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# Vector optimization with cone semilocally preinvex functions

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**Abstract.** In this paper we introduce cone semilocally preinvex, cone semilocally quasi preinvex and cone semilocally pseudo preinvex functions and study their properties. These functions are further used to establish necessary and sufficient optimality conditions for a vector minimization problem over cones. A Mond-Weir type dual is formulated for the vector optimization problem and various duality theorems are proved.

**Keywords:** Vector optimization, semilocally preinvex functions, cones, optimality, duality.
**AMS Classification:** 90C30; 90C25, 90C46

## 1. Introduction

The concept of semilocally convex functions was introduced by Ewing [1] who applied the notion to provide sufficient optimality conditions in variational and control problems. These functions have some important properties such as local minimum of a semilocally convex function defined on a locally star shaped set is a global minimum and non-negative linear combination of semilocally convex functions is also semilocally convex. Kaul and Kaur [3] defined semilocally quasi convex and semilocally pseudo convex functions. Suneja and Gupta [14] defined the (strict) semilocally pseudo convexity at a point with respect to a set.

By using these concepts Kaul and Kaur [4, 5] Kaur [7] and Suneja and Gupta [14] obtained optimality conditions and duality results for a class of non-linear programming problems. Gupta and Vartak [8] defined $\rho$-semilocally convex and related functions and studied sufficient optimality conditions for a non-linear program involving these functions. Mukherjee and Mishra [9] and Preda [10] discussed optimality results for a multiobjective programming problem using semilocally convex functions. Weir [16] introduced cone-semilocally convex functions and studied optimality conditions and duality theorems for vector optimization problems over cones.

Preda and Stancu-Minasian [12] discussed the Fritz-John and Karush-Kuhn-Tucker type optimality conditions for weak vector minima using semilocally preinvex functions. Stancu-Minasian [13] established optimality and duality results for a non-linear fractional programming problem where the functions involved were semilocally preinvex, semilocally quasi preinvex and semilocally pseudo preinvex. Preda [11] studied optimality and duality for a multiobjective fractional programming problem involving semilocally preinvex functions. Suneja et al. [15] introduced $\rho$-semilocally preinvex, $\rho$-semilocally quasi preinvex and $\rho$-semilocally pseudo preinvex functions and proved optimality conditions and duality results for a multiobjective non-linear programming problem using the above defined functions.

In this paper we introduce $K$-semilocally preinvex, $K$-semilocally naturally quasi preinvex, $K$-semilocally quasi preinvex and $K$-

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semilocally pseudo preinvex functions where $K$ is a closed convex cone with nonempty interior. Their properties and interrelations are established. Necessary and sufficient optimality conditions are obtained for a vector optimization problem over cones by using the above defined functions. A Mond-Weir type dual is associated with the optimization problem and duality results are studied.

## 2. Preliminaries and Definitions

Let $S \subseteq R^n$ be a nonempty set and $\eta : S \times S \to R^n$ be a vector valued function.

**Definition 2.1 [11]** The set $S$ is said to be $\eta$-locally star shaped at $\bar{x} \in S$ if for each $x \in S$, there exists a positive number $a_\eta(x,\bar{x}) \le 1$ such that $\bar{x} + \lambda\eta(x,\bar{x}) \in S$, for $0 < \lambda < a_\eta(x,\bar{x})$.

If $\eta(x,\bar{x}) = x - \bar{x}$ then $\eta$-locally starshaped set reduces to locally star shaped set [1].

Let $S \subseteq R^n$ be an $\eta$-locally star shaped at $\bar{x} \in S$, $K \subseteq R^m$ be a closed convex cone with nonempty interior and let $f : S \to R^m$ be a vector valued function.

**Definition 2.2.** The function $f$ is said to be ***K*-semilocally preinvex** ($K$-slpi) at $\bar{x} \in S$ with respect to $\eta$ if corresponding to each $x \in S$, there exists a positive number $d_\eta(x,\bar{x}) \le a_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x,\bar{x}).$$

$f$ is said to be $K$-slpi on $S$ if it is $K$-slpi at each $\bar{x} \in S$.

The following theorem gives a characterization of cone semilocally preinvex functions.

**Theorem 2.1.** The function $f$ is $K$-semilocally preinvex with respect to $\eta$ if and only if its epigraph

$$\text{Epi}(f) = \{(x, y) : x \in S,\ y \in f(x) + K\} \subseteq R^n \times R^m$$

is $\eta$-locally star shaped in the first component and locally star shaped in the second component.

**Proof.** First suppose that $f$ is $K$-slpi on $S$, with respect to $\eta$.

Let $(x_1, y_1)$ and $(x_2, y_2) \in \text{Epi}(f)$, then $x_1, x_2 \in S$, $y_1 \in f(x_1) + K$ and $y_2 \in f(x_2) + K$,

which implies that $y_1 = f(x_1) + k_1$ and $y_2 = f(x_2) + k_2$ where $k_1, k_2 \in K$.

Since $f$ is $K$-slpi on $S$, there exists a positive number $a_\eta(x_1,x_2) \le 1$ such that $x_2 + t\eta(x_1,x_2) \in S$ for $0 < t < a_\eta(x_1,x_2)$ and there exists a positive number $d_\eta(x_1,x_2) \le a_\eta(x_1,x_2)$ such that

$$tf(x_1) + (1-t)f(x_2) - f(x_2 + t\eta(x_1,x_2)) \in K, \quad \text{for } 0 < t < d_\eta(x_1,x_2)$$

$$\Rightarrow \quad t(y_1 - k_1) + (1-t)(y_2 - k_2) - f(x_2 + t\eta(x_1,x_2)) \in K$$

$$\Rightarrow \quad ty_1 + (1-t)y_2 - f(x_2 + t\eta(x_1,x_2)) \in K + (tk_1 + (1-t)k_2) = K$$

$$\Rightarrow \quad ty_1 + (1-t)y_2 \in f(x_2 + t\eta(x_1,x_2)) + K$$

$$\Rightarrow \quad \left(x_2 + t\eta(x_1,x_2), ty_1 + (1-t)y_2\right) \in \text{Epi}(f),$$

for $0 < t < d_\eta(x_1,x_2)$

$\Rightarrow$ Epi($f$) is $\eta$-locally star shaped in first component and locally star shaped in second component.

