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RESEARCH ARTICLE

Witte's conditions for uniqueness of solutions to a class of Fractal-Fractional ordinary differential equations

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ABSTRACT

In this paper, Witte's conditions for the uniqueness solution of nonlinear differential equations with integer and non-integer order derivatives are investigated. We present a detailed analysis of the uniqueness solutions of four classes of nonlinear differential equations with nonlocal operators. These classes include classical and fractional ordinary differential equations in fractal calculus. For each case, theorems and lemmas and their proofs are presented in detail.

1. Introduction

Nonlinear differential equations are powerful mathematical tools used to model real-world problems arising in several fields of study [1, 2]. The analysis of their solutions is of great importance, as they are for comparison with the collected data [3]. It is worth noting that, most of the time, obtaining their exact solutions is sometimes impossible. Researchers have therefore developed different approaches to help guarantee the existence and uniqueness of these solutions [4–7]. We note that several researchers have provided different conditions in the case of uniqueness in the last decades. For existence, many iterative approaches have been suggested, for example, Picard, Toneli, and others. For uniqueness, Witte provided several conditions that can be tested to conclude that a given nonlinear ordinary differential equation with a classical derivative has a unique solution. Several other researchers, like Caratheordory, Nagumo, and others, have also provided some important conditions [8, 9]. While several works have been published for ordinary differential equations with integer-order derivatives, much attention has not been devoted to classical and fractional nonlinear ordinary differential equations in fractal calculus [10, 11]. Fractional calculus and fractal calculus are interconnected fields, primarily through their shared focus on non-integer dimensions and scales. Fractional calculus extends the concept of differentiation and integration to non-integer orders, allowing for more flexible mathematical modeling of complex systems. A key connection is that fractional calculus provides the mathematical tools needed to describe the dynamics of processes on fractal structures. For example, the study by Metzler and Klafter [12] titled "The random walk's guide

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to anomalous diffusion: a fractional dynamics approach" discusses how fractional calculus can be applied to model diffusion processes on fractal media . Whereas these equations are suitable for the depiction of several complex real-world problems that cannot be modeled using classical ordinary differential equations. In this paper, we shall consider four classes of nonlinear ordinary differential equations, including those with classical differentiation in fractal calculus, those with power law, exponential decay, and generalized Mittag-Leffler kernels in fractal calculus. For each case, we will find conditions of uniqueness based on the framework of Witte [9].

2. Preliminaries

We shall provide some definitions that will be used in this paper.

$$\frac{df(t)}{dt^\beta} = \lim_{t_1\to t}\frac{f(t_1)-f(t)}{t_1^\beta - t^\beta},\ \beta>0, \tag{1}$$

which the fractal derivative of the function f with respect to a fractal measure t with scaling indice β [11]. We note that if f is differentiable then,

$$\frac{df(t)}{dt^\beta} = \frac{f'(t)}{\beta t^{\beta-1}}. \tag{2}$$

Fractal-fractional derivatives of the function f with power law, exponential decay and Mittag-Leffler kernel are given below respectively [10].

$${}_{t_0}^{FFP}D_t^{\alpha,\beta}f(t) = \frac{d}{dt^\beta}\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t} f(\tau)(t-\tau)^{-\alpha}d\tau, \tag{3}$$

$${}_{t_0}^{FFE}D_t^{\alpha,\beta}f(t) = \frac{d}{dt^\beta}\frac{1}{(1-\alpha)}\int_{t_0}^{t} f(\tau)\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right)d\tau, \tag{4}$$

$${}_{t_0}^{FFM}D_t^{\alpha,\beta}f(t) = \frac{d}{dt^\beta}\frac{1}{(1-\alpha)}\int_{t_0}^{t} f(\tau)E_\alpha\left(\frac{-\alpha}{1-\alpha}(t-\tau)^\alpha\right)d\tau, \tag{5}$$

where $(\alpha,\beta)\in(0,1]$.

Their respective integrals are given as below:

$${}_{t_0}^{FFP}J_t^{\alpha,\beta}f(t) = \frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\tau^{\beta-1}f(\tau)d\tau, \tag{6}$$

$${}_{t_0}^{FFE}J_t^{\alpha,\beta}f(t) = (1-\alpha)\beta t^{\beta-1}f(t) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}f(\tau)d\tau, \tag{7}$$

$${}_{t_0}^{FFM}D_t^{\alpha,\beta}f(t) = (1-\alpha)\beta t^{\beta-1}f(t) + \frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\tau^{\beta-1}f(\tau)d\tau. \tag{8}$$

We note that, when $\beta=1$, we recover all the fractional differential and integral operators.

3. The Witte's uniqueness conditions for classical fractal ordinary differential equations

In this section, we are interested in the following general fractal differential equation.

$$\begin{cases} {}_{t_0}^{F}D_t^{\alpha}y(t) = f(t,y(t)), & t>t_0. \\ y(t_0)=y_0, \end{cases} \tag{9}$$

The aim is to establish uniqueness conditions based on the Witte's uniqueness.

Theorem 1. *Let assume that $f(t,y)$ is continuous in $S_+ = \{(t,y)\mid t_0<t\le a,\ |y|<\infty\}$ and satisfies*

i) $\forall(t,y),(t,\overline{y})\in S_+$

$$|f(t,y)-f(t,\overline{y})| \le h(t)\,|y-\overline{y}|, \tag{10}$$

ii) $|f(t,y)| \le \varphi(t)h(t)\exp\left(\int_a^t h(\tau)d\tau\right)$ *in* S_+,

where $h(t)>0$ is continuous in $[t_0,a]$ and $\varphi(t)$ is continuous in $[t_0,a]$ and $\varphi(t_0)=0$.

Then the considered equation has almost one solution.

Proof. To proof the above, we shall first provide the proof of the following Lemma. □

Lemma 1. *Let $\Omega(t)$ be a nonnegative continuous function on $[t_0,a]$ and let*

i) $h(t)>0$ be continuous functions in $[t_0,a]$,

ii) There exists a function $H(t)$ in $[t_0,a]$ such that $H'(t)=h(t)$ for almost all $t\in[t_0,a]$ and $\lim_{t\to t_0^+}H(t)$ exists, it can be finite,

iii) $\Omega(t) \le \int_{t_0}^{t} h(\tau)\Omega(\tau)d\tau,\ t\in[t_0,a]$,

iv) $\Omega(t) = o\left(\exp\left(t^\alpha H(t)\right)\right)$ *as* $t\to t_0^+$. *Then* $\Omega(t)=0$.

Proof. Let the mentioned conditions hold, then

$$\Psi(t)=\alpha\int_{t_0}^{t}\tau^{\alpha-1}h(\tau)\Omega(\tau)d\tau. \tag{11}$$

Thank to the hypothesis of the Lemma $\Psi(t)$ exists and is continuous on $[t_0,a]$. Then

$$ {}_{t_0}^{F}D_t^{\alpha}\Psi(t)=\frac{1}{\alpha t^{\alpha-1}}\frac{d}{dt}\left[\alpha\int_{t_0}^{t}\tau^{\alpha-1}h(\tau)\Omega(\tau)d\tau\right], \tag{12}$$

$$=\frac{1}{\alpha t^{\alpha-1}}\left[\alpha t^{\alpha-1}h(t)\Omega(t)\right],$$
$$=h(t)\Omega(t)\le h(t)\Psi(t).$$

We define

$$F(t)=\exp\left(-t^{\alpha}H(t)\right)\Psi(t). \tag{13}$$

$${}_{t_0}^{F}D_t^{\alpha}F(t) \tag{14}$$

$$=\frac{1}{\alpha t^{\alpha-1}}\frac{d}{dt}\int_{t_0}^{t}F(\tau)d\tau=\frac{1}{\alpha t^{\alpha-1}}F'(t),$$

$$=\frac{1}{\alpha t^{\alpha-1}}$$

$$\times\left[\begin{array}{c}\Psi'(t)\exp\left(-t^{\alpha}H(t)\right)\\ -\Psi(t)\begin{pmatrix}-\alpha t^{\alpha-1}H(t)\\ -t^{\alpha}h(t)\end{pmatrix}\exp(-t^{\alpha}H(t))\end{array}\right],$$

$$=\frac{1}{\alpha t^{\alpha-1}}\exp(-t^{\alpha}H(t))$$

$$\times\left[\begin{array}{c}\Psi'(t)\\ -\Psi(t)\left[\begin{array}{c}\alpha t^{\alpha-1}H(t)\\ -t^{\alpha}h(t)\end{array}\right]\end{array}\right],$$

