An International Journal of Optimization
and Control: Theories & Applications
Vol.4, No.1, pp.35-44 (2014) © IJOCTA
ISSN: 2146-0957eISSN: 2146-5703
DOI: 10.11121/ijocta.01.2014.00160
http://www.ijocta.com

Vector variational inequalities and their relations with vector optimization

Surjeet Kaur Suneja [a] and Bhawna Kohli [b]

[a] Department of Mathematics, Miranda House, University of Delhi, Delhi-110007, India
Email: surjeetsuneja@gmail.com

[b] Department of Mathematics, University of Delhi, Delhi-110007, India
Email: bhawna_kohli@rediffmail.com

(*Received February 25, 2013; in final form December 05, 2013*)

Abstract. In this paper, K - ∂^c quasiconvex, K - ∂^c pseudoconvex and other related functions have been introduced in terms of their Clarke subdifferentials, where K is an arbitrary closed convex, pointed cone with nonempty interior. The (strict, weakly) K -pseudomonotonicity, (strict) K -naturally quasimonotonicity and K -quasimonotonicity of Clarke subdifferential maps have also been defined. Further, we introduce Minty weak (MVVIP) and Stampacchia weak (SVVIP) vector variational inequalities over arbitrary cones. Under regularity assumption, we have proved that a weak minimum solution of vector optimization problem (VOP) is a solution of (SVVIP) and under the condition of K - ∂^c pseudoconvexity we have obtained the converse for MVVIP (SVVIP). In the end we study the interrelations between these with the help of strict K -naturally quasimonotonicity of Clarke subdifferential map.

Keywords: Generalized nonsmooth cone convexity; generalized cone monotonicity; vector optimization problem; vector variational inequality problem.
AMS Classification: 90C30

1. Introduction

The Variational inequality problem was first introduced by Hartman and Stampacchia [1] in their seminal paper. As it has many applications in fundamental sciences as well as in economics and management sciences, it has become very popular among researchers. Giannessi [2] introduced vector variational inequality problem and since then a great deal of research started in this area by various authors like Chen and Yang [3], Chen [4], Yao [5], Giannessi [6,7,8], Komlosi [9], Yang et al. [10], Lee and Lee [11] etc. Giannessi [6] has shown equivalence between efficient solutions of differentiable, convex vector optimization problem and solutions of variational inequality of Minty type. Yang et al. [10] established some relations between a solution of a Minty vector variational inequality (VVI) problem and an efficient solution of a differentiable nonconvex vector optimization problem under the assumptions of pseudoconvexity of functions or pseudomonotonicity of their gradients. Komlosi [9] studied Stampacchia and Minty vector variational inequality problems and discussed the solution concepts of these problems and vector optimization problem. Vector variational-like inequality problems have been studied by many authors like Mishra and Wang [12], Chinaie et al. [13] etc. Recently, Rezaie and Zafarani [14] have studied relations between vector optimization problem and Minty and Stampacchia vector variational-like inequalities over cones contained in $\mathbb{R}^n_+ \setminus \{0\}$ for nondifferentiable functions under generalized invexity or generalized monotonicity assumptions.

Corresponding Author. Email: bhawna_kohli@rediffmail.com

Generalized monotonicity plays a central role in the study of the existence of solution of variational inequality problems and their relations with vector optimization problems. Monotonicity (generalized) concepts have been related to convexity (generalized) of functions in case of gradient maps by various authors like Karamardian and Schaible [15], Hadjisavvas and Schaible [16] and Schaible [17]. Various authors like Cambini [18], Cambini and Martein [19], Vani [20] etc. have extended generalized convexity and / or generalized monotonicity concepts from scalar case to the vector valued functions.

In this paper, we introduce the notions of K-∂^c quasiconvex, weakly K-∂^c quasiconvex, K-∂^c pseudoconvex and strict K-∂^c pseudoconvex functions in terms of Clarke subdifferentials. The interrelations between above mentioned functions have been given. The concepts of (strict, weakly) K-pseudomonotonicity, (strict) K-naturally quasimonotonicity and K-quasimonotonicity of Clarke subdifferential maps have been defined. Further Minty weak and Stampacchia weak vector variational inequalities over arbitrary cones have been introduced. Their relations with vector optimization problem over cones have been studied with the help of K-∂^c pseudoconvex functions. We end the paper by presenting interrelations among Minty weak and Stampacchia weak vector variational inequalities over cones under the assumption of strict K-naturally quasimonotonicity of Clarke subdifferential map.