**Conversely,** let Epi($f$) be $\eta$-locally star shaped in the first component and locally star shaped in the second component. Let $x_1, x_2 \in S$, then

$$(x_1, f(x_1)), (x_2, f(x_2)) \in \text{Epi}(f).$$

Thus there exists a positive number $a_\eta(x_1,x_2) \le 1$ such that

$$(x_2 + t\eta(x_1,x_2), tf(x_1) + (1-t)f(x_2)) \in \text{Epi}(f), \quad \text{for } 0 < t < a_\eta(x_1,x_2)$$

$$\Rightarrow \quad tf(x_1) + (1-t)f(x_2) \in f(x_2 + t\eta(x_1,x_2)) + K$$

$$\Rightarrow \quad tf(x_1) + (1-t)f(x_2) - f(x_2 + t\eta(x_1,x_2)) \in K,$$

for $0 < t < a_\eta(x_1,x_2)$.

Hence $f$ is $K$-slpi on $S$.

□

**Remark 2.1.** If $m = n$ and $K = R^n_+$ then $K$-semilocally preinvex functions reduce to semilocally preinvex functions defined by Preda [11]. We now give an example of a function which is $K$-slpi but fails to be slpi.

**Example 2.1.** Consider the set $S = R \setminus E$ where $E = \left[\frac{-1}{2}, \frac{1}{2}\right] \cup \{2\}$. Thus the set $S$ is $\eta$-locally starshaped, where

$$\eta(x,\bar{x}) = \begin{cases} x-\bar{x}, & x,\bar{x} > \frac{1}{2}, x \neq 2, \bar{x} \neq 2 \\ & \text{or } x, \bar{x} < -\frac{1}{2} \\ \bar{x}-x, & x > \frac{1}{2}, x \neq 2, \bar{x} < -\frac{1}{2} \\ & \text{or } \bar{x} > \frac{1}{2}, \bar{x} \neq 2\ x < -\frac{1}{2} \end{cases}$$

and

$$a_\eta(x,\bar{x}) = \begin{cases} \left|\frac{2-\bar{x}}{x-\bar{x}}\right|, & \text{if } \frac{1}{2} < \bar{x} < 2,\ 2 < x \\ & \text{or } \frac{1}{2} < \bar{x} < 2,\ x < -\frac{1}{2} \\ \frac{\bar{x}-2}{\bar{x}-x}, & \text{if } 2 < \bar{x},\ \frac{1}{2} < x < 2 \\ 1, & \text{elsewhere} \end{cases}$$

Consider the function $f : S \to R^2$ defined by

$$f(x) = \begin{cases} (x,0), & \frac{1}{2} < x, x \neq 2 \\ (0,-x), & x < -\frac{1}{2} \end{cases}$$

Let $K = \{(x, y): y \leq 0,\ y \leq -x\}$

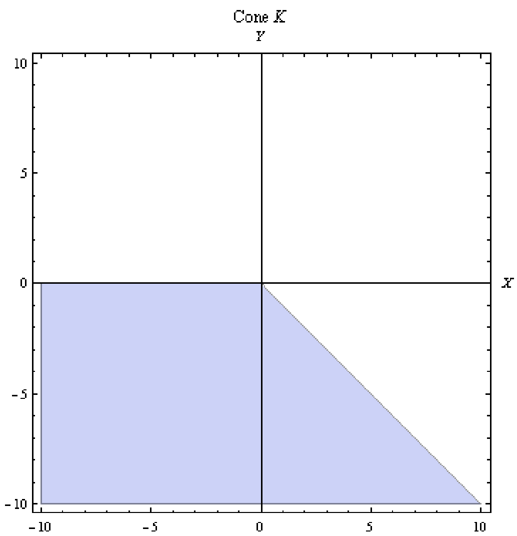


**Figure 1**

Thus, $f$ is $K$-slpi at $\bar{x} = -1$ because

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K,$$

$$\text{for } 0 < t < d_\eta(x,\bar{x}) = a_\eta(x,\bar{x}).$$

The function $f$ fails to be slpi at $\bar{x} = -1$ because for $x = 1$, there does not exist any positive number $d_\eta(x,\bar{x}) \leq a_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x}))$$

$\geqq 0$ for $0 < t < d_\eta(x,\bar{x})$.

□

**Remark 2.2** If $\eta(x,\bar{x}) = x - \bar{x}$ then $K$-semilocally preinvex functions reduce to $K$-semilocally convex functions defined by Weir [16].

We now give an example of a $K$-slpi function which fails to be $K$-semilocally convex.

**Example 2.2** The function $f$ considered in Example 2.1 is $K$-slpi at $\bar{x} = -1$ but it fails to be $K$-semilocally convex at $\bar{x} = -1$ because for $x = 1$, there does not exist any positive real number $d < 1$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(tx + (1-t)\bar{x}) \in K$$

for $0 < t < d$.

**Theorem 2.2** Let $f : S \to R^m$ be $K$-slpi on an $\eta$-locally star shaped set $S \subseteq R^n$. Then the set $Z = f(S) + K$ is locally star shaped.