$$\le\frac{1}{\alpha t^{\alpha-1}}\exp(-t^{\alpha}H(t))$$

$$\left[\begin{array}{c}h(t)\Psi(t)\alpha t^{\alpha-1}\\ -\Psi(t)\left[\begin{array}{c}\alpha t^{\alpha-1}H(t)\\ -t^{\alpha}h(t)\end{array}\right]\end{array}\right],$$

$$\le\frac{\Psi(t)\exp(-t^{\alpha}H(t))}{\alpha t^{\alpha-1}}\left[\begin{array}{c}\alpha t^{\alpha-1}h(t)\\ -\alpha t^{\alpha-1}H(t)-t^{\alpha}h(t)\end{array}\right],$$

$$\le\frac{\Psi(t)\exp(-t^{\alpha}H(t))}{\alpha t^{\alpha-1}}\left[\begin{array}{c}t^{\alpha}h(t)\\ -\alpha t^{\alpha-1}H(t)-t^{\alpha}h(t)\end{array}\right],$$

$$\le -H(t)\Psi(t)\exp(-t^{\alpha}H(t)),$$
$$\le -H(t)F(t)\le 0.$$

We can say that $\forall t\in[t_0,a]$, $\Psi(t)\exp(-t^{\alpha}H(t))$ is decreasing. We now choose $\varepsilon>0$ with t small enough

$$\Psi(t)\exp(-t^{\alpha}H(t)) \tag{15}$$

$$=\exp(-t^{\alpha}H(t))\int_{t_0}^{t}\alpha\tau^{\alpha-1}h(\tau)\Omega(\tau)d\tau,$$

$$\le\varepsilon\exp(-t^{\alpha}H(t))\alpha\int_{t_0}^{t}\tau^{\alpha-1}h(\tau)\exp(\tau^{\alpha}H(\tau))d\tau,$$

$$\le\varepsilon\exp(-t^{\alpha}H(t))\alpha\int_{t_0}^{t}\tau^{\alpha}h(\tau)\exp(\tau^{\alpha}H(\tau))d\tau,$$

$$\le\varepsilon\exp(-t^{\alpha}H(t))\alpha\int_{t_0}^{t}\begin{pmatrix}\tau^{\alpha}h(\tau)\\ +\alpha\tau^{\alpha-1}H(\tau)\end{pmatrix}$$
$$\times\exp(\tau^{\alpha}H(\tau))d\tau,$$
$$=\varepsilon\alpha\exp(-t^{\alpha}H(t))\exp(t^{\alpha}H(t)),$$
$$=\varepsilon\alpha.$$

$$\lim_{t\to t_0^{+}}\exp(-t^{\alpha}H(t))\Psi(t)=0, \tag{16}$$

thus

$$\exp(-t^{\alpha}H(t))\Psi(t)\le 0 \text{ for } t>0, \tag{17}$$

this also implies that

$$\alpha\int_{t_0}^{t}\tau^{\alpha-1}h(\tau)\Omega(\tau)d\tau\le 0. \tag{18}$$

Therefore we should have

$$\Omega(t)=0. \tag{19}$$

□

The new uniqueness criteria will be presented below. This is more general that the previous condition of the theorem.

Theorem 2. *Let $f(t,y)$ be continuous in $\overline{S}_+$ in addition to the hypothesis in theorem 1, we have*

$$|f(t,y)-f(t,\overline{y})|=o\left(\exp(t^{\alpha}H(t))\right), \tag{20}$$

as $t\to t_0^{+}$ uniformly with respect to $y,\overline{y}\in[-\lambda,\lambda]$, $\lambda>0$ arbitrary with $h(t)$ and $H(t)$ the same like in Lemma 1. Then the considered equation has almost one solution in $[t_0,a]$.

Proof. Let $y(t)$ and $\overline{y}(t)$ be two different solutions of one equation

$$y(t)=y(t_0)+\alpha\int_{t_0}^{t}\tau^{\alpha-1}f(\tau,y(\tau))d\tau, \tag{21}$$

$$|y(t)-\overline{y}(t)|\le\alpha\int_{t_0}^{t}\tau^{\alpha-1}\left|f(\tau,y(\tau))-f(\tau,\overline{y}(\tau))\right|d\tau,$$

$$\le\alpha\int_{t_0}^{t}\tau^{\alpha-1}h(\tau)\left|y-\overline{y}\right|d\tau,$$

$$\leq \alpha \int_{t_0}^{t} \left(\tau^{\alpha-1}H(\tau)+\tau^{\alpha}h(\tau)\right)\exp\left(\tau^{\alpha}H(\tau)\right)d\tau,$$
$$\leq \varepsilon \exp\left(t^{\alpha}H(t)\right).$$

From the Lemma, the result is obtained. □

Corollary 1. *Let f satisfies the following conditions; $\forall(t,\overline{y}),(t,y)\in S_{+}$, $\beta\in(1,2]$ and $\alpha\in(0,1]$:*

i) $\left(f(t,\overline{y})-f(t,y)\right)(\overline{y}-y)^{\overline{\beta}-1}\leq \frac{\overline{\beta}}{\alpha}th(t)\left(\overline{y}-y\right)^{\overline{\beta}}$,

ii) $f(t,\overline{y})-f(t,y)=o\left(\exp(t^{\alpha}H(t))\right)$,

as $t\to t_0^{+}$ uniformly with respect to $y,\overline{y}\in[-\delta,\delta]$, $\delta>0$ arbitrary. Then the considered equation has almost one solution.

Proof. Let $\overline{y}$ and y be different solutions in $\overline{S}_{+}$. Let put $\Phi(t)=\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}}$ then we have that,

$$\begin{aligned}
{}_{t_0}^{F}D_t^{\alpha}\Phi(t) &= \frac{1}{\alpha t^{\alpha-1}}\frac{d}{dt}\left[\Phi(t)\right],\\
&= \frac{1}{\alpha t^{\alpha-1}}\left(\overline{\beta}\left(\overline{y}(t)-y(t)\right)\right)'\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}-1},\\
&= \overline{\beta}\left({}_{t_0}^{F}D_t^{\alpha}\overline{y}(t)-{}_{t_0}^{F}D_t^{\alpha}y(t)\right)\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}-1},\\
&= \overline{\beta}\left(f\left(t,\overline{y}(t)\right)-f(t,y(t))\right)\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}-1}.
\end{aligned} \tag{22}$$

By the hypothesis (i), we have that

$$\begin{aligned}
{}_{t_0}^{F}D_t^{\alpha}\Phi(t) &\leq \overline{\beta}h(t)\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}},\\
&= \frac{\overline{\beta}}{\alpha}th(t)\Phi(t).
\end{aligned} \tag{23}$$

Therefore

$${}_{t_0}^{F}D_t^{\alpha}\Phi(t)\leq \frac{\overline{\beta}}{\alpha}h(t)\Phi(t)t. \tag{24}$$

Note that

$$\begin{aligned}
&{}_{t_0}^{F}D_t^{\alpha}\left(\Phi(t)\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\right)\\
&= \frac{1}{\alpha t^{\alpha-1}}\left[\begin{array}{c}\Phi'(t)\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\\ +\Phi(t)\left[\begin{array}{c}-\overline{\beta}t^{\alpha}h(t)\\ -\overline{\beta}\alpha t^{\alpha-1}H(t)\end{array}\right]\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\end{array}\right],\\
&= \exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\left[\begin{array}{c}{}_{t_0}^{F}D_t^{\alpha}\Phi(t)\\ -\Phi(t)\left(\begin{array}{c}\frac{\overline{\beta}}{\alpha}th(t)\\ +\overline{\beta}H(t)\end{array}\right)\end{array}\right],\\
&\leq \exp\left(-\overline{\beta}t^{\alpha}h(t)\right)\left[\begin{array}{c}{}_{t_0}^{F}D_t^{\alpha}\Phi(t)\\ -\overline{\beta}\frac{t}{\alpha}h(t)\Phi(t)\end{array}\right],\\
&\leq 0.
\end{aligned} \tag{25}$$

Since

$${}_{t_0}^{F}D_t^{\alpha}\Phi(t)-\overline{\beta}\frac{t}{\alpha}h(t)\Phi(t)\leq 0, \tag{26}$$

$${}_{t_0}^{F}D_t^{\alpha}\left(\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\Phi(t)\right)\leq 0. \tag{27}$$

The conclusion is that the function $\exp\left(-\overline{\beta}t^{\alpha}\right)\Phi(t)$ is non increasing for almost $\forall t\in[t_0,a]$. On the other hand we have that

$$\begin{aligned}
&\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\Phi(t)\\
&= \exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}},\\
&= \exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\\
&\times\left(\alpha\int_{t_0}^{t}\tau^{\alpha-1}\left(f\left(\tau,\overline{y}\right)-f\left(\tau,y\right)\right)d\tau\right)^{\overline{\beta}}.
\end{aligned} \tag{28}$$

