi^2}{\gamma} = 0.$ (3.75)

(by using (3.50))

Also from (3.63), (3.64) and (3.53), we have

$\hat{y}^1 = \dfrac{\beta^1}{\gamma} \geq 0,\ \hat{y}^2 = \dfrac{\beta^2}{\gamma} \geq 0.$ (3.76)

Hence from (3.51), (3.52) and from (3.71) to (3.76), we get

$(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{\lambda}, \hat{a}_i^1, \hat{a}_i^2, \hat{q}_i^1 = 0, \hat{q}_i^2 = 0)$ is a feasible solution of (SMSD).

Let $\dfrac{2\tau_i}{\alpha_i} = t$, then $t \geq 0$. From (3.37), (3.38), (3.63) and (3.64), we get

$C_i^1 \hat{y}^1 = t C_i^1 \hat{w}_i^1, C_i^2 \hat{y}^2 = t C_i^2 \hat{w}_i^2.$ (3.77)

This is a condition of Schwartz Inequality.

Therefore $(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}} (\hat{w}_i^{1T} C_i^1 \hat{w}_i^1)^{\frac{1}{2}}$ (3.78)

and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}} (\hat{w}_i^{2T} C_i^2 \hat{w}_i^2)^{\frac{1}{2}}.$ (3.79)

In case $\tau_i > 0$, from (3.45) and (3.46), we get $\hat{w}_i^{1T} C_i^1 \hat{w}_i^1 = 1, \hat{w}_i^{2T} C_i^2 \hat{w}_i^2 = 1$ and

so we get $(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}}$ and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}}$.

In case $\tau_i = 0$ we get $\frac{2\tau_i}{\alpha_i} = t = 0.$ So (3.77) implies $C_i^1 \hat{y}^1 = C_i^2 \hat{y}^2 = 0.$

Hence $(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}} = 0$ and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}} = 0.$

Thus in either case

$$(\hat{y}^1)^T C_i^1 \hat{w}_i^1 = ((\hat{y}^1)^T C_i^1 \hat{y}^1)^{\frac{1}{2}} \quad (3.80)$$

and $(\hat{y}^2)^T C_i^2 \hat{w}_i^2 = ((\hat{y}^2)^T C_i^2 \hat{y}^2)^{\frac{1}{2}}$. (3.81)

Therefore using (3.39), (3.40), (3.78) and (3.79), we get

$$f_i(\hat{x}^1,\hat{y}^1)+g_i(\hat{x}^2,\hat{y}^2)+(\hat{x}^{1T}E_i^1\hat{x}^1)^{\frac{1}{2}}+(\hat{x}^{2T}E_i^2\hat{x}^2)^{\frac{1}{2}}$$

$$-\hat{y}^{1T}C_i^1\hat{w}_i^1-\hat{y}^{2T}C_i^2\hat{w}_i^2$$

$$=f_i(\hat{x}^1,\hat{y}^1)+g_i(\hat{x}^2,\hat{y}^2)+(\hat{x}^{1T}E_i^1\hat{a}_i^1)+(\hat{x}^{2T}E_i^2\hat{a}_i^2)$$

$$-(\hat{y}^{1T}C_i^1\hat{y}^1)^{\frac{1}{2}}-(\hat{y}^{2T}C_i^2\hat{y}^2)^{\frac{1}{2}}. \quad (3.82)$$

for each $i=1,2,3,\dots,r$;

or $H_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2,\hat{p}^1=0,\hat{p}^2=0)=$

$$G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\hat{q}^1=0,\hat{q}^2=0)$$

for each $i=1,2,3,\dots\dots r$,

or $H(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2,0,0)=G(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)$. (3.83)

So the objective values of both problems are equal.

Now we claim that $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\lambda,\hat{q}^1=0,\hat{q}^2=0)$ is properly efficient solution for (SMSD).

First we have to show it is an efficient solution of (SMSD). If this would not be the case, then there would exist a feasible solution $(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,\hat{\lambda},\hat{q}^1=0,\hat{q}^2=0)$ of (SMSD) such that $G(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\hat{q}^1=0,\hat{q}^2=0)$

$$\le G(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,\hat{q}^1=0,\hat{q}^2=0).$$

This by (3.83) gives

$$H(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2,\hat{p}^1=0,\hat{p}^2=0)$$

$$\le G(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,\hat{q}^1=0,\hat{q}^2=0).$$

This is a contradiction to Theorem 3.1.