**Proof.** Let $z,\ \bar{z} \in Z$, then there exist $x,\ \bar{x} \in S,\ k,\ \bar{k} \in K$ such that

$$z = f(x) + k, \quad \bar{z} = f(\bar{x}) + \bar{k}. \tag{2.1}$$

Since $S$ is an $\eta$-locally star shaped set and $x, \bar{x} \in S$, therefore there exists a maximum positive number $a_\eta(x, \bar{x}) \leq 1$ such that

$$\bar{x} + t\eta(x, \bar{x}) \in S \text{ for } 0 < t < a_\eta(x, \bar{x}).$$

As $f$ is $K$-slpi on $S$, there exists a positive number $d_\eta(x, \bar{x}) \leq a_\eta(x, \bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x, \bar{x})) \in K,$$
$$\text{for } 0 < t < d_\eta(x, \bar{x})$$

$$\Rightarrow \quad tf(x) + (1-t)f(\bar{x}) \in f(\bar{x} + t\eta(x, \bar{x})) + K.$$

On using (2.1) we have,

$$t(z-k) + (1-t)(\bar{z} - \bar{k}) \in f(S) + K$$

$$\Rightarrow \quad tz + (1-t)\bar{z} \in f(S) + tk + (1-t)\bar{k} + K$$

$$\Rightarrow \quad tz + (1-t)\bar{z} \in f(S) + K$$

$$\text{for } 0 < t < d_\eta(x, \bar{x}),$$

as $K$ is a convex cone.

Hence $Z$ is locally star shaped.

□

**Theorem 2.3.** Let $f$ be $K$-slpi on $S$ with respect to $\eta$ then for each $y \in R^m$, the set $S_f(y) = \{x \in S : y \in f(x) + K\}$ is $\eta$-locally star shaped.

**Proof.** Let $x,\ \bar{x} \in S_f(y), y \in R^m$, then $x,\ \bar{x} \in S$ such that

$$y \in f(x) + K \text{ and } y \in f(\bar{x}) + K,$$

Thus there exist $k, \bar{k} \in K$ such that

$$y = f(x) + k \text{ and } y = f(\bar{x}) + \bar{k}. \qquad (2.2)$$

Since $f$ is $K$-slpi with respect to $\eta$ therefore there exists a positive number $d_\eta(x,\bar{x}) \le a_\eta(x,\bar{x})$ where $\eta : S \times S \to R^n$ such that

$$tf(x) + (1-t) f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad 0 < t < d_\eta(x,\bar{x})$$

On using (2.2) we get

$$t(y-k) + (1-t)(y-\bar{k}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad 0 < t < d_\eta(x,\bar{x})$$

$$\Rightarrow \quad y - (tk + (1-t)\bar{k}) - f(\bar{x} + t\eta(x,\bar{x})) \in K$$

$$\Rightarrow \quad y - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x,\bar{x})$$

$$\Rightarrow \quad \bar{x} + t\eta(x,\bar{x}) \in S_f(y), \quad \text{for } 0 < t < d_\eta(x,\bar{x})$$

Hence $S_f(y)$ is $\eta$-locally star shaped.

□

We now give the definition of $\eta$-semi differentiable function.

**Definition 2.3.** The function $f : S \to R^m$ is said to be ***η*-semi differentiable** at $\bar{x} \in S$ if

$$(df)^+(\bar{x}, \eta(x,\bar{x})) = \lim_{t \to 0^+} \frac{1}{t}[f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})]$$

exists for each $x \in S$.

**Remark 2.3.**

(1) If $\eta(x,\bar{x}) = x - \bar{x}$ then $\eta$-semi differentiability of $f$ reduces to (one sided) directional differentiability of $f$ at $\bar{x}$ in the direction $x - \bar{x}$, as considered by Weir [16].

(2) If $m = 1$ and $\eta(x,\bar{x}) = x - \bar{x}$, then $\eta$-semi differentiability reduces to semi differentiability [6].

Let $f : S \to R^m$ be $\eta$-semi differentiable at $\bar{x} \in S$.

In the following result we give another property of $K$-slpi functions.

**Theorem 2.4.** If $f$ is $K$-slpi at $\bar{x}$ then $f(x) - f(\bar{x}) - (df)^+(\bar{x}, \eta(x,\bar{x})) \in K \quad \forall\ x \in S$.

**Proof.** Since the function $f$ is $K$-slpi at $\bar{x}$ with respect to $\eta$, therefore corresponding to each $x \in S$ there exists a positive number $d_\eta(x,\bar{x}) \le a_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t) f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x,\bar{x})$$

which implies

$$f(x) - f(\bar{x}) - \frac{1}{t}[f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})] \in K, \quad 0 < t < d_\eta(x,\bar{x})$$

Since $K$ is a closed cone, therefore taking limit as $t \to 0^+$, we get

$$f(x) - f(\bar{x}) - (df)^+(\bar{x}, \eta(x,\bar{x})) \in K, \ \forall\ x \in S.$$

□

We now introduce semilocally naturally quasi preinvex functions over cones.

**Definition 2.4.** The function $f$ is said to be ***K*-semilocally naturally quasi preinvex** ($K$-slnqpi) at $\bar{x}$, with respect to $\eta$ if

$$-(f(x) - f(\bar{x})) \in K$$

$$\Rightarrow \quad -(df)^+(\bar{x}, \eta(x,\bar{x})) \in K.$$

**Remark 2.4.** If $m = n$ and cone $K = R^n_+$, $K$-slnqpi functions reduce to slqpi functions defined by Preda [11].

**Theorem 2.5.** If the set $S_f(y) = \{x \in S : y \in f(x) + K\}$ is $\eta$-locally star shaped for each $y \in R^m$, then $f$ is $K$-semilocally naturally quasi preinvex on $S$ with respect to same $\eta$.

**Proof.** Let $S_f(y)$ be $\eta$-locally star shaped for each $y \in R^m$.

Let $x,\ \bar{x} \in S$ such that $-(f(x) - f(\bar{x})) \in K$.

Denoting $y = f(\bar{x})$, we get $-(f(x) - y) \in K$.

$$\Rightarrow \quad y \in f(x) + K,$$

$$\Rightarrow \quad x \in S_f(y)$$

Also $\bar{x} \in S_f(y)$ as $0 \in K$.