However by hypothesis (ii), we can find $\varepsilon>0$ small enough such that

$$\begin{aligned}
&\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\Phi(t)\\
&\leq \exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\alpha^{\overline{\beta}}\\
&\times\left(\int_{t_0}^{t}\varepsilon\beta\left(\begin{array}{c}\alpha\tau^{\alpha-1}H(\tau)\\ +\tau^{\alpha}h(\tau)\end{array}\right)\exp\left(\tau^{\alpha}\overline{\beta}H(\tau)\right)d\tau\right)^{\overline{\beta}},\\
&\leq \exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\alpha^{\overline{\beta}}\varepsilon^{\overline{\beta}}\\
&\times\left(\int_{t_0}^{t}\exp\left(\tau^{\alpha}\overline{\beta}H(\tau)\right)'d\tau\right),\\
&\leq \exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\alpha^{\overline{\beta}}\varepsilon^{\overline{\beta}}\exp\left(t^{\alpha}\overline{\beta}H(t)\right),\\
&= \alpha^{\overline{\beta}}\varepsilon^{\overline{\beta}}=(\alpha\varepsilon)^{\overline{\beta}},
\end{aligned} \tag{29}$$

and then

$$\lim_{t\to t_0^{+}}\exp\left(-\overline{\beta}t^{\alpha}H(t)\right)\Phi(t)=0. \tag{30}$$

Therefore $\Phi(t)=0$ so we get

$$\overline{y}(t)=y(t), \tag{31}$$

which completes the proof. □

We shall now evaluation the above condition in the case of the fractal fractional with power law.This will be acheived in the next section

4. The Witte's uniqueness conditions for Fractal-Fractional ordinary differential equations with exponential kernel

We shall consider in this section, the following fractal-fractional differential equation

$$\begin{cases} {}_{t_0}^{FFE}D_t^{\alpha,\beta}y(t) = f(t,y(t)), & \text{if } t > t_0, \\ y(t_0) = y_0, & \text{if } t = t_0. \end{cases} \tag{32}$$

that under the witte's condition $\alpha, \beta \in (0,1]$. The aim of this section is to show that under the Witte's condition equation has a unique solution if such solution exists in $[t_0, a]$. We will start our investigation on with the following lemma.

Lemma 2. *Let $f(t,y(t))$, $h(t)$ and $H(t)$ satisfy the properties presented before*

i) $\Phi(t) \leq (1-\alpha)\beta t^{\beta-1}h(t)\Phi(t) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}h(\tau)\Phi(\tau)d\tau,$

ii) $\Phi(t) = o(\exp(H(t)))$ *as* $t \to t_0^+$, *then* $\Phi(t) = 0$ *in* $[t_0, a]$.

Proof. Let set

$$\Omega(t) = (1-\alpha)\beta h(t)t^{\beta-1}\Phi(t) + \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}h(\tau)\Phi(\tau)d\tau. \tag{33}$$

From the hypothesis, we have that $\Omega(t)$ exists and is continuous in $[t_0, a]$. We recall that

$${}_{t_0}^{FFE}D_t^{\alpha,\beta}\left({}_{t_0}^{FFE}J_t^{\alpha}u(t)\right) = u(t). \tag{34}$$

Thus applying ${}_{t_0}^{FFE}D_t^{\alpha,\beta}$ on both sides yields

$${}_{t_0}^{FFE}D_t^{\alpha,\beta}\Omega(t) = h(t)\Phi(t) \leq h(t)\Omega(t). \tag{35}$$

Now, we shall find the sign of the

$$\begin{aligned} &{}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\Omega(t)\exp(-H(t))\right] \\ &= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{CFR}D_t^{\alpha}\left[\Omega(t)\exp(-H(t))\right], \\ &= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{CF}D_t^{\alpha}\left[\Omega(t)\exp(-H(t))\right]. \end{aligned} \tag{36}$$

Since $\Omega(t_0) = 0$, therefore, we have that

$${}_{t_0}^{CFR}D_t^{\alpha}\Omega(t) = {}_{t_0}^{CF}D_t^{\alpha}\Omega(t). \tag{37}$$

Therefore

$$\begin{aligned} &{}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\Omega(t)\exp(-H(t))\right] \\ &= \frac{1}{\beta t^{\beta-1}}\left[\begin{array}{c} \frac{1}{1-\alpha}\int_{t_0}^{t}\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right) \\ \times\left[\begin{array}{c}\Omega'(\tau)\exp(-H(\tau)) \\ -h(\tau)\Omega(\tau)\exp(-H(\tau))\end{array}\right]\end{array}\right]d\tau, \\ &= \frac{1}{\beta t^{\beta-1}}\left[\begin{array}{c} \frac{1}{1-\alpha}\int_{t_0}^{t}\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right) \\ \times\left[\Omega'(\tau) - h(\tau)\Omega(\tau)\right]\exp(-H(\tau))\end{array}\right]d\tau, \\ &\leq 0. \end{aligned} \tag{38}$$

In reference [9] it was shown that under the condition prescribed here

$$\Omega'(t) - h(t)\Omega(t) \leq 0, \tag{39}$$

therefore

$${}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\exp(-H(t))\Omega(t)\right] \leq 0. \tag{40}$$

Since by the hypothesis the integral is positive therefore

$${}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\exp(-H(t))\Omega(t)\right] \leq 0, \tag{41}$$

almost every where in $[t_0, a]$.

$$\exp(-H(t))\Omega(t) = \exp(-H(t))\left[\begin{array}{c}(1-\alpha)\beta h(t)\Phi(t)t^{\beta-1} \\ +\beta\alpha\int_{t_0}^{t}\tau^{\beta-1}h(\tau)\Phi(\tau)d\tau\end{array}\right].$$

For a sufficient small t, we choose $\varepsilon > 0$ such that in the view of (iv), we get

$$\begin{aligned} \Omega(t)\exp(-H(t)) &\leq \exp(-H(t)) \\ &\times\left[\begin{array}{c}(1-\alpha)\beta h(t)t^{\beta-1}\exp(H(t))\varepsilon' \\ +\beta\alpha\varepsilon'\int_{t_0}^{t}\tau^{\beta-1}h(\tau)\exp(H(\tau))d\tau\end{array}\right], \\ &\leq \exp(-H(t)) \\ &\times\left[\begin{array}{c}(1-\alpha)\beta h(t)(\bar{t}_0)^{\beta-1}\exp(H(t))\varepsilon' \\ +(\bar{t}_0)^{\beta-1}\beta\alpha\varepsilon'\exp(H(t))\end{array}\right], \\ &\leq (1-\alpha)\beta h(t)(\bar{t}_0)^{\beta-1}\varepsilon' + (\bar{t}_0)^{\beta-1}\beta\alpha\varepsilon'. \end{aligned}$$

Using the continuity of $h(t)$ in $[t_0, a]$. $\exists t_1 \in [t_0, a]$ such that $\forall t \in [t_0, a]$

$$h(t_1) \geq h(t), \tag{42}$$

therefore

$$\exp\left(-H(t)\right)\Omega(t) \leq \left(\bar{t}_0\right)^{\beta-1}\beta\left(h\left(t_1\right)\varepsilon'\left(1-\alpha\right)+\alpha\varepsilon'\right), \\ \leq \mu\varepsilon' = \frac{\mu}{\mu}\varepsilon = \varepsilon. \tag{43}$$

where

$$\varepsilon' = \frac{\varepsilon}{\mu} = \frac{\varepsilon}{\left(\bar{t}_0\right)^{\beta-1}\beta\left(h\left(t_1\right)\left(1-\alpha\right)+\alpha\beta\right)}. \tag{44}$$

Therefore

$$\Phi(t) = 0. \tag{45}$$

□

Theorem 3. *Let $f(t,y)$ be continuous in $\overline{S}_+$ in addition to Theorem 2 and Lemma 2 we have*

$\forall\varepsilon' > 0$,

$$\varepsilon' = \frac{\varepsilon}{\left(1-\alpha\right)\beta h\left(t\right)\left(\bar{t}_0\right)^{\beta-1}+\left(\bar{t}_0\right)^{\beta-1}\beta\alpha}. \tag{46}$$

Then the initial value problem (32) has almost one solution.