2. Generalized Nonsmooth Cone Convexity and Generalized Cone Monotonicity

We begin this section with the following definitions.

Definition 1[21] Let $\phi : R^n \to R$ be a locally Lipschitz function on R^n. Then the Clarke generalized subdifferential of ϕ at $x \in R^n$ is given as

$$\partial^c\phi(x) = \{\xi \in R^n \mid \phi^0(x,v) \geq \langle \xi, v\rangle, \forall v \in R^n\}$$

where $\phi^0(x,v)$ is Clarke generalized directional derivative of ϕ at $x \in R^n$ in direction v and is given by

$$\phi^0(x,v) = \mathop{Lim\,sup}_{y\to x, t\to 0^+} \frac{\phi(y+tv)-\phi(y)}{t},$$

where $y \in R^n$ and $t > 0$.

$\partial^c\phi(x)$ is nonempty, convex and compact set for each $x \in R^n$.

$$\phi^0(x,v) = \sup\{\langle \xi, v\rangle : \xi \in \partial^c f(x)\}.$$

The function ϕ is said to be regular [22] if

(i) $\phi'(x,v)$ exists for all $x \in R^n$ and every direction $v \in R^n$.

(ii) $\phi'(x,v) = \phi^\circ(x,v).$

Let D be a nonempty open subset of R^n.

Let $f : D \to R^k$ be a vector valued function. Then f is said to be locally Lipschitz on R^n, if, each f_i is locally Lipschitz on R^n. The Clarke generalized directional derivative of locally Lipschitz function f at x in the direction $y-x$ is given by $f^0(x, y-x) = ((f_1)^0(x,y-x),\ldots,(f_k)^0(x,y-x))$ where $(f_i)^0(x,y-x)$ is Clarke generalized directional derivative of f_i at x in the direction $y-x$. The Clarke generalized jacobian of f at x is given by $\partial^c f(x) = \partial^c f_1(x) \times \partial^c f_2(x) \times \ldots \times \partial^c f_k(x)$, where $\partial^c f_i(x)$ is Clarke generalized subdifferential of f_i at x and $f = (f_1, f_2, \ldots, f_k)$.

Let $K \subseteq R^k$ be a closed convex, pointed cone with non empty interior and let int K and $\overline{K}$ denote the interior and closure of K respectively. The positive dual cone K^+ (Swaragi, Nakayama and Tanino [23]) is defined as

$$K^+ = \{y^* \in \mathrm{R}^k \mid \langle y, y^*\rangle \geq 0, \text{ for all } y \in K\}.$$

Let $A \subseteq R^n$ be a nonempty subset. Then the convex hull of A is denoted by coA.

Jahn [24] defined K- quasiconvex function as given below.

Definition 2. The function $f : D \to R^k$ is said to be K - quasiconvex on D, if for all $x, y \in D$

$$f(x) - f(y) \in -K$$
$$\Rightarrow \quad f(y + \lambda(x - y)) - f(y) \in -K,$$

for all $\lambda \in [0,1]$.

We now introduce two important classes of generalized convex functions (with respect to cones) using the concept of Clarke subdifferential namely K -∂^c quasiconvex and weakly K -∂^c quasiconvex functions.

Following definition has been introduced on the lines of Cambini [18] and Jahn [24].

Definition 3. The function $f : D \to R^k$ is said to be K -∂^c quasiconvex on D, if for all $x, y \in D$

$$f(x) - f(y) \in -K \Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K,$$

where

$$\left\langle \partial^c f(y), x - y \right\rangle = \left\langle \partial^c f_1(y), x - y \right\rangle \times \left\langle \partial^c f_2(y), x - y \right\rangle \times \ldots \times \left\langle \partial^c f_k(y), x - y \right\rangle.$$

Remark 1 (i) If k=1, $K = R_+$, then the above definition reduces to the definition of ∂^c -quasiconvex function given by Bector, Chandra and Dutta [22].

(ii) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to the definition of quasiconvex function.