Since $S_f(y)$ is $\eta$-locally star shaped, therefore there exists a maximum positive number $a_\eta(x,\bar{x}) \le 1$ such that

$$\bar{x} + t\eta(x,\bar{x}) \in S_f(y), \text{ for } 0 < t < a_\eta(x, \bar{x})$$

which implies,

$$y \in f(\bar{x} + t\eta(x,\bar{x})) + K \text{ for } 0 < t < a_\eta(x, \bar{x}).$$

Thus,

$$f(\bar{x}) \in f(\bar{x} + t\eta(x,\bar{x})) + K, \quad \text{for } 0 < t < a_\eta(x,\bar{x})$$

$$\Rightarrow \quad -(f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x,\bar{x}).$$

$$\Rightarrow \quad \frac{-1}{t}(f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})) \in K, \quad 0 < t < d_\eta(x,\bar{x})$$

Since $K$ is a closed cone, therefore taking limit as $t \to 0^+$, we get

$$-(df)^+(\bar{x},\eta(x,\bar{x})) \in K$$

Thus $\quad -(f(x) - f(\bar{x})) \in K$

$$\Rightarrow \quad -(df)^+(\bar{x},\eta(x,\bar{x})) \in K.$$

Hence $f$ is $K$-slnqpi on $S$.

□

**Theorem 2.6.** If $f$ is $K$-slpi at $\bar{x} \in S$ with respect to $\eta$ then $f$ is $K$-slnqpi at $\bar{x}$, with respect to same $\eta$.

**Proof.** Let $f$ be $K$-slpi at $\bar{x}$, then there exists a positive number $d_\eta(x,\bar{x}) \le a_\eta(x,\bar{x})$ such that

$$tf(x) + (1-t)f(\bar{x}) - f(\bar{x} + t\eta(x,\bar{x})) \in K, \quad \text{for } 0 < t < d_\eta(x,\bar{x}). \tag{2.3}$$

Suppose that,

$$-(f(x) - f(\bar{x})) \in K$$

then,

$$-t(f(x) - f(\bar{x})) \in K, \quad \text{for } t > 0. \tag{2.4}$$

Adding (2.3) and (2.4) we have

$$-[f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})] \in K, \quad \text{for } 0 < t < d_\eta(x,\bar{x}).$$

$$\Rightarrow \quad \frac{-1}{t}(f(\bar{x} + t\eta(x,\bar{x})) - f(\bar{x})) \in K, \quad 0 < t < d_\eta(x,\bar{x}).$$

Since $K$ is a closed cone, therefore taking limit as $t \to 0^+$, we get

$$-(df)^+(\bar{x},\eta(x,\bar{x})) \in K.$$

Thus

$$-(f(x) - f(\bar{x})) \in K$$

$$\Rightarrow \quad -(df)^+(\bar{x},\eta(x,\bar{x})) \in K.$$

Hence $f$ is $K$-slnqpi at $\bar{x}$ with respect to same $\eta$.

□

The converse of the above theorem may not hold as can be seen from the following example.

**Example 2.3.** Consider the set $S = R \setminus E$, where $E = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{2\}$. Then as discussed in Example 2.1, $S$ is $\eta$-locally starshaped.

Consider the function $f : S \to R^2$ defined by

$$f(x) = \begin{cases} (-x^2, 0), & x < -\frac{1}{2} \\ (0, -x), & x > \frac{1}{2},\ x \ne 2. \end{cases}$$

and $K = \{(x,y) : y \le 0, y \ge x\}$

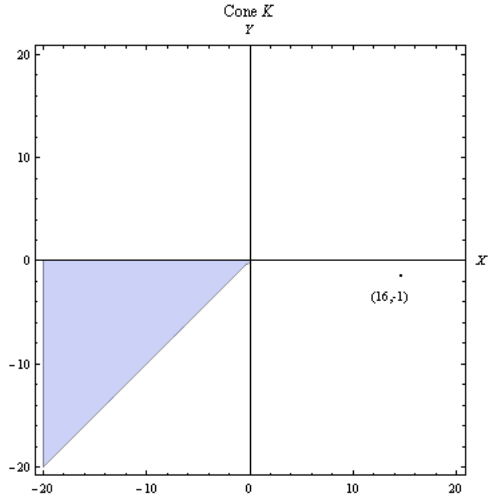


**Figure 2**

The function $f$ is $K$-slnqpi at $\bar{x} = -2$ because

$$-(f(x) - f(\bar{x})) \in K$$

$$\Rightarrow \quad -2 \le x < -\frac{1}{2}$$

$$\Rightarrow \quad -(df)^+(\bar{x},\eta(x,\bar{x})) = (-4(x+2), 0) \in K.$$

The function $f$ fails to be $K$-slpi at $\bar{x} = -2$ by Theorem 2.4, because for $x = 1$

$$f(x) - f(\bar{x}) - (df)^+(\bar{x},\eta(x,\bar{x})) = (16,-1) \notin K.$$

**Definition 2.5.** The function $f : S \to R^m$ is said to be ***K*-semilocally quasi preinvex** ($K$-slqpi) at $\bar{x}$ with respect to $\eta$, if

$$(f(x) - f(\bar{x})) \notin \operatorname{int} K$$

$$\Rightarrow \quad -(df)^+(\bar{x},\eta(x,\bar{x})) \in K.$$

**Theorem 2.7.** If $K$ is a pointed cone and $f$ is $K$-slqpi at $\bar{x}$ then $f$ is $K$-slnqpi at $\bar{x}$ with respect to same $\eta$.

**Proof.** Let $K$ be a pointed cone and $f$ be $K$-slqpi at $\bar{x}$ with respect to $\eta$, then,

$$(f(x) - f(\bar{x})) \notin \operatorname{int} K$$
$$\Rightarrow \quad -(df)^{+}(\bar{x}, \eta(x, \bar{x})) \in K. \tag{2.5}$$

Suppose that

$$-(f(x) - f(\bar{x})) \in K. \tag{2.6}$$

Since $K$ is pointed, $K \cap (-K) = \{0\}$

$\Rightarrow \quad \operatorname{int} K \cap (-K) = \phi$

$\Rightarrow \quad -K \subset R^m \setminus \operatorname{int} K$.