Proof. Let $y(t)$ and $\overline{y}(t)$ be two different solutions of our equation, then

$$\begin{aligned} |\Phi(t)| = |\overline{y}(t)-y(t)| &\leq (1-\alpha)\beta t^{\beta-1}\left|f(t,\overline{y}(t))-f(t,y(t))\right| \\ &+ \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left|f(\tau,\overline{y}(\tau))-f(\tau,y(\tau))\right|d\tau, \\ &\leq (1-\alpha)\beta t^{\beta-1}h\left(t\right)\Phi\left(t\right) \\ &+ \alpha\beta\int_{t_0}^{t}\tau^{\beta-1}h\left(\tau\right)\Phi(\tau)d\tau, \\ &\leq \left(1-\alpha\right)\beta h\left(t\right)\left(\bar{t}_0\right)^{\beta-1}\varepsilon'\exp\left(H(t)\right) \\ &+ \left(\bar{t}_0\right)^{\beta-1}\beta\alpha\varepsilon'\exp\left(H(t)\right), \\ &\leq \begin{pmatrix} \left(1-\alpha\right)\beta h\left(t\right)\left(\bar{t}_0\right)^{\beta-1} \\ +\left(\bar{t}_0\right)^{\beta-1}\beta\alpha \end{pmatrix}\varepsilon'\exp\left(H(t)\right), \\ &\leq \mu\varepsilon'\exp\left(H(t)\right) = \varepsilon\exp\left(H(t)\right). \end{aligned} \tag{47}$$

□

Theorem 4. *Let $f(t,y)$ satisfies all the condition described in Theorem 3.*

Proof. Let $y(t)$ and $\overline{y}(t)$ be two different solution of equation (32). We set as before

$$\Psi(t) = \left(\overline{y}-y\right)^{\overline{\beta}}. \tag{48}$$

We have that $\Psi(t_0) = 0$, thus

$$\begin{aligned} &{}^{FFE}_{t_0}D_t^{\alpha,\beta}\Psi(t) \\ &= \frac{1}{\beta t^{\beta-1}}\,{}^{CFR}_{t_0}D_t^{\alpha}\Psi(t) = \frac{1}{\beta t^{\beta-1}}\,{}^{CF}_{t_0}D_t^{\alpha}\Psi(t), \\ &= \frac{1}{\beta t^{\beta-1}}\frac{1}{1-\alpha}\int_{t_0}^{t}\Psi'(\tau)\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right)d\tau, \\ &= \frac{1}{\beta t^{\beta-1}}\frac{1}{1-\alpha}\int_{t_0}^{t}\left[\overline{\beta}\left(\overline{y}-y\right)'\left(\overline{y}-y\right)^{\overline{\beta}-1}\right] \\ &\times\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right)d\tau, \\ &\leq \overline{\beta}\delta\left[{}^{FFE}_{t_0}D_t^{\alpha,\beta}\overline{y} - {}^{FFE}_{t_0}D_t^{\alpha,\beta}y\right], \\ &\leq \overline{\beta}\delta\left|f(t,\overline{y}(t))-f(t,y(t))\right|, \\ &\leq \overline{\beta}\delta h(t)\Psi(t), \end{aligned} \tag{49}$$

here

$$\delta = \begin{cases} \max\limits_{t\in[t_0,a]}\left|\overline{y}-y\right|^{\overline{\beta}-1}, & \text{if} \quad y'-\overline{y}' > 0, \\ \min\limits_{t\in[t_0,a]}\left|\overline{y}-y\right|^{\overline{\beta}-1}, & \text{if} \quad y'-\overline{y}' < 0. \end{cases}$$

In the view of the first hypothesis. Thus

$${}^{FFE}_{t_0}D_t^{\alpha,\beta}\Psi(t) \leq \overline{\Delta}\Psi(t), \tag{50}$$

almost every where in $[t_0,a]$.

$$\begin{aligned} &{}^{FFE}_{t_0}D_t^{\alpha,\beta}\left[\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\right] \\ &= \frac{1}{(1-\alpha)\beta t^{\beta-1}}\frac{d}{dt}\int_{t_0}^{t}\exp\left(-\frac{\alpha}{1-\alpha}\left(t-\tau\right)\right) \\ &\times\exp\left(-\overline{\beta}H(\tau)\right)\Psi(\tau)d\tau, \\ &= \frac{1}{(1-\alpha)\beta t^{\beta-1}}\int_{t_0}^{t}\exp\left(-\frac{\alpha}{1-\alpha}\left(t-\tau\right)\right) \\ &\times\left(\Psi(\tau)\exp\left(-\overline{\beta}H(\tau)\right)\right)'d\tau \\ &- \frac{1}{\beta t^{\beta-1}}\frac{1}{1-\alpha}\Psi(t_0)\exp\left(-\overline{\beta}H(t_0)\right)\exp\left(\frac{-\alpha}{1-\alpha}t\right). \end{aligned} \tag{51}$$

But

$$\Psi(t_0) = 0, \tag{52}$$

therefore

$$
\begin{aligned}
&{}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\right] \\
&= \frac{1}{(1-\alpha)\beta t^{\beta-1}}\int_{t_0}^{t}\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \\
&\times\left(\exp\left(-\overline{\beta}H(\tau)\right)\Psi(\tau)\right)' d\tau, \\
&= \frac{1}{(1-\alpha)\beta t^{\beta-1}}\int_{t_0}^{t}\exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right) \\
&\times\left(\begin{array}{c}\Psi'(\tau)\exp\left(-\overline{\beta}H(\tau)\right) \\ -\overline{\beta}H(\tau)\exp\left(-\overline{\beta}H(\tau)\right)\Psi(\tau)\end{array}\right)d\tau.
\end{aligned}
\tag{53}
$$

In reference [9], it was shown that

$$
\Psi'(t)\exp\left(-\overline{\beta}H(t)\right)-\overline{\beta}H(t)\exp\left(-\overline{\beta}H(t)\right)\Psi(t)<0. \tag{54}
$$

Therefore

$$
{}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\Psi(t)\exp\left(-\overline{\beta}H(t)\right)\right]\leq 0.
$$

$$
\begin{aligned}
&{}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\right] \\
&= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{CFR}D_t^{\alpha}\left[\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\right], \\
&= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{CF}D_t^{\alpha}\left[\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\right], \\
&= \frac{1}{\beta t^{\beta-1}}\frac{1}{1-\alpha}\int_{t_0}^{t}\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right) \\
&\times\left[\begin{array}{c}-\Psi'(\tau)\overline{\beta}\exp\left(-\overline{\beta}H(\tau)\right) \\ -\overline{\beta}h(\tau)\exp\left(-\overline{\beta}H(\tau)\right)\Psi(\tau)\end{array}\right]d\tau, \\
&= \frac{1}{\beta t^{\beta-1}}\frac{1}{1-\alpha}\int_{t_0}^{t}\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right) \\
&\times\left[\left[\Psi'(\tau)+h(\tau)\Psi(\tau)\right]\exp\left(-\overline{\beta}H(\tau)\right)\right]d\tau, \\
&= \frac{1}{\beta t^{\beta}}\frac{1}{1-\alpha}\int_{t_0}^{t}\exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right) \\
&\times\left[\left[\Psi'(\tau)-h(\tau)\Psi(\tau)\right]\exp\left(-\overline{\beta}H(\tau)\right)\right]d\tau, \\
&\leq 0.
\end{aligned}
\tag{55}
$$

Therefore, we have

$$
{}_{t_0}^{FFE}D_t^{\alpha,\beta}\left[\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\right]<0. \tag{56}
$$

Following the routine presented earlier we shall have for ε'

$$
\begin{aligned}
&\exp\left(-\overline{\beta}H(t)\right)\Psi(t) \\
&= \exp\left(-\overline{\beta}H(t)\right)\left(\overline{y}(t)-y(t)\right)^{\overline{\beta}}, \\
&= \exp\left(-\overline{\beta}H(t)\right)\left(\begin{array}{c}(1-\alpha)\beta t^{\beta-1}\left(f(t,\overline{y}(t))-f(t,y(t))\right) \\ +\alpha\beta\int_{t_0}^{t}\tau^{\beta-1}\left(f(\tau,\overline{y}(\tau))-f(\tau,y(\tau))\right)d\tau\end{array}\right)^{\overline{\beta}}, \\
&\leq \exp\left(-\overline{\beta}H(t)\right)\left(\begin{array}{c}(1-\alpha)\beta t^{\beta-1}\varepsilon'\exp\left(H(t)\right)h(t) \\ +\alpha\beta\int_{t_0}^{t}h(\tau)\tau^{\beta-1}\varepsilon'\exp\left(H(\tau)\right)d\tau\end{array}\right)^{\overline{\beta}}, \\
&\leq \exp\left(-\overline{\beta}H(t)\right)\left(\begin{array}{c}(1-\alpha)\beta\left(\overline{t}_0\right)^{\beta-1}\varepsilon'\exp\left(H(t)\right)h(t) \\ +\left(\overline{t}_0\right)^{\beta-1}\alpha\beta\varepsilon'\int_{t_0}^{t}h(\tau)\exp\left(H(\tau)\right)d\tau\end{array}\right)^{\overline{\beta}}, \\
&\leq \exp\left(-\overline{\beta}H(t)\right)\left(\begin{array}{c}(1-\alpha)\overline{\beta}\left(\overline{t}_0\right)^{\beta-1}\varepsilon'\exp\left(H(t)\right)h(t) \\ +\left(\overline{t}_0\right)^{\beta-1}\alpha\overline{\beta}\varepsilon'\exp\left(H(t)\right)\end{array}\right)^{\overline{\beta}}, \\
&\leq \exp\left(-\overline{\beta}H(t)\right)\left(\begin{array}{c}(1-\alpha)\beta\left(\overline{t}_0\right)^{\beta-1}\varepsilon'\exp\left(H(t)\right)h(t_1) \\ +\left(\overline{t}_0\right)^{\beta-1}\alpha\beta\varepsilon'\exp\left(H(t)\right)\end{array}\right)^{\overline{\beta}}, \\
&\leq \exp\left(-\overline{\beta}H(t)\right)\exp\left(\beta H(t)\right)\left(\varepsilon'\right)^{\overline{\beta}}\left((1-\alpha)\beta\left(\overline{t}_0\right)^{\beta-1}+\left(\overline{t}_0\right)^{\beta-1}\alpha\beta\right)^{\overline{\beta}}, \\
&\leq \left(\varepsilon'\right)^{\overline{\beta}}\mu^{\overline{\beta}},\ \mu=\left((1-\alpha)\beta+\alpha\beta\right)\left(\overline{t}_0\right)^{\beta-1}.
\end{aligned}
\tag{57}
$$