On the lines of Cambini [18], we give below the definition of weakly K -∂^c quasiconvex function.

Definition 4. The function $f : D \to R^k$ is said to be weakly K -∂^c quasiconvex on D, if for all $x, y \in D$

$$f(x) - f(y) \in -\operatorname{int} K$$
$$\Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K$$

Remark 2 (i) If k=1, $K = R_+$, then the above definition reduces to the following

$$f(x) < f(y)$$
$$\Rightarrow \quad \left\langle \xi, x - y \right\rangle \leq 0,$$

for all $\xi \in \partial^c f(y)$.

Then by Theorem 5.4.1 of Bector, Chandra and Dutta [22], f is quasiconvex function. Thus it characterizes Lipschitz quasiconvex function.

We now provide an example of K -∂^c quasiconvex function.

Example 1 Let $f : D \to R^2$ be a function defined by $f = (f_1, f_2)$ where

$f_1 : D \to R,\ f_2 : D \to R$ are defined as

$$f_1(x) = \begin{cases} x, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad f_2(x) = \begin{cases} -x^2, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

and $D = (-5, 5)$

$$\partial^c f(x) = \begin{cases} (1, -2x), & x > 0 \\ \{(t, l) : t \in [0,1], l \in [-1, 0]\}, & x = 0 \\ (0, -1), & x < 0 \end{cases}$$

Let $K = \{(x, y) \in R^2 \mid x \leq 0, y \geq 0\}$.

Then f is K -∂^c quasiconvex as

$$f(x) - f(y) \in -K$$
$$\Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K,$$

We now introduce the following definition of K -∂^c pseudoconvex and strict K -∂^c pseudoconvex functions on the lines of Cambini [18].

Definition 5. The function $f : D \to R^k$ is said to be K -∂^c pseudoconvex on D, if for all $x, y \in D$

$$f(x) - f(y) \in -\operatorname{int} K$$
$$\Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -\operatorname{int} K.$$

Remark 3 (i) If k=1, $K = R_+$, then the above definition reduces to the definition of ∂^c -pseudoconvex function given by Bector, Chandra and Dutta [22].

(ii) If f is continuously differentiable function then the above definition reduces to the definition of K -pseudoconvex function given by Aggarwal [25] and Cambini and Martein [19] as $\partial^c f(y) = J_f(y)$.

(iii) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to the definition of pseudoconvex function.

Definition 6. The function $f : D \to R^k$ is said to be strict K -∂^c pseudoconvex on D, if for all $x, y \in D$

$$f(x) - f(y) \in -K$$
$$\Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -\operatorname{int} K.$$

Remark 4 (i) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to the definition of strict pseudoconvex function.

(ii) If f is continuously differentiable function then the above definition reduces to the definition of strict K-pseudoconvex function given by Cambini and Martein [19] as $\partial^c f(y) = J_f(y)$.

We now give below an example of K-∂^c pseudoconvex function.

Example 2 Let $f : D \to R^2$ be a function defined by $f = (f_1, f_2)$ where

$f_1 : D \to R,\ f_2 : D \to R$ are defined as

$$f_1(x) = \begin{cases} -x, & x \geq 0 \\ 3x, & x < 0 \end{cases},\quad f_2(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

and $D = (-5, 5)$

$$\partial^c f(x) = \begin{cases} (-1,1), & x > 0 \\ \{(t,l) : t \in [-1,3], l \in [-1,1]\}, & x = 0 \\ (3,-1), & x < 0 \end{cases}$$

Let $K = \left\{(x, y) \in R^2 \mid x \leq 0, y \geq 0, y \geq \frac{-x}{2}\right\}$.

Then f is K-∂^c pseudoconvex as

$$f(x) - f(y) \in -\operatorname{int} K \quad \Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -\operatorname{int} K.$$

We now present interrelations between the above defined functions in the form of following remarks.

Remark 5 Every K-∂^c quasiconvex function is weakly K-∂^c quasiconvex function.

Remark 6 Every strict K-∂^c pseudoconvex function is K-∂^c pseudoconvex function.

Remark 7 Every strict K-∂^c pseudoconvex function is K-∂^c quasiconvex function.

On the lines of Rezaie and Zafarani [14], we now give the definitions of generalized monotone set valued maps over arbitrary closed convex and pointed cones with non empty interior.