In view of (2.5) we get $f(x) - f(\bar{x}) \notin \operatorname{int} K$.

Thus by (2.6) we have $-(df)^{+}(\bar{x}, \eta(x, \bar{x})) \in K$.

Hence $f$ is $K$-slnqpi.

□

The converse of the above theorem may not hold, as can be seen by the following example.

**Example 2.4.** The function $f$ considered in Example 2.3 is $K$-slnqpi at $\bar{x} = -2$. But $f$ fails to be $K$-slqpi at $\bar{x} = -2$ because for $x = 1$

$$f(x) - f(\bar{x}) = (4, -1) \notin \operatorname{int} K$$

whereas $-(df)^{+}(\bar{x}, \eta(x, \bar{x})) = (12, 0) \notin K$.

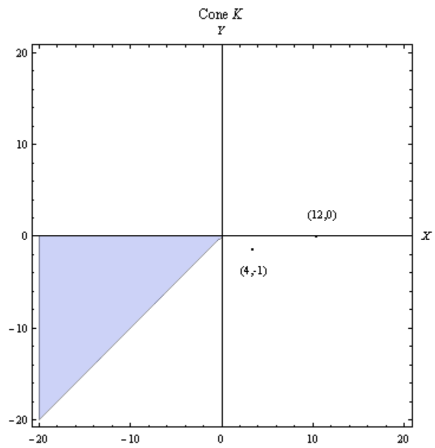


**Figure 3**

**Remark 2.5.** The following diagram illustrates the relation between $K$-slpi, $K$-slnqpi and $K$-slqpi functions.

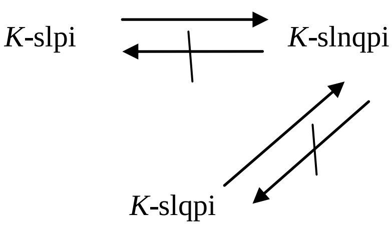


**Figure 4**

It can be seen from Example 2.1 that $f$ fails to be $K$-slqpi because for $\bar{x} = -1, x = -2$, $f(x) - f(\bar{x}) = (0,1) \notin \operatorname{int} K$ however $-(df) + (\bar{x}, \eta(x, \bar{x})) = (0,1) \notin K$.

We now give an example to show that a $K$-slqpi function need not be $K$-slpi.

**Example 2.5.** Consider the set $S = R \setminus E$, where $E = \left[-\frac{1}{2}, \frac{1}{2}\right] \cup \{2\}$. Then as discussed in Example 2.1, $S$ is $\eta$-locally starshaped.

Consider the function $f : S \to R^2$ defined by

$$f(x) = \begin{cases} (-x^2, 0), & x < -\frac{1}{2} \\ (0, -x), & x > \frac{1}{2},\ x \neq 2. \end{cases}$$

and $K = \{(x, y) : y \leq x,\ x \geq 0\}$.

The function $f$ is $K$-slqpi at $\bar{x} = -2$ because

$$(f(x) - f(\bar{x})) \notin \operatorname{int} K$$
$$\Rightarrow \quad x \leq -2$$
$$\Rightarrow \quad -(df)^{+}(\bar{x}, \eta(x, \bar{x})) = (-4(x+2), 0) \in K.$$

The function $f$ fails to be $K$-slpi at $\bar{x} = -2$ by Theorem 2.4, because for $x = -3$

$$f(x) - f(\bar{x}) - (df)^{+}(\bar{x}, \eta(x, \bar{x})) = (-1, 0) \notin K.$$

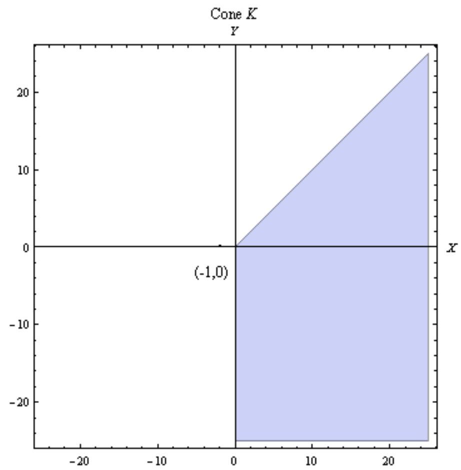


**Figure 5**

The next definition introduces cone semilocally pseudo preinvex functions.

**Definition 2.6.** The function $f : S \to R^m$ is said to be **K-semilocally pseudo preinvex** (K-slppi) at $\bar{x}$, with respect to $\eta$ if

$$-(df)^{+}(\bar{x}, \eta(x, \bar{x})) \notin \operatorname{int} K$$
$$\Rightarrow \quad -(f(x) - f(\bar{x})) \notin \operatorname{int} K$$

## 3. Optimality Conditions

We consider the vector optimization problem

**(VOP)** $K$-minimize $f(x)$

subject to $-g(x) \in Q$

$h(x) \in O$

where $f : S \to R^m$, $g : S \to R^p$ and $h : S \to R^k$ are $\eta$-semi differentiable functions with respect to same $\eta$ and $S \subseteq R^n$ is a nonempty $\eta$-locally star shaped set and $O = \{0_{R^k}\}$.

Let $K \subseteq R^m$ and $Q \subseteq R^p$ be closed convex cones having nonempty interior and let $X_0 = \{x \in S : -g(x) \in Q,\ h(x) \in O\}$ be the set of all feasible solutions of (VOP).

Let $F_0$: $(f, g, h) : S \to S'$ where $S \subseteq R^n$ is a nonempty set and $S' = R_m \times R_p \times R_k$ and $K_0 = (K \times Q \times O)$. If $F_0$ is $K_0$-slpi on $S$, that is $f$ is $K$-slpi, $g$ is $Q$-slpi and $h$ is $O$-slpi with respect to same $\eta$ then by Theorem 2.2, $F_0(S) + K_0$ is locally starshaped. If we assume, $F_0(S) + K_0$ to be a closed set, then, $F_0(S) + K_0$ becomes convex and the following alternative theorem follows on the lines of Illés and Kassay [2].