We choose

$$
\varepsilon'=\frac{\varepsilon}{\mu}, \tag{58}
$$

such that

$$
\exp\left(-\overline{\beta}H(t)\right)\Psi(t)\leq\varepsilon^2. \tag{59}
$$

Therefore

$$
\lim_{t\to 0^+}\exp\left(-\overline{\beta}H(t)\right)\Psi(t)=0. \tag{60}
$$

So we conclude that

$$
\begin{aligned}
\Psi(t) &= 0 \\
&\Rightarrow \overline{y}(t)=y(t),
\end{aligned}
\tag{61}
$$

which concludes the proof. □

5. The Witte's uniqueness conditions for Fractal-Fractional ordinary differential equations with power-law kernel

In this section, we shall consider the following differential equation

$$
\begin{cases}
{}_{t_0}^{FFP}D_t^{\alpha,\beta}y(t)=f(t,y(t)), & \text{if } t>t_0, \\
y(t_0)=y_0, & \text{if } t=t_0.
\end{cases}
\tag{62}
$$

We aim to show that if the solution of the above equation exists in $S_+=\{(t,y)\mid t_0<t\leq a,\ |y|<\infty\}$, $\alpha\in(0,1]$, $\beta\in(0,1]$ then it is unique.

Lemma 3. *Let $\Phi(t)$ be a non negative continuous in $(t_0, a]$ such that $\Phi(t_0) = 0$. Let*

i) $h(t) > 0$ be continuous function in $(t_0, a]$,

ii) We can find a function $H(t)$ in $(t_0, a]$ such that $H'(t) = h(t)$ for almost all $t \in (t_0, a]$ and $\lim\limits_{t \to t_0^+} H(t)$ exists,

iii) $\Phi(t) \le \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} h(\tau)\Phi(\tau)d\tau$, $\forall t \in (t_0, a]$ and

iv) $\Phi(t) = o(\exp(H(t)))$ as $t \to t_0^+$. Then

$$\Phi(t) = 0, \tag{63}$$

in $(t_0, a]$.

Proof. Let

$$\Omega(t) = \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1} (t-\tau)^{\alpha-1} h(\tau)\Phi(\tau)d\tau. \tag{64}$$

The existence and the continuty of the function $\Omega(t)$ is assumed since the hypothesis of the Lemma. Therefore we have that

$${}_{t_0}^{FFP}D_t^{\alpha,\beta}\Omega(t) = h(t)\Phi(t) \le h(t)\Phi(t). \tag{65}$$

We note that

$$\begin{aligned}
&{}_{t_0}^{FFP}D_t^{\alpha,\beta}\Omega(t) \\
&= \frac{\beta}{\Gamma(\alpha)}\, {}_{t_0}^{FFP}D_t^{\alpha,\beta} \left[\int_{t_0}^{t} \tau^{\beta-1}(t-\tau)^{\alpha-1} h(\tau)\Phi(\tau)d\tau \right], \\
&= \frac{\beta}{\Gamma(\alpha)\beta t^{\beta-1}} \frac{d}{dt} \frac{1}{\Gamma(1-\alpha)} \\
&\times \left(\int_{t_0}^{t} \tau^{\beta-1}(t-\tau)^{-\alpha} \left[\int_{t_0}^{\tau} l^{\beta-1}(\tau-l)^{\alpha-1} h(l)\Phi(l)dl \right] d\tau \right), \\
&= \frac{\beta}{\beta t^{\beta-1}}\, {}_{t_0}^{RL}D_t^{\alpha} \left[{}_{t_0}^{RL}J_t^{\alpha} \left(t^{\beta-1}h(t)\Phi(t) \right) \right], \\
&= \frac{\beta t^{\beta-1}h(t)\Phi(t)}{\beta t^{\beta-1}}, \\
&= h(t)\Phi(t).
\end{aligned} \tag{66}$$

We recall that $\Omega(t_0) = 0$, then

$${}_{t_0}^{RL}D_t^{\alpha}\Omega(t) = {}_{t_0}^{C} D_t^{\alpha}\Omega(t). \tag{67}$$

$$\begin{aligned}
&{}_{t_0}^{FFP}D_t^{\alpha,\beta} [\exp(-H(t))\Omega(t)] \\
&= \frac{1}{\beta t^{\beta-1}}\, {}_{t_0}^{RL}D_t^{\alpha} [\exp(-H(t))\Omega(t)], \\
&= \frac{1}{\beta t^{\beta-1}}\, {}_{t_0}^{C}D_t^{\alpha} [\exp(-H(t))\Omega(t)], \\
&= \frac{1}{\beta t^{\beta}} \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \begin{bmatrix} \Omega'(\tau)\exp(-H(\tau)) \\ -h(\tau)\Omega(\tau)\exp(-H(\tau)) \end{bmatrix} d\tau, \\
&= \frac{1}{\beta t^{\beta}} \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \exp(-H(\tau)) [\Omega'(\tau) - h(\tau)\Omega(\tau)] d\tau,
\end{aligned} \tag{68}$$

We have due to reference [9] that

$$\Omega'(\tau) - h(\tau)\Omega(\tau) \le 0. \tag{69}$$

Therefore

$${}_{t_0}^{FFP}D_t^{\alpha,\beta} [\exp(-H(t))\Omega(t)] \le 0. \tag{70}$$

We can now have for a small t

$$\begin{aligned}
&\exp(-H(t))\Omega(t) \\
&= \exp(-H(t)) \frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1}(t-\tau)^{\alpha-1} h(\tau)\Phi(\tau)d\tau, \\
&\le \frac{\exp(-H(t))\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta}(t-\tau)^{\alpha} h(\tau)\Phi(\tau)d\tau, \\
&\le \frac{\exp(-H(t))\beta a^{\beta+\alpha}}{\Gamma(\alpha)} \int_{t_0}^{t} h(\tau)\Phi(\tau)d\tau.
\end{aligned} \tag{71}$$

By hypothesis (iv), we have

$$\begin{aligned}
\exp(-H(t))\Omega(t) &\le \frac{\exp(-H(\tau))\beta a^{\beta+\alpha}}{\Gamma(\alpha)} \varepsilon \exp\left(H(\tau) \times \frac{\Gamma(\alpha)}{\beta a^{\beta+\alpha}} \right), \\
&\le \varepsilon.
\end{aligned} \tag{72}$$

$$\lim_{t \to 0^+} \exp(-H(t))\Omega(t) = 0. \tag{73}$$

This leads to

$$\exp(-H(t))\Omega(t) \le 0, \ \forall t > t_0, \tag{74}$$

which implies

$$\frac{\beta}{\Gamma(\alpha)} \int_{t_0}^{t} \tau^{\beta-1}(t-\tau)^{\alpha-1} h(\tau)\Phi(\tau)d\tau \le 0, \tag{75}$$

which is a contradiction therefore

$$\Phi(t) = 0. \tag{76}$$

□

Theorem 5. *Let f be continuous in $S_+ = \{(t,y) \mid t_0 < t \leq a,\ |y| < \infty\}$ such that $\forall(t,y),(t,\overline{y}) \in S_+$*

i) $|f(t,y) - f(t,\overline{y})| \leq h(t)\,|y - \overline{y}|$,

ii) $f(t,y) - f(t,\overline{y}) = o\left(\exp(H(t))\right)$,

as $t \to t_0^+$ uniformly with respect to $y,\overline{y} \in [-\delta,\delta]$, $\delta > 0$ arbitrary, where $h(t) = H'(t)$ are the same as in above. Then the considered equation has almost one solution.