Hence $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\lambda,\hat{q}^1=0,\hat{q}^2=0)$ is an efficient solution of (SMSD).

Now we have to claim $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\lambda,\hat{q}^1=0,\hat{q}^2=0)$ is properly efficient for (SMSD).

For that rewriting the objective function of (SMSD) into minimization form we get

$$\min G_i^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\hat{q}^1=0,\hat{q}^2=0)$$

If $(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,\lambda,\hat{q}^1=0,\hat{q}^2=0)$ were not properly efficient for (SMSD), then for every scalar $M>0$, there exist a feasible solution

$$(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,\hat{\lambda},\hat{q}^1=0,\hat{q}^2=0)$$

of (SMSD) and an index i such that

$$\{G_i^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)$$

$$-G_i^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)\}$$

$$<M\begin{Bmatrix}G_j^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)\\-G_j^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)\end{Bmatrix} \quad (3.84)$$

and for all j satisfying

$$G_j^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{w}^1,\hat{w}^2,0,0)$$

$$<G_j^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0), \quad (3.85)$$

whenever $G_i^*(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)$

$$<G_i^*(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0) \quad (3.86)$$

This implies that

$$G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)-G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)$$

$$>M\begin{pmatrix}G_j(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)\\-G_j(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)\end{pmatrix}, \quad (3.87)$$

for all j satisfying

$$G_j(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)>G_j(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0), \quad (3.88)$$

whenever $G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)$

$$>G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0) \quad (3.89)$$

Since $M>0$ and using (3.88) in (3.87), we get

$$G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)$$

$$-G_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)>0. \quad (3.90)$$

Using (3.83) in (3.90), we obtain

$$G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)$$

$$-H_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0)>0$$

$$\Rightarrow G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)>$$

$$H_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0).$$

For each $\hat{\lambda}_i>0$, we have

$$\sum_{i=1}^{r}\hat{\lambda}_i\{G_i(\hat{u}^1,\hat{u}^2,\hat{v}^1,\hat{v}^2,\hat{a}^1,\hat{a}^2,0,0)\}>$$

$$\sum_{i=1}^{r}\hat{\lambda}_iH_i(\hat{x}^1,\hat{x}^2,\hat{y}^1,\hat{y}^2,\hat{a}^1,\hat{a}^2,0,0).$$

This again contradicts Theorem 3.1.

Hence
$(\hat{x}^1, \hat{x}^2, \hat{y}^1, \hat{y}^2, \hat{a}^1, \hat{a}^2, \lambda, \hat{q}^1 = 0, \hat{q}^2 = 0)$ is properly efficient solution of (SMSD).

Theorem 3.3 (Converse Duality)

Let $f_i : R^{|J_1|} \times R^{|K_1|} \to R$ *and* $g_i : R^{|J_2|} \times R^{|K_2|} \to R$ *be thrice differentiable functions and let* $(\hat{u}^1, \hat{u}^2, \hat{v}^1, \hat{v}^2, \hat{\lambda}, \hat{a}_i^1, \hat{a}_i^2, \hat{q}_i^1, \hat{q}_i^2)$ *be a weak efficient solution for (SMSD). Let* $\hat{\lambda} = \lambda$ *be fixed in (SMSD). Assume that (i) the Hessian matrices* $\nabla_{x^1x^1} f_i(\hat{x}^1, \hat{y}^1)$ *and* $\nabla_{x^2x^2} g_i(\hat{x}^2, \hat{y}^2)$ *are nonsingular for all* $i = 1,2,.....r$. *(ii) The matrix* $\sum_{i=1}^{r} \lambda_i \nabla_{x^1}(\nabla_{x^1x^1} f_i \hat{q}_i^1)$ *and* $\sum_{i=1}^{r} \lambda_i \nabla_{x^1}(\nabla_{x^1x^1} g_i \hat{q}_i^2)$ *are positive definite or negative definite and*

(iii) The set

$\{\nabla_{x^1} f_1 + E_1^1 \hat{a}_1^1 + \nabla_{x^1x^1} f_1 \hat{q}_1^1, ..., \nabla_{x^1} f_r + E_r^1 \hat{a}_r^1 + \nabla_{x^1x^1} f_r \hat{q}_r^1\}$ *and*

$\{\nabla_{x^2} g_1 + E_1^2 \hat{a}_1^2 + \nabla_{x^2x^2} g_1 \hat{q}_1^2, ..., \nabla_{x^2} g_r + E_r^2 \hat{a}_r^2 + \nabla_{x^2x^2} g_r \hat{q}_r^2\}$

are linearly independent.