**Theorem 3.1 (Theorem of Alternative).** Let $F_0$ be $K_0$-slpi on $S$ such that $F_0(S) + K_0$ is closed with nonempty interior then exactly one of the following holds

(i) there exists $x \in S$ such that $-f(x) \in \operatorname{int} K$, $-g(x) \in \operatorname{int} Q$ and $-h(x) \in O$

(ii) there exists $\lambda \in K^+$, $\mu \in Q^+$ and $\nu \in R^k$ such that $\lambda^T f(x) + \mu^T g(x) + \nu^T h(x) \geq 0$, $(\lambda, \mu, \nu) \neq (0, 0, 0)$ for all $x \in S$.

We shall be using the following constraint qualification to prove the necessary optimality conditions for (VOP).

**Definition 3.1.** The constraint pair $(g, h)$ is said to satisfy **generalized Slater type constraint qualification** at $\bar{x}$ if there exists $x^* \in S$ such that

$$-g(x^*) \in \operatorname{int} Q \text{ and } h(x^*) \in O .$$

We now establish the necessary optimality conditions for (VOP).

**Theorem 3.2 (Necessary Optimality Conditions).** Let $F_1(x) = (f(x) - f(\bar{x}), g(x), h(x)) \quad \forall\ x \in S$ and $F_1(S) + (K \times Q \times O)$ be closed with nonempty interior. Let $\bar{x} \in X_0$ be a weak minimum of (VOP), $f$ be $K$-slpi, $g$ be $Q$-slpi and $h$ be $O$-slpi with respect to same $\eta$. Suppose that the pair $(g, h)$ satisfies generalized Slater type constraint qualification and $\eta(\bar{x}, \bar{x}) = 0$, then there exist $0 \neq \bar{\lambda} \in K^+$, $\bar{\mu} \in Q^+$ and $\bar{\nu} \in O^+$ such that

$$\bar{\lambda}^T (df)^+(\bar{x}, \eta(x, \bar{x})) + \bar{\mu}^T (dg)^+(\bar{x}, \eta(x, \bar{x})) + \bar{\nu}^T (dh)^+(\bar{x}, \eta(x, \bar{x})) \geq 0 , \quad \text{for all } x \in S \tag{3.1}$$

and
$$\bar{\mu}^T g(\bar{x}) = 0 . \tag{3.2}$$

**Proof.** Since $\bar{x}$ is a weak minimum of (VOP), therefore there does not exist any $x \in S$ such that

$$-F_1(x) \in \operatorname{int}(K \times Q) \times O$$

that is,

$$-\left[(f(x) - f(\bar{x})), g(x), h(x)\right] \in \operatorname{int}(K \times Q) \times O .$$

By Theorem 3.1, there exist $\bar{\lambda} \in K^+$, $\bar{\mu} \in Q^+$, $\bar{\nu} \in O^+$, $(\bar{\lambda}, \bar{\mu}, \bar{\nu}) \neq 0$ such that

$$\bar{\lambda}^T (f(x) - f(\bar{x})) + \bar{\mu}^T g(x) + \bar{\nu}^T h(x) \geq 0, \quad \text{for all } x \in S$$

$$\Rightarrow \quad \bar{\lambda}^T f(x) + \bar{\mu}^T g(x) + \bar{\nu}^T h(x) \geq \bar{\lambda}^T f(\bar{x}) \quad \text{for all } x \in S \tag{3.3}$$

Now, $\bar{\mu} \in Q^+$, and $-g(\bar{x}) \in Q$, $\bar{\mu}^T g(\bar{x}) \leq 0$.

By taking $x = \bar{x}$ in (3.3) and using $h(\bar{x}) = 0$, we get $\bar{\mu}^T g(\bar{x}) \geq 0$.

Thus

$$\bar{\mu}^T g(\bar{x}) = 0 . \tag{3.4}$$

From (3.2), (3.3) and $h(\bar{x}) = 0$, we have

$$\left(\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h\right)(x) - (\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h)(\bar{x}) \geq 0 \text{ for all } x \in S$$

As $\bar{x} + t\eta(x, \bar{x}) \in S$, for $0 < t < a_\eta(x, \bar{x})$ we have

$$\left(\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h\right)\left(\bar{x} + t\eta(x, \bar{x})\right) - (\bar{\lambda}^T f + \bar{\mu}^T g + \bar{\nu}^T h)(\bar{x}) \geq 0$$

which can be rewritten as,

$$\bar{\lambda}^T (f(\bar{x} + t\eta(x, \bar{x})) - f(\bar{x})) + \bar{\mu}^T (g(\bar{x} + t\eta(x, \bar{x})) - g(\bar{x}))$$

$$+\bar{\nu}^T(h(\bar{x}+t\eta(x,\bar{x}))-h(\bar{x}))\geq 0$$

Dividing by $t>0$ and taking limit as $t\to 0^+$ we get

$$\bar{\lambda}^T(df)^+(\bar{x},\eta(x,\bar{x}))+\bar{\mu}^T(dg)^+(\bar{x},\eta(x,\bar{x}))$$
$$+\bar{\nu}^T(dh)^+(\bar{x},\eta(x,\bar{x}))\geq 0,$$
$$\text{for all } x\in S. \tag{3.5}$$

Next, let if possible $\bar{\lambda}=0$, then (3.5) reduces to,

$$\bar{\mu}^T(dg)^+(\bar{x},\eta(x,\bar{x}))+\bar{\nu}^T(dh)^+(\bar{x},\eta(x,\bar{x}))\geq 0$$
$$\text{for all } x\in S. \tag{3.6}$$