Proof. Let $\overline{y}(t)$ and $y(t)$ be two different solutions, we have that

$$\begin{aligned}
|\overline{y}(t) - y(t)| &\leq \frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\left|\begin{array}{c} f(\tau,\overline{y}(\tau)) \\ -f(\tau,y(\tau)) \end{array}\right| d\tau, \\
&\leq \frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\,|y-\overline{y}|\,d\tau, \\
&\leq \frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha}h(\tau)\,|y-\overline{y}|\,d\tau.
\end{aligned} \tag{77}$$

In the view of (ii), we have

$$\begin{aligned}
|\overline{y}(t) - y(t)| &\leq \varepsilon\frac{\beta\Gamma(\alpha)}{\beta a^{\beta+\alpha}\Gamma(\alpha)}\int_{t_0}^{t}a^{\beta+\alpha}h(\tau)\exp(H(\tau))d\tau, \\
&\leq \varepsilon\exp(H(t)).
\end{aligned} \tag{78}$$

The result of the previous lemma leads to

$$\overline{y}(t) = y(t). \tag{79}$$

□

Theorem 6. *Let f be continuous in $\overline{S}_+ = \{(t,y) \mid t_0 < t \leq a,\ |y| < \infty\}$ such that $\overline{\beta} \in (1,2]$, $\alpha,\beta \in (0,1]$,$\forall(t,y),(t,\overline{y}) \in \overline{S}_+$, we have*

i) $\left(f(t,\overline{y}) - f(t,y)\right)(\overline{y} - y) \leq h(t)\,(\overline{y} - y)^{\overline{\beta}}$,

ii) $f(t,\overline{y}) - f(t,y) = o\left(h(t)\exp\left(H(t)\right)\right)$,

uniformly with respect to $y,\overline{y} \in [-\delta,\delta]$, $\delta > 0$ arbitrary then

$$\overline{y}(t) = y(t). \tag{80}$$

Proof. Let $\overline{y}$ and y be two solutions, we put $\Phi(t) = (\overline{y}(t) - y(t))^{\overline{\beta}}$. We have that at $t = t_0$, $\Phi(t_0) = 0$ initial condition then we will have that

$${}_{t_0}^{C}D_t^{\alpha}\Phi(t) = {}_{t_0}^{RL}D_t^{\alpha}\Phi(t). \tag{81}$$

However,

$$\begin{aligned}
&{}_{t_0}^{FFP}D_t^{\alpha,\beta}\Phi(t) \\
&= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{RL}D_t^{\alpha}\Phi(t) = \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{C}D_t^{\alpha}\Phi(t), \\
&= \frac{1}{\beta t^{\beta-1}}\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\Phi'(\tau)d\tau, \\
&= \frac{1}{\beta t^{\beta-1}}\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\left[\overline{\beta}\,(\overline{y}-y)'\,(\overline{y}-y)^{\overline{\beta}-1}\right]d\tau, \\
&= \frac{1}{\beta t^{\beta-1}}\left[\begin{array}{c} \frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\overline{\beta}\,(\overline{y})'\,(\overline{y}-y)^{\overline{\beta}-1}(t-\tau)^{-\alpha}d\tau \\ -\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}\overline{\beta}y'\,(\overline{y}-y)^{\overline{\beta}-1}(t-\tau)^{-\alpha}d\tau \end{array}\right], \\
&\leq \overline{\beta}\Lambda\left({}_{t_0}^{FFP}D_t^{\alpha,\beta}\overline{y}' - {}_{t_0}^{FFP}D_t^{\alpha,\beta}y\right), \\
&\leq \overline{\beta}\Lambda\,|f(t,\overline{y}(t)) - f(t,y(t))|, \\
&\leq \overline{\beta}\Lambda h(t)\,(\overline{y}(t) - y(t))^{\overline{\beta}}.
\end{aligned} \tag{82}$$

here

$$\Lambda = \begin{cases} \max\limits_{t\in[t_0,a]}|\overline{y}-y|^{\overline{\beta}-1}, & \text{if} \quad y' - \overline{y}' > 0, \\ \min\limits_{t\in[t_0,a]}|\overline{y}-y|^{\overline{\beta}-1}, & \text{if} \quad y' - \overline{y}' < 0. \end{cases}$$

By the hypothesis (i), thus

$$\begin{aligned}
{}_{t_0}^{FFP}D_t^{\alpha,\beta}\Phi(t) &\leq \overline{\beta}\Lambda h(t)\Phi(t), \\
&\leq \overline{\beta\Lambda}h(t)\Phi(t).
\end{aligned} \tag{83}$$

$${}_{t_0}^{FFP}D_t^{\alpha,\beta}\Phi(t) - \overline{\beta\Lambda}h(t)\Phi(t) \leq 0. \tag{84}$$

$$\begin{aligned}
&{}_{t_0}^{FFP}D_t^{\alpha,\beta}\left[\exp(-\overline{\beta}H(t))\Phi(t)\right] \\
&= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{RL}D_t^{\alpha}\left[\exp(-\overline{\beta}H(t))\Phi(t)\right], \\
&= \frac{1}{\beta t^{\beta-1}}\,{}_{t_0}^{C}D_t^{\alpha}\left[\exp(-\overline{\beta}H(t))\Phi(t)\right], \\
&= \frac{1}{\beta t^{\beta-1}}\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\left(\exp(-\overline{\beta}H(\tau))\Phi(\tau)\right)'d\tau, \\
&= \frac{1}{\beta t^{\beta-1}}\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\left[\begin{array}{c} -\overline{\beta}h(\tau)\Phi(\tau) \\ -\beta\,(\overline{y}-y)'\,(\overline{y}-y)^{\overline{\beta}-1}H(\tau) \end{array}\right]d\tau, \\
&= \frac{-1}{\beta t^{\beta-1}}\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\left[\begin{array}{c} \overline{\beta}h(\tau)\Phi(\tau) \\ +\beta\,(\overline{y}-y)'\,(\overline{y}-y)^{\overline{\beta}-1}H(\tau) \end{array}\right]d\tau, \\
&\leq 0,
\end{aligned} \tag{85}$$

for almost all $t \in [0,a]$. This shows that $\exp(-\overline{\beta}H(t))\Phi(t)$ is non increasing for a small t.

$$\begin{aligned}
\exp(-\overline{\beta}H(t))\Phi(t) &= \exp(-\beta H(t))\,(\overline{y}-y)^{\overline{\beta}}, \\
&= \exp(-\beta H(t)) \\
&\quad\times\left(\frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\left|\begin{array}{c} f(\tau,\overline{y}(\tau)) \\ -f(\tau,y(\tau)) \end{array}\right|d\tau\right)^{\overline{\beta}}.
\end{aligned} \tag{86}$$

In the view of the second hypothesis, we will have that

$$\begin{aligned}&\exp(-\overline{\beta}H(t))\Phi(t)\\&\leq\left(\frac{\beta}{\Gamma(\alpha)}\frac{\Gamma(\alpha)a^{\beta+\alpha}}{\beta a^{\beta+\alpha}}\right)^{\overline{\beta}}\varepsilon^{\overline{\beta}}\exp(\overline{\beta}H(t))\exp(-\overline{\beta}H(t)),\\&=\varepsilon^{\overline{\beta}}.\end{aligned} \tag{87}$$

Therefore

$$\exp(-\overline{\beta}H(t))\Phi(t)\leq\varepsilon^{\overline{\beta}}, \tag{88}$$

$$\lim_{t\to0^+}\exp(-\beta H(t))\Phi(t)=0.$$

Therefore

$$\Phi(t)=0, \tag{89}$$

$$\Longrightarrow\overline{y}(t)=y(t)\text{ in }[t_0,a].$$

□

6. The Witte's uniqueness conditions for Fractal-Fractional ordinary differential equations with the Mittag Leffler kernel

In this section, we will consider the following fractal-fractional differential equation

$$\begin{cases}{}^{FFM}_{t_0}D_t^{\alpha,\beta}y(t)=f(t,y(t)), & \text{if } t>t_0,\\ y(t_0)=y_0, & \text{if } t=t_0.\end{cases} \tag{90}$$

Assuming the existence of the solution $y(t)$, we shall show that $y(t)$ is unique.