Let $f : D \to R^k$ be a vector valued function.

Definition 7. The set valued map $\partial^c f$ is K-pseudomonotone on D if for every pair of distinct points $x, y \in D$

$$\left\langle \partial^c f(x), y - x \right\rangle \subseteq K \quad \Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K.$$

Remark 8 (i) If $K \subseteq R^n_+ \setminus \{0\}$ and $y - x = \eta(y, x)$, then the above definition becomes the definition of K-pseudomonotone map given by Rezaie and Zafarani [14].

(ii) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to the following definition of pseudomonotonicity of the map $\partial^c f = \nabla f$ (Karamardian and Schaible [15])

$$\left\langle \nabla f(x), y - x \right\rangle \geq 0 \quad \Rightarrow \quad \left\langle \nabla f(y), y - x \right\rangle \geq 0$$

Definition 8. The set valued map $\partial^c f$ is strict K-pseudomonotone on D if for every pair of distinct points $x, y \in D$

$$\left\langle \partial^c f(x), y - x \right\rangle \subseteq K \quad \Rightarrow \quad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -\operatorname{int} K.$$

Remark 9 (i) If $K \subseteq R^n_+ \setminus \{0\}$ and $y - x = \eta(y, x)$, then the above definition becomes the definition of strict K-pseudomonotone map given by Rezaie and Zafarani [14].

(ii) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to the following definition of strict pseudomonotonicity of the map $\partial^c f = \nabla f$ (Karamardian and Schaible [15])

$$\left\langle \nabla f(x), y - x \right\rangle \geq 0 \quad \Rightarrow \quad \left\langle \nabla f(y), y - x \right\rangle > 0.$$

Definition 9. The set valued map $\partial^c f$ is weakly K-pseudomonotone on D if for distinct points $x, y \in D$ there exists $\xi \in \partial^c f(x)$ such that

$$\left\langle \xi, y - x \right\rangle \in \operatorname{int} K \quad \Rightarrow \quad \left\langle \partial^c f(y), y - x \right\rangle \subseteq \operatorname{int} K.$$

Definition 10. The set valued map $\partial^c f$ is K-naturally quasimonotone on D if for every pair of distinct points $x, y \in D$

$$\left\langle \partial^c f(x), y - x \right\rangle \subseteq \operatorname{int} K$$

$$\Rightarrow \qquad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K.$$

Remark 10 (i) If $K \subseteq R_+^n \setminus \{0\}$ and $y - x = \eta(y, x)$, then the above definition becomes the definition of K - quasimonotone map given by Rezaie and Zafarani [14].

(ii) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to the following definition of quasimonotonicity of the map $\partial^c f = \nabla f$ (Karamardian and Schaible [15])

$$\left\langle \nabla f(x), y - x \right\rangle > 0 \quad \Rightarrow \quad \left\langle \nabla f(y), y - x \right\rangle \geq 0.$$

Definition 11. The set valued map $\partial^c f$ is strict K -naturally quasimonotone on D if for every pair of distinct points $x, y \in D$

$$\left\langle \partial^c f(x), y - x \right\rangle \subseteq \operatorname{int} K$$

$$\Rightarrow \qquad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -\operatorname{int} K.$$

Remark 11(i) If $K \subseteq R_+^n \setminus \{0\}$ and $y - x = \eta(y, x)$, then the above definition becomes the definition of strict K - quasimonotone map given by Rezaie and Zafarani [14].

(ii) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to pseudomonotonicity of the map $\partial^c f = \nabla f$ (Karamardian and Schaible [15])

$$\left\langle \nabla f(x), y - x \right\rangle > 0 \quad \Rightarrow \quad \left\langle \nabla f(y), y - x \right\rangle > 0.$$

Definition 12. The set valued map $\partial^c f$ is K - quasimonotone on D if for every pair of distinct points $x, y \in D$

$$\left\langle \partial^c f(x), x - y \right\rangle \not\subseteq K$$

$$\Rightarrow \qquad \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K.$$

Remark 12(i) If k=1, $K = R_+$ and f is continuously differentiable then the above definition reduces to pseudomonotonicity of the map $\partial^c f = \nabla f$ (Karamardian and Schaible [15])

$$\left\langle \nabla f(x), x - y \right\rangle < 0 \quad \Rightarrow \quad \left\langle \nabla f(y), x - y \right\rangle \leq 0.$$