Since $(g, h)$ is $(Q\times O)$-slpi at $\bar{x}$, therefore we have for every $x\in S$,

$$g(x)-g(\bar{x})-(dg)^+(\bar{x},\eta(x,\bar{x}))\in Q$$

and

$$h(x)-h(\bar{x})-(dh)^+(\bar{x},\eta(x,\bar{x}))\in O$$

$$\Rightarrow \quad \bar{\mu}^T g(x)-\bar{\mu}^T g(\bar{x})-\bar{\mu}^T(dg)^+(\bar{x},\eta(x,\bar{x}))\geq 0 \tag{3.7}$$

and

$$\bar{\nu}^T h(x)-\bar{\nu}^T h(\bar{x})-\bar{\nu}^T(dh)^+(\bar{x},\eta(x,\bar{x}))=0\,. \tag{3.8}$$

Adding (3.7), (3.8) and using (3.4) and $h(\bar{x})=0_k$, we get

$$\bar{\mu}^T g(x)+\bar{\nu}^T h(x)-\bar{\mu}^T(dg)^+(\bar{x},\eta(x,\bar{x}))$$
$$-\bar{\nu}^T(dh)^+(\bar{x},\eta(x,\bar{x}))\geq 0,$$
$$\text{for all } x\in S. \tag{3.9}$$

On using (3.6) we obtain,

$$\bar{\mu}^T g(x)+\bar{\nu}^T h(x)\geq 0,\ \text{for all } x\in S. \tag{3.10}$$

Again by generalized Slater type constraint qualification, there exists $x^*\in S$ such that,

$$-g(x^*)\in \operatorname{int} Q \quad \text{and} \quad h(x^*)\in O$$

which implies $\bar{\mu}^T g(x^*)<0$ and $\bar{\nu}^T h(x^*)=0$.

Adding the above we have

$$\bar{\mu}^T g(x^*)+\bar{\nu}^T h(x^*)<0$$

which is a contradiction to (3.10).

Hence $\bar{\lambda}\neq 0$.

□

The following theorem establishes sufficiency result for (VOP).

**Theorem 3.3 (Sufficient Optimality Conditions).** Let $\bar{x}\in X_0$ and $f$ be $K$-slppi, $g$ be $Q$-slqpi and $h$ be $O$-slnqpi at $\bar{x}$ with respect to same $\eta$. If there exist $0\neq\bar{\lambda}\in K^+$, $\bar{\mu}\in Q^+$ and $\bar{\nu}\in O^+$ such that (3.1) and (3.2) hold $\forall\ x\in X_0$ then $\bar{x}$ is a weak minimum of (VOP).

**Proof.** Let $x$ be feasible for (VOP) then $-g(x)\in Q$.

On using $\bar{\mu}\in Q^+$, we get

$$\bar{\mu}^T g(x)\leq 0. \tag{3.11}$$

In view of (3.2), (3.11) can be written as

$$\bar{\mu}^T(g(x)-g(\bar{x}))\leq 0 \tag{3.12}$$

If $\bar{\mu}\neq 0$, then from (3.12), $g(x)-g(\bar{x})\notin \operatorname{int} Q$.

Since $g$ is $Q$-slqpi at $\bar{x}$, we get

$$-(dg)^+(\bar{x},\eta(x,\bar{x}))\in Q$$

which gives,

$$\bar{\mu}^T(dg)^+(\bar{x},\eta(x,\bar{x}))\leq 0\,. \tag{3.13}$$

If $\bar{\mu}=0$ then (3.13) holds trivially.

Again for $x\in X_0$, $h(x)\in O=\{0_{R^k}\}$, therefore

$$-(h(x)-h(\bar{x}))\in O$$

Since $h$ is $O$-slnqpi at $\bar{x}$ we have

$$-(dh)^+(\bar{x},\eta(x,\bar{x}))\in O\,.$$

Therefore

$$\bar{\nu}^T(dh)^+(\bar{x},\eta(x,\bar{x}))=0 \tag{3.14}$$

Adding (3.13) and (3.14) and using (3.1) we get $\bar{\lambda}^T(df)^+(\bar{x},\eta(x,\bar{x}))\geq 0$.

Since $\bar{\lambda}\neq 0$, we obtain

$$-(df)^+(\bar{x},\eta(x,\bar{x}))\notin \operatorname{int} K\,.$$

As $f$ is $K$-slppi we get

$$-(f(x)-f(\bar{x}))\notin \operatorname{int} K\,,$$

that is

$$f(\bar{x})-f(x)\notin \operatorname{int} K\,.$$

Since $x\in X_0$ is arbitrarily chosen, therefore $\bar{x}$ is a weak minimum of (VOP).

□

## 4. Duality

The following Mond-Weir type dual is associated with the primal problem (VOP).

**(VOD)** K-maximize $f(u)$
subject to

$$\lambda^T(df)^+(u,\eta(x,u))+\mu^T(dg)^+(u,\eta(x,u))$$
$$+\nu^T(dh)^+(u,\eta(x,u))\geq 0,\forall x\in X_0 \tag{4.1}$$

$$\mu^T g(u)\geq 0 \tag{4.2}$$

$$h(u)=0_{R^k} \tag{4.3}$$

$0\neq\lambda\in K^+$, $\mu\in Q^+$ and $\nu\in R^k$, $u\in S$

**Theorem 4.1 (Weak Duality).** Let $x$ be feasible for (VOP) and $(u,\lambda,\mu,\nu)$ be feasible for (VOD). Let $f$ be $K$-slppi, $g$ be $Q$-slqpi and $h$ be $O$-slnqpi at $u$, with respect to same $\eta$. Then

$$f(u) - f(x) \notin \text{int } K.$$

**Proof.** Since $x$ is feasible for (VOP) and $(u,\lambda,\mu,\nu)$ feasible for (VOD), therefore we have,

$$\mu^T g(x) \leq 0 \text{ and } \mu^T g(u) \geq 0$$

which implies

$$\mu^T (g(x) - g(u)) \leq 0 .$$

If $\mu \neq 0$, then the above inequality results in $g(x) - g(u) \notin \text{int } Q$ .

Since $g$ is $Q$-slqpi we have,

$$-(dg)^+(u,\eta(x,u)) \in Q$$

That is,

$$\mu^T (dg)^+(u,\eta(x,u)) \leq 0 . \tag{4.4}$$

also holds if $\mu = 0$ .