Lemma 4. *Let $\Phi(t)$ be a nonnegative continuous in $[t_0,a]$ and*
i) Let $h(t)>0$ be a continuous function in $(t_0,a]$ such that $1-zh(t_1)>0$,
ii) $\Phi(t)\leq(1-\alpha)t^{\beta-1}\beta h(t)\Phi(t)+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau,$
iii) and $\frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)d\tau$, *exists Then*

$$\Phi(t)=0. \tag{91}$$

in $[t_0,a]$.

Proof. Let $\Phi(t)$ and $h(t)$ satisfy the condition of the theorem, then, we set

$$\Omega(t)=(1-\alpha)t^{\beta-1}\beta h(t)\Phi(t)+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau. \tag{92}$$

We have from the fundamental theorem of fractal-fractional calculus that

$${}^{FFM}_{t_0}D_t^{\alpha,\beta}\left({}^{FFM}_{t_0}J_t^{\alpha,\beta}f(t)\right)=f(t). \tag{93}$$

Therefore

$${}^{FFM}_{t_0}D_t^{\alpha,\beta}\Omega(t)={}^{FFM}_{t_0}D_t^{\alpha,\beta}\left({}^{FFM}_{t_0}J_t^{\alpha,\beta}(h(t)\Phi(t))\right)=h(t)\Phi(t), \tag{94}$$

which produces

$${}^{FFM}_{t_0}D_t^{\alpha,\beta}\Omega(t)\leq h(t)\Omega(t). \tag{95}$$

Then, we obtain $\Omega(t)$ as

$$\begin{aligned}\Omega(t)&\leq(1-\alpha)\beta t^{\beta-1}h(t)\Omega(t)\\&\quad+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Omega(\tau)d\tau,\\&\leq(1-\alpha)\beta(\bar{t}_0)^{\beta-1}h(t_1)\Omega(t)\\&\quad+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Omega(\tau)d\tau,\\\Omega(t)&\leq\frac{\alpha\beta}{\Gamma(\alpha)(1-zh(t_1))}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Omega(\tau)d\tau.\end{aligned} \tag{96}$$

We put

$$\Delta=\frac{\alpha\beta}{\Gamma(\alpha)(1-zh(t_1))}. \tag{97}$$

By the Gronwall inequality

$$\Omega(t)\leq o\exp\left(\Delta\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)d\tau\right), \tag{98}$$

$$\begin{aligned}&=o\exp\left(\frac{\alpha}{(1-zh(t_1))}\,{}^{FFM}_{t_0}J_t^{\alpha,\beta}h(t)\right),\\&=0.\end{aligned}$$

$z=(1-\alpha)\beta(\bar{t}_0)^{\beta-1}$, which is contraction. Therefore

$$\Omega(t)=0\Rightarrow\Phi(t)=0, \tag{99}$$

in $[t_0,a]$. □

Lemma 5. *Let $\Phi(t)$, $h(t)$ and $H(t)$ be the same like before and $\Phi(t_0)=0$.*
i) $\Phi(t)=o\left(\exp\left(t^{\beta}H(t)\right)\right)$ as $t\to t_0^+$ then

$$\Phi(t)=0,\ \forall t\in[t_0,a]. \tag{100}$$

Proof. Let $\Phi(t)$ and $h(t)$ satisfy the condition above, then

$$\begin{aligned}\Omega(t)&=(1-\alpha)\beta t^{\beta-1}h(t)\Phi(t)\\&\quad+\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau,\end{aligned}$$

exists

$$ {}^{FFM}_{t_0}D_t^{\alpha,\beta}\Omega(t) \le h(t)\Omega(t). \quad (101) $$

$$ \begin{aligned} &{}^{FFM}_{t_0}D_t^{\alpha,\beta}\left(\Omega(t)\exp\left(-H(t)\right)\right) \\ &= \frac{1}{(1-\alpha)\beta t^{\beta-1}}\frac{d}{dt}\int_{t_0}^{t} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\Omega(\tau)\exp\left(-H(\tau)\right)d\tau. \end{aligned} \quad (102) $$

Since $\Phi(t_0)=0$, we will have $\Omega(t_0)$ therefore,

$$ \begin{aligned} &{}^{FFM}_{t_0}D_t^{\alpha,\beta}\left(\Omega(t)\exp\left(-H(t)\right)\right) \\ &= \frac{1}{(1-\alpha)\beta t^{\beta-1}}\frac{d}{dt}\int_{t_0}^{t} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\Omega(\tau)\exp\left(-H(\tau)\right)d\tau, \\ &= \frac{t^{1-\beta}}{(1-\alpha)\beta}\int_{t_0}^{t} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\begin{bmatrix} \Omega'(\tau)\exp\left(-H(\tau)\right) \\ -h(\tau)\Omega(\tau)\exp\left(-H(\tau)\right)\end{bmatrix}d\tau, \\ &= \frac{t^{1-\beta}}{(1-\alpha)\beta}\int_{t_0}^{t} E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right)\exp\left(-H(\tau)\right)\begin{bmatrix} \Omega'(\tau) \\ -h(\tau)\Omega(\tau)\end{bmatrix}d\tau, \end{aligned} \quad (103) $$

whereas from [9], we have that

$$ \Omega(\tau) - h(\tau)\Omega(\tau) \le 0. \quad (104) $$

Therefore since $E_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\tau)^\alpha\right) > 0$, we concluded that

$$ {}^{FFM}_{t_0}D_t^{\alpha,\beta}\left(\Omega(t)\exp\left(-H(t)\right)\right) < 0. \quad (105) $$

$$ \begin{aligned} &\exp\left(-t^\beta H(t)\right)\Omega(t) \\ &= \exp\left(-t^\beta H(t)\right)\begin{bmatrix} (1-\alpha)\beta t^{\beta-1}h(t)\Phi(t) \\ +\frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau \end{bmatrix}, \\ &= \exp\left(-t^\beta H(t)\right)(1-\alpha)\beta t^{\beta-1}h(t)\Phi(t) \\ &+ \frac{\alpha\beta\exp\left(-t^\beta H(t)\right)}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau, \\ &\le \exp\left(-t^\beta H(t)\right)(1-\alpha)\beta t^{\beta}h(t)\Phi(t) \\ &+ \frac{\alpha\beta\exp\left(-t^\beta H(t)\right)}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau, \\ &\le \exp\left(-t^\beta H(t)\right)(1-\alpha)\beta t^{\beta}h(t)\Phi(t) \\ &+ \frac{\exp\left(-t^\beta H(t)\right)\alpha\beta\varepsilon}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}\begin{bmatrix}\beta\tau^{\beta-1}H(\tau) \\ +\tau^\beta h(\tau)\end{bmatrix} \\ &\qquad \times\exp\left(-\tau^\beta H(\tau)\right)d\tau. \end{aligned} \quad (106) $$

In the view (i)

$$ \begin{aligned} &\le \exp\left(-t^\beta H(t)\right)(1-\alpha)\varepsilon'\beta t^\beta h(t)\exp\left(t^\beta H(t)\right) \\ &+ \frac{\alpha\beta\varepsilon' a^\alpha}{\Gamma(\alpha)}\exp\left(t^\beta H(t)\right)\exp\left(-t^\beta H(t)\right), \\ &\le \varepsilon'\begin{pmatrix}(1-\alpha)\beta a^\alpha h(t_1) \\ +\frac{\alpha\beta a^\alpha}{\Gamma(\alpha)}\end{pmatrix}, \\ &\le \frac{\varepsilon}{\left((1-\alpha)\beta a^\alpha h(t_1)+\frac{\alpha\beta a^\alpha}{\Gamma(\alpha)}\right)}\begin{pmatrix}(1-\alpha)\beta a^\alpha h(t_1) \\ +\frac{\alpha\beta a^\alpha}{\Gamma(\alpha)}\end{pmatrix}, \\ &\le \varepsilon. \end{aligned} \quad (107) $$

Therefore

$$ \lim_{t\to t_0^+}\exp\left(-t^\beta H(t)\right)\Omega(t) = 0. \quad (108) $$

Thus

$$ \exp\left(-t^\beta H(t)\right)\Omega(t) \le 0 \text{ for } t > t_0, \quad (109) $$

which implies

$$ \begin{aligned} &(1-\alpha)\beta t^{\beta-1}h(t)\Phi(t) \\ &+ \frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\Phi(\tau)d\tau, \\ &\le 0. \end{aligned} \quad (110) $$

contradiction, thus

$$ \Phi(t) = 0. \quad (111) $$

□

Theorem 7. *Let $f(t,y(t))$ be as presented before and exp $\left(t^\beta H(t)\right)$ as $t\to t_0^+$ uniformly with respect to y, $\overline{y}\in[-\delta,\delta]$, $\delta>0$ arbitrary $h(t)$ and $H(t)$ are the same as previously. Then equation (92) has a unique solution.*