Now we give an example of strict K – pseudomonotone map as follows:

Example 3 Let $f : D \to R^2$ be a function defined by $f = (f_1, f_2)$ where $f_1 : D \to R$, $f_2 : D \to R$ are defined as

$$f_1(x) = \begin{cases} x^2 + x, & x \geq 0 \\ x, & x < 0 \end{cases}, \quad f_2(x) = \begin{cases} -2x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

and $D = (-5, 5)$

$$\partial^c f(x) = \begin{cases} (2x + 1, -2), & x > 0 \\ (1, l) : l \in [-2, -1] \ , & x = 0 \\ (1, -1), & x < 0 \end{cases}$$

Let $K = \ (x, y) \in R^2 \mid x \leq 0, y \geq 0, y \geq \frac{-x}{2} \ .$

Then $\partial^c f$ is strict K - pseudomonotone as

$$\left\langle \partial^c f(x), y - x \right\rangle \subseteq K \ \Rightarrow \left\langle \partial^c f(y), x - y \right\rangle \subseteq -\operatorname{int} K.$$

The section proceeds further by presenting interrelationships between above mentioned generalized monotone maps.

Remark 13 Every strict K -pseudomonotone map is K - pseudomonotone but converse may not be true as can be seen from the following example:

Example 4 Let $f : D \to R^2$ be a function defined by $f = (f_1, f_2)$ where $f_1 : D \to R$, $f_2 : D \to R$ are defined as

$$f_1(x) = \begin{matrix} x, & x \geq 0 \\ 0, & x < 0 \end{matrix}, \quad f_2(x) = \begin{matrix} -x, & x \geq 0 \\ -1, & x < 0 \end{matrix}$$

and $D = (-5, 5)$

$$\partial^c f(x) = \begin{cases} (1, -1), & x > 0 \\ (t, l) : t \in [0, 1], l \in [-1, 0] \ , & x = 0 \\ (0, 0), & x < 0 \end{cases}$$

Let $K = \ (x, y) \in R^2 \mid x \geq 0, y \leq 0, -y \leq x \ .$

Then $\partial^c f$ is K -pseudomonotone as

$$\left\langle \partial^c f(x), y - x \right\rangle \subseteq K \quad \Rightarrow \left\langle \partial^c f(y), x - y \right\rangle \subseteq -K$$

but $\partial^c f$ is not strict K -pseudomonotone because for $x=-1, y=0$

$\left\langle \partial^c f(x), y-x \right\rangle \subseteq K$ but $\left\langle \partial^c f(y), x-y \right\rangle \not\subseteq -\operatorname{int} K$.

Remark 14 Every strict K-naturally quasimonotone map is K-naturally quasimonotone but converse may not be true as can be seen from the following example:

Example 5 Let $f: D \to R^2$ be a function defined by $f=(f_1, f_2)$ where

$f_1: D \to R,\ f_2: D \to R$ are defined as

$$f_1(x)=\begin{cases} x, & x \geq 0 \\ 0, & x<0 \end{cases},\quad f_2(x)=\begin{cases} -x^2, & x \geq 0 \\ -x, & x<0 \end{cases}$$

and $D=(-5,5)$

$$\partial^c f(x)=\begin{cases}(1,-2x), & x>0 \\ \{(t,l): t \in [0,1], l \in [-1,0]\}, & x=0 \\ (0,-1), & x<0\end{cases}$$

Let $K=\{(x,y) \in R^2 \mid x \leq 0, y \geq 0\}$.

Then $\partial^c f$ is K- naturally quasimonotone as

$\left\langle \partial^c f(x), y-x \right\rangle \subseteq \operatorname{int} K \Rightarrow \left\langle \partial^c f(y), x-y \right\rangle \subseteq -K$

but $\partial^c f$ is not strict K- naturally quasimonotone because for $x=1, y=0$ $\left\langle \partial^c f(x), y-x \right\rangle \subseteq \operatorname{int} K$

but $\left\langle \partial^c f(y), x-y \right\rangle \not\subseteq -\operatorname{int} K$.