Again using feasibility of $x$ and $u$, we have

$$-(h(x) - h(u)) \in O .$$

As $h$ is $O$-slnqpi at $u$, we get

$$-(dh)^+(u,\eta(x,u)) \in O$$

therefore,

$$\nu^T (dh)^+(u,\eta(x,u)) = 0 . \tag{4.5}$$

Using (4.4) and (4.5) in (4.1) we obtain

$$\lambda^T (df)^+(u,\eta(x,u)) \geq 0 .$$

As, $0 \neq \lambda \in K^+$ we have

$$-(df)^+(u,\eta(x,u)) \notin \text{int } K,$$

Because $f$ is $K$-slppi at $u$, we get

$$-(f(x) - f(u)) \notin \text{int } K$$

which gives

$$-(f(u) - f(x)) \notin \text{int } K .$$

□

**Theorem 4.2 (Strong Duality).** Let $f$ be $K$-slpi, $g$ be $Q$-slpi and $h$ be $O$-slpi with respect to same $\eta$. Let $F_1(S) + (K \times Q \times O)$ be closed with nonempty interior. Suppose that the pair $(g, h)$ satisfies generalized Slater type constraint qualification. If $\bar{x}$ is a weak minimum of (VOP) and $\eta(\bar{x},\bar{x}) = 0$, then there exist $0 \neq \lambda^* \in K^+$, $\mu^* \in Q^+$ and $\nu^* \in O^+$ such that $(\bar{x},\lambda^*,\mu^*,\nu^*)$ is a feasible solution of (VOD). Moreover if the conditions of Weak Duality Theorem 4.1 are satisfied for all feasible solutions of (VOP) and (VOD) then $(\bar{x},\lambda^*,\mu^*,\nu^*)$ is a weak maximum of (VOD).

**Proof.** Since $\bar{x}$ is a weak minimum of (VOP), therefore by Theorem 3.2, there exist $0 \neq \lambda^* \in K^+$, $\mu^* \in Q^+$, $\nu^* \in O^+$ such that

$$\lambda^{*T} (df)^+(\bar{x},\eta(x,\bar{x})) + \mu^{*T} (dg)^+(\bar{x},\eta(x,\bar{x}))$$
$$+ \nu^{*T} (dh)^+(\bar{x},\eta(x,\bar{x})) \geq 0 , \quad \forall \quad x \in S$$

and

$$\mu^{*T} g(\bar{x}) = 0 .$$

Thus $(\bar{x},\lambda^*,\mu^*,\nu^*)$ is feasible for (VOD). Let if possible, $(\bar{x},\lambda^*,\mu^*,\nu^*)$ be not a weak maximum of (VOD). Then there exists $(u,\lambda,\mu,\nu)$ feasible for (VOD) such that $f(u) - f(\bar{x}) \in \text{int } K$ which contradicts Weak Duality Theorem (Theorem 4.1) as $\bar{x}$ is feasible for (VOP).

□

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## References

[1] Ewing G.M., Sufficient Conditions for Global Minima of Suitable Convex Functional from Variational and Control Theory. *SIAM Review* 19, 202-220 (1977).

[2] Illes, T., Kassay, G., Theorems of the Alternative and Optimality Conditions for Convexlike and General Convexlike Programming. *Journal of Optimization Theory and Applications* 101(2), 243-257 (1999).

[3] Kaul, R.N., Kaur, S., Generalization of Convex and Related Functions. *European Journal of Operational Research* 9, 369-377 (1982).

[4] Kaul, R.N., Kaur, S., Sufficient Optimality Conditions using Generalized Convex Functions. *Opsearch* 19, 212-224 (1982).

[5] Kaul, R.N., Kaur, S., Generalized Convex Functions, Properties, Optimality and Duality. *Technical Report, SOL.*, 84-4, Stanford University, California (1984).

[6] Kaul, R.N., Lyall, V., Kaur, S., Semilocal Pseudolinearity and Efficiency. *European*

*Journal of Operational Research* 36, 402-409 (1988).

[7] Kaur, S., *Theoretical Studies in Mathematical Programming*. Ph.D. Thesis, University of Delhi, Delhi, India (1983).

[8] Gupta, I., Vartak, M.N., Kuhn-Tucker and Fritz John Type Sufficient Optimality Conditions for Generalized Semilocally Convex Programs. *Opsearch* 26, 11-27 (1989).

[9] Mukherjee, R.N., Mishra, S.K., Multiobjective Programming with Semilocally Convex Functions. *Journal of Mathematical Analysis and Applications*. 199, 409-424 (1996).

[10] Preda, V., Optimality Conditions and Duality in Multiobjective Programming Involving Semilocally Convex and Related Functions. *Optimization* 36, 219-230 (1996).

[11] Preda, V., Optimality and Duality in Fractional Multiobjective Programming involving Semilocally Preinvex and Related Functions. *Journal of Mathematical Analysis and Applications* 288, 365-382 (2003).

[12] Preda V., Stancu-Minasian, J.M., Duality in Multiobjective Programming involving Semilocally Preinvex and Related Functions. *Glas. Math*. 32, 153-165 (1997).

[13] Stancu-Minasian, J.M., Optimality and Duality in Fractional Programming involving Semilocally Preinvex and Related Functions. *Journal of Information and Optimization Science* 23, 185-201 (2002).

[14] Suneja, S.K., Gupta, S., Duality in Nonlinear Programming involving Semilocally Convex and Related Functions. *Optimization* 28, 17-29 (1993).

[15] Suneja, S.K., Gupta, S., Vani, Optimality and Duality in Multiobjective Nonlinear programming involving $\rho$-Semilocally Preinvex and Related Functions. *Opsearch* 44(1), 27-40 (2007).

[16] Weir, T., Programming with Semilocally Convex Functions. *Journal of Mathematical Analysis and Applications* 168 (1-2), 1-12 (1992).

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