Proof. Let $y(t)$ and $\overline{y}(t)$ be solutions of equation (92).

$$ \begin{aligned} &|y(t)-\overline{y}(t)| \le (1-\alpha)\beta t^{\beta-1}\left|f(t,y(t))-f(t,\overline{y}(t))\right| \\ &+ \frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\left|f(\tau,y(\tau))-f(\tau,\overline{y}(\tau))\right|d\tau, \\ &\le (1-\alpha)\beta t^{\beta-1}\left|f(t,y(t))-f(t,\overline{y}(t))\right| \\ &+ \frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\left|y-\overline{y}\right|d\tau, \\ &\le (1-\alpha)\beta t^{\beta-1}\left|f(t,y(t))-f(t,\overline{y}(t))\right| \\ &+ \frac{\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha}h(\tau)\left|y-\overline{y}\right|d\tau. \end{aligned} \quad (112) $$

In the view of (i), we get

$$|y(t)-\overline{y}(t)| \leq (1-\alpha)\beta t^{\beta-1}h(t)\,|y(t)-\overline{y}(t)| + \frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\,|y(\tau)-\overline{y}(\tau)|\,d\tau,$$
$$\leq (1-\alpha)\beta t^{\beta-1}h(t)\,|y(t)-\overline{y}(t)| + \frac{\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta}(t-\tau)^{\alpha}h(\tau)\,|y(\tau)-\overline{y}(\tau)|\,d\tau,$$
$$\leq (1-\alpha)\beta\varepsilon' h(t)t^{\beta}\exp\left(t^{\beta}H(t)\right) + \frac{\alpha\beta\varepsilon'}{\Gamma(\alpha)}\int_{t_0}^{t}a^{\alpha}\tau^{\beta}h(\tau)\exp\left(\tau^{\beta}H(\tau)\right)d\tau. \tag{113}$$

$$|y(t)-\overline{y}(t)| \leq (1-\alpha)\beta\varepsilon' h(t)a^{\beta}\exp\left(t^{\beta}H(t)\right) + \frac{\alpha\beta\varepsilon' a^{\alpha}}{\Gamma(\alpha)}\int_{t_0}^{t}\left(\tau^{\beta}h(\tau)+\beta\tau^{\beta-1}H(\tau)\right)\exp\left(\tau^{\beta}H(\tau)\right)d\tau,$$
$$\leq \left((1-\alpha)\beta\varepsilon' h(t)a^{\beta}+\frac{\alpha\beta\varepsilon' a^{\alpha}}{\Gamma(\alpha)}\right)\exp\left(t^{\beta}H(t)\right),$$
$$\leq \left((1-\alpha)\beta\varepsilon' h(t_1)a^{\beta}+\frac{\alpha\beta\varepsilon' a^{\alpha}}{\Gamma(\alpha)}\right)\exp\left(t^{\beta}H(t)\right),$$
$$\leq \frac{\varepsilon\left((1-\alpha)\beta h(t_1)a^{\beta}+\frac{\alpha\beta a^{\alpha}}{\Gamma(\alpha)}\right)}{\left((1-\alpha)\beta h(t_1)a^{\beta}+\frac{\alpha\beta a^{\alpha}}{\Gamma(\alpha)}\right)}\exp\left(t^{\beta}H(t)\right),$$
$$\leq \varepsilon\exp\left(t^{\beta}H(t)\right). \tag{114}$$

From the above Lemma

$$y(t)-\overline{y}(t)=0. \tag{115}$$

□

Corollary 2. *Let the condition in above theorem hold, then*

i) $(f(t,y)-f(t,\overline{y}))(y-\overline{y}) \leq h(t)(y-\overline{y})^{\overline{\beta}}$, $\forall(t,\overline{y}),(t,y)\in S_+,\ \beta\in(1,2]$.
ii) $h(t_1)=\max\limits_{t\in(t_0,a)}h(t)$,

$$1-\overline{\beta}(1-\alpha)\beta\left(\overline{t}_0\right)^{\beta-1}\Lambda>0,$$

iii) ${}^{FFM}_{t_0}J^{\alpha,\beta}_{t}h(t)$ *exists.*

Proof. Let $y(t)$ and $\overline{y}(t)$ be two solutions of equation (92) then, let set $\overline{\Omega}(t_0)=0$ then we get

$${}^{FFM}_{t_0}D^{\alpha,\beta}_{t}\overline{\Omega}(t)=\frac{1}{\beta t^{\beta-1}}\,{}^{ABC}_{t_0}D^{\alpha}_{t}\overline{\Omega}(t),$$
$$=\frac{1}{\beta t^{\beta-1}(1-\alpha)}\int_{t_0}^{t}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)\overline{\Omega}'d\tau,$$
$$=\frac{1}{\beta t^{\beta-1}(1-\alpha)}\int_{t_0}^{t}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)\times\left[\overline{\beta}(y-\overline{y})^{\overline{\beta}}(y-\overline{y})'\right]d\tau. \tag{116}$$

Let $m=\min\limits_{\overline{t}\in[t_0,t]}|y-\overline{y}|^{\beta-1}$, $M=\max\limits_{\overline{t}\in[t_0,t]}|y(\overline{t})-\overline{y}(\overline{t})|$. We define

$$\Lambda=\begin{cases} m, & \text{if } y'-\overline{y}'<0,\\ M, & \text{if } y'-\overline{y}'>0.\end{cases} \tag{117}$$

Therefore

$${}^{FFM}_{t_0}D^{\alpha,\beta}_{t}\overline{\Omega}(t)\leq\frac{\overline{\beta}\Lambda}{\beta t^{\beta-1}(1-\alpha)}\int_{t_0}^{t}E_{\alpha}\left(-\frac{\alpha}{1-\alpha}(t-\tau)^{\alpha}\right)(y-\overline{y})'d\tau,$$
$$\leq\overline{\beta}\Lambda\left({}^{FFM}_{t_0}D^{\alpha,\beta}_{t}y(t)-{}^{FFM}_{t_0}D^{\alpha,\beta}_{t}\overline{y}(t)\right),$$
$$\leq\overline{\beta}\Lambda\left(f(t,y)-f(t,\overline{y})\right),$$
$$\leq\overline{\beta}\Lambda h(t)(y-\overline{y})^{\overline{\beta}}. \tag{118}$$

$${}^{RL}_{t_0}D^{\alpha}_{t}\overline{\Omega}(t)=\overline{\beta}\Lambda\beta t^{\beta-1}, \tag{119}$$
$$\leq\overline{\beta}\Lambda h(t)\overline{\Omega}(t),$$
$${}^{ABR}_{t_0}D^{\alpha}_{t}\overline{\Omega}(t)\leq\beta t^{\beta-1}\Lambda h(t)\overline{\Omega}(t)\overline{\beta}.$$

$$\overline{\Omega}(t)\leq\overline{\beta}(1-\alpha)\beta t^{\beta-1}\Lambda h(t)\overline{\Omega}(t) + \frac{\alpha\beta\overline{\beta}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}\Lambda h(\tau)\overline{\Omega}(\tau)d\tau,$$
$$\leq\overline{\beta}(1-\alpha)\beta\left(\overline{t}_0\right)^{\beta-1}\Lambda h(t_1)\overline{\Omega}(t) + \frac{\Lambda\alpha\beta\overline{\beta}}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\overline{\Omega}(\tau)d\tau. \tag{120}$$

If we take as

$$A=\frac{\Lambda\overline{\beta}}{1-\overline{\beta}(1-\alpha)\beta\left(\overline{t}_0\right)^{\beta-1}\Lambda}, \tag{121}$$

$$\overline{\Omega}(t)\leq\frac{A\alpha\beta}{\Gamma(\alpha)}\int_{t_0}^{t}\tau^{\beta-1}(t-\tau)^{\alpha-1}h(\tau)\overline{\Omega}(\tau)d\tau. \tag{122}$$

By the Gronwall inequality we set

$$\overline{\Omega}(t) \leq o \exp\left(A^{FFP}_{t_0} J^{\alpha,\beta}_{t} h(t)\right), \tag{123}$$
$$\overline{\Omega}(t) \leq 0.$$

Therefore we have $\overline{\Omega}(t) \leq 0$ which is a contraction therefore

$$\overline{\Omega}(t) = 0 \Rightarrow y(t) = \overline{y}(t),\ \forall t \in [t_0, a]\,. \tag{124}$$

□

7. Conclusion

Witte provided a set of conditions under which a given nonlinear ordinary differential equation admits unique solutions. This was established when the differential operator was in integer order. Based on the framework of Witte, we have presented a detailed analysis of the uniqueness of nonlinear ordinary differential equations with fractal-fractional derivatives.

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