Then there exist $w_i^1 \in R^{|K_1|}, w_i^2 \in R^{|K_2|}$ *such that* $(\hat{u}^1, \hat{u}^2, \hat{v}^1, \hat{v}^2, \hat{\lambda}, \hat{w}_i^1, \hat{w}_i^2, \hat{q}_i^1 = 0, \hat{q}_i^2 = 0)$ *is feasible solution of (SMSD) and the two objective values are equal. Also if the hypotheses of theorem 3.1 are satisfied for all feasible solution of (SMSP) and (SMSD), then* $(\hat{u}^1, \hat{u}^2, \hat{v}^1, \hat{v}^2, \hat{\lambda}, \hat{w}_i^1, \hat{w}_i^2, \hat{q}_i^1 = 0, \hat{q}_i^2 = 0)$ *is properly efficient solution of (SMSP).*

Proof: It follows on the lines of theorem 3.2.

4. Special Case

1. If $|J_2| = 0, |K_2| = 0$, then our problem reduces to a pair of (MP) and (MD) given by Thakur et al. [18].

2. If $\phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ with $\rho = 0$, $b = 1, \psi = I, C_i^j = \{E_i^j x^j; x^{jT} E_i^j x^j \le 1\}$, $D_i^j = \{C_i'^j y^j; y^{jT} C_i'^j y^j \le 1\}$, $(x^{jT} E_i^j x^j)^{\frac{1}{2}} = s(x^j \mid C_i^j)$ and $(y^{jT} C_i'^j y^j)^{\frac{1}{2}} = s(y^j \mid D_i^j)$, where E_i^j and $C_i'^j$ are positive semi definite, $C_i^j w_i^j = z_i^j, E_i^j a_i^j = w_i^j$, $i = 1,2,...,r;\ j = 1,2,$ then our problem (MSP) and (MSD) reduces to the pair of dual and dual results given by Li and Gao [11], Mishra et al. [14, 15].

3. If $\phi(x,u;(\nabla f(u),\rho)) = F(x,u;\nabla f(u))$ with $\rho = 0$, $b = 1, \psi = I, C_i^j = \{E_i^j x^j; x^{jT} E_i^j x^j \le 1\}$, $D_i^j = \{C_i'^j y^j; y^{jT} C_i'^j y^j \le 1\}$, $(x^{jT} E_i^j x^j)^{\frac{1}{2}} = s(x^j \mid C_i^j)$ and $(y^{jT} C_i'^j y^j)^{\frac{1}{2}} = s(y^j \mid D_i^j)$, where E_i^j and $C_i'^j$ are positive semi definite matrices, $C_i^j w_i^j = z_i^j, E_i^j a_i^j = w_i^j$, $i = 1,2,...,r;\ j = 1,2,$ $C_i^1 w_i^1 = \nabla_{y^1} f(x^1, y^1)$ $E_i^1 a_i^1 = \nabla_{x^1} f(u^1, v^1)$ and then our problem (MSP) and (MSD) reduce to the problem (MP) and (MD) given by Agarwal et al. [1].

4. If $b = 1, \psi = I, |J_2| = 0, |K_2| = 0, \rho = 0$, $\phi(x,u;(\nabla_u f(u) + \nabla_{uu} f(u)q, \rho)) = F(x,u;\nabla_u f(u) + \nabla_{uu} f(u)q)$ and, $C_i = E_i = 0, i = 1,2,...,r$, then the problem (SMSP) and (SMSD) reduces to a pair of problems (MP) and (MD) and the results studied by Suneja and Lalita [17].

5. If $b = 1, \psi = I, |J_2| = 0, |K_2| = 0, \rho = 0$, $\phi(x,u;(\nabla_u f(u) + \nabla_{uu} f(u)q, \rho)) = F(x,u;\nabla_u f(u) + \nabla_{uu} f(u)q)$, in (SMSP) and (SMSD), then we obtain a pair of nondifferentiable second order symmetric dual in multiobjective program considered by Ahmad and Husain [4].