3. Vector Variational Inequalities

In this section we consider the following vector optimization problem over cones and study its relation with associated vector variational inequalities:

(VOP) K-minimize $f(x)$

subject to $x \in C$

where $f: D \to R^k$, $f=(f_1, f_2, \ldots, f_k)$, each f_i, $i=1,2,\ldots,k$ is locally Lipschitz, K is closed convex and pointed cone with nonempty interior in R^k and C is nonempty, convex subset of D.

Definition 13. A vector $x \in C$ is said to be a weak minimum of (VOP) if

$f(x)-f(y) \notin \operatorname{int} K$, for all $y \in C$.

We now consider the following Minty weak vector variational inequality problem(MVVIP) over cones.

(MVVIP) Find $x \in C$ such that $\left\langle \partial^c f(y), y-x \right\rangle \not\subseteq -\operatorname{int} K$, for all $y \in C$.

Remark 15(i) If $k=n$, $K \subseteq R^n_+ \setminus \{0\}$ and $y-x=\eta(y,x)$, then the above problem becomes Minty weak vector variational - like inequality problem (MWVLI) given by Rezaie and Zafarani [14] as follows:

Find $x \in C$ such that

$\langle \partial^c f(y), \eta(y,x) \rangle \not\subseteq -\operatorname{int} K(x)$, for all $y \in C$,

where $\{K(x): x \in C\}$ is a family of convex and pointed cones of R^n such that $K(x) \subseteq R^n_+ \setminus \{0\}$, for all $x \in C$.

(ii) If $K=R^k_+$, $k=p$, C is closed and f is convex then the above problem reduces to the following variational inequality problem (WMVVI) considered by Lee and Lee [11].

Find $x \in C$ such that

for all $\xi_i \in \partial f_i(y)$, $i=1,2,\ldots,k$,

$\left(\left\langle \xi_1, y-x \right\rangle, \ldots, \left\langle \xi_k, y-x \right\rangle\right) \notin -\operatorname{int} R^k_+$, for all $y \in C$,

where $\partial f_i(y)$ denotes subdifferential of f_i, $i=1,2,\ldots,k$ at y.

Further, we consider the following Stampacchia weak vector variational inequality problem(SVVIP) over cones.

(SVVIP) Find $x \in C$ such that

$\langle \partial^c f(x), y-x \rangle \not\subseteq -\operatorname{int} K$, for all $y \in C$.

Remark 16 (i) If $k=n$, $K \subseteq R^n_+ \setminus \{0\}$ and $y-x=\eta(y,x)$, then the above problem becomes Stampacchia weak vector variational-like inequality problem (SWVLI) given by Rezaie and Zafarani [14] as follows:

Find $x \in C$ such that

$\langle \partial^c f(x), \eta(y,x) \rangle \not\subseteq -\operatorname{int} K(x)$, for all $y \in C$,

where $\{K(x): x \in C\}$ is a family of convex and pointed cones of R^n such that $K(x) \subseteq R^n_+ \setminus \{0\}$, for all $x \in C$.

(ii) If $K = R^k_+$, $k = p, C$ is closed and f is continuously differentiable then the above problem reduces to the following variational inequality problem (WVVI)$_\nabla$ considered by Giannessi [8].

Find $x \in C$ such that

$$\left\langle \nabla f_1(x), y-x \right\rangle, ..., \left\langle \nabla f_k(x), y-x \right\rangle \notin -\operatorname{int} R^k_+,$$

for all $y \in C$.

(iii) If $K = R^k_+, k = p$, C is closed and f is convex then the above problem reduces to the following variational inequality problem (WVVI)$_1$ considered by Lee and Lee [11].

Find $x \in C$ such that for all $\xi_i \in \partial f_i(x)$, $i = 1, 2, ..., k$,

$$\left\langle \xi_1, y-x \right\rangle, ..., \left\langle \xi_k, y-x \right\rangle \notin -\operatorname{int} R^k_+, \quad \text{for all } y \in C.$$

We now present relations between weak minimum solution of (VOP) and solution of MVVIP(SVVIP).

Theorem 1. If $x \in C$ is a solution of (MVVIP) and f is K-∂^c pseudoconvex then x is a weak minimum for (VOP).

Proof Let $x \in C$ be a solution of (MVVIP). Then for all $y \in C$,

$$\langle \partial^c f(y), y-x \rangle \not\subseteq -\operatorname{int} K. \tag{1}$$

Let $x(\lambda) = x + \lambda(y-x)$, $0 < \lambda < 1$.

Since C is convex, $x(\lambda) \in C$.

Replacing y by $x(\lambda)$ in (1) we get

$\langle \partial^c f(x(\lambda)), x(\lambda) - x \rangle \not\subseteq -\operatorname{int} K$,

which implies that there exists at least one $\xi_\lambda \in \partial^c f(x(\lambda))$ such that

$$\langle \xi_\lambda, x(\lambda) - x \rangle \notin -\operatorname{int} K. \tag{2}$$

(2) gives $\langle \xi_\lambda, y - x \rangle \notin -\operatorname{int} K$.

As $\partial^c f$ is locally bounded at x, there exist a neighbourhood of x and a constant $k' > 0$ such that for each z in this neighbourhood and $\xi \in \partial^c f(z)$ we have $\| \xi_\lambda \| \le k'$.

As $x(\lambda) \to x$ when $\lambda \to 0^+$, thus for $\lambda > 0$ small enough $\| \xi_\lambda \| \le k'$.

Without loss of generality we may assume that $\xi_\lambda \to \xi'$.

Since $\partial^c f$ is closed, $\xi' \in \partial^c f(x)$, therefore for $x \in C$ there exists $\xi' \in \partial^c f(x)$ such that $\langle \xi', y-x \rangle \notin -\operatorname{int} K$

which gives $\left\langle \partial^c f(x), y-x \right\rangle \not\subseteq -\operatorname{int} K$.

Since f is K-∂^c pseudoconvex, we have

$f(y) - f(x) \notin -\operatorname{int} K$, for all $y \in C$.

Hence x is a weak minimum for (VOP). □

Theorem 2. If $x \in C$ is a weak minimum for (VOP) and each $f_i, i = 1, 2, ..., k$ is continuous, regular, then x is a solution of (SVVIP).

Proof Since $x \in C$ is a weak minimum for (VOP)

$$f(x) - f(y) \notin \operatorname{int} K \text{ for all } y \in C. \tag{3}$$

Suppose on contrary x is not a solution of (SVVIP). Then there exist $y \in C$ such that

$$\left\langle \partial^c f(x), y-x \right\rangle \subseteq -\operatorname{int} K.$$

That is, $\langle \xi, y-x \rangle \in -\operatorname{int} K$ for all $\xi \in \partial^c f(x)$, where $\xi = (\xi_1, \xi_2, ..., \xi_k)$, $\xi_i \in \partial^c f_i(x)$, $i = 1, 2, ..., k$.

Also, we have

$$(f_i)^0(x, y-x) = \sup_{\xi_i \in \partial^c f_i(x)} \left\langle \xi_i, y-x \right\rangle,$$

$i = 1, 2, ..., k$.

Since each $\partial^c f_i(x)$, $i = 1, 2, ..., k$ is compact, choose $\overline{\xi}_i \in \partial^c f_i(x)$ such that

$$\sup_{\xi_i \in \partial^c f_i(x)} \left\langle \xi_i, y-x \right\rangle = \left\langle \overline{\xi}_i, y-x \right\rangle, \quad i = 1, 2, ..., k.$$

Then,

$$(f_i)^0(x, y-x) = \left\langle \overline{\xi}_i, y-x \right\rangle, \ i = 1, 2, ..., k. \tag{4}$$

Let $\overline{\xi} = (\overline{\xi}_1, \overline{\xi}_2, ..., \overline{\xi}_k)$, then $\overline{\xi} \in \partial^c f(x)$.

Thus for some $y \in C$, there exist $\overline{\xi} \in \partial^c f(x)$ such that

$$\langle \overline{\xi}, y-x \rangle \in -\operatorname{int} K \tag{5}$$

Then, (4) gives $(f)^0(x, y-x) = \left\langle \overline{\xi}, y-x \right\rangle$.

Using (5), we get $(f)^0(x, y-x) \in -\operatorname{int} K$, which implies for all $\tau \in K^+ / \{0