5. Conclusion

In this article, a new pair of nondifferentiable multiobjective second order mixed symmetric dual programs is presented and duality relations between primal and dual problems are established. The results developed in this paper improve and generalize a number of existing results in the literature. The results discussed in this paper can be extended to higher order as well as to other generalized convexity assumptions.

These results can be extended to the case of continuous – time problems as well.

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References

[1] Agarwal, R.P., Ahmad, I., Gupta, S. K. and Kailey, N., Generalized second order mixed symmetric duality in non differentiable mathematical programming, *Abstract and Applied Analysis*, Hindawi Publishing Corporation, Article ID. 103587, doi: 10, 1155/2011/103587 (2011).

[2] Aghezzaf, B., Second order mixed type duality in multiobjective programming problems, *Journal of Mathematical Analysis and Applications*, 285, 97-106 (2003).

[3] Ahmad, I. and Husain, Z., Multiobjective mixed symmetric duality involving cones, *Computers & Mathematics with Applications*, 59 (1), 319-326 (2010).

[4] Ahmad, I. and Husain, Z., Nondifferentiable second order duality in multiobjective programming, *Applied Mathematics Letters*, 18, 721-728 (2005).

[5] Ahmad, I. and Sharma, S., Second order duality for non differentiable multiobjective programming, *Numerical Functional Analysis and Optimization*, 28 (9), 975-988 (2007).

[6] Bector, C. R. and Chandra, S., Second order symmetric and self dual programs, *Opsearch*, 23, 89-95 (1986).

[7] Bector, C. R., Chandra, S. and Abha, On mixed symmetric duality in Mathematical programming, *Journal of Mathematical Analysis and Applications*, 259, 346-356 (2001).

[8] Bector, C. R., Chandra, S. and Goyal, A., On mixed symmetric duality in multiobjective programming, *Opsearch*, 36, 399-407 (1999).

[9] Chandra, S., Husain, I. and Abha, On mixed symmetric duality in mathematical programming, *Opsearch*, 36 (2), 165-171 (1999).

[10] Kailey, N., Gupta, S. and Danger, D., Mixed second order multiobjective symmetric duality with cone constraints, *Nonlinear Analysis: Real World Applications*, 12, 3373-3383 (2011).

[11] Li, J. and Gao, Y., Nondifferentiable multiobjective mixed symmetric duality under generalized convexity, *Journal of Inequalities and Applications*, doi: 10.1186/1029-24x-2011-23 (2011).

[12] Mangasarian, O.L., 'Second order and higher order duality in nonlinear programming, *Journal of Mathematical Analysis and Applications*, 51, 607-620 (1975).

[13] Mishra, S.K., Second order mixed symmetric duality in nondifferentiable multiobjective mathematical program-ming, *Journal of Applied Analysis*, 13 (1), 117-132 (2007).

[14] Mishra, S. K., Wang, S. Y. and Lai, K. K., Mond-Weir type mixed symmetric first and second order duality in nondifferentiable mathematical program-ming, *Journal of Nonlinear and Convex Analysis*, 7 (3), 189-198 (2006).

[15] Mishra, S. K., Wang, S. Y., Lai, K. K. and Yang, F. M., Mixed symmetric duality in nondifferentiable mathematical programming, *European Journal of Operational Research*, 181, 1-9 (2007).

[16] Mond, B., Second order duality for nonlinear programs, *Opsearch*, 11, 90-99 (1974).

[17] Suneja, S. K., Lalita, C.S. and Khurana, S., Second order symmetric dual in multiobjective programming, *European Journal of Operational Research*, 144, 492-500 (2003).

[18] Thakur, G. K. and Priya, B., Second order duality for nondifferentiable multiobjective programming involving (Φ,ρ)-univexity, *Kathmandu University Journal of Science, Engineering and Technology*, 7 (1), 99-104 (2011).

[19] Xu, Z., Mixed type duality in multiobjective programming problems, *Journal of Mathematical Analysis and Applications*, 198, 621-635 (1996).

[20] Yang, X. M., Teo, K. L. and Yang, X. Q., Mixed symmetric duality in nondifferentiable Mathematical programming, *Indian Journal of Pure and Applied Mathematics*, 34 (5), 805-815 (2003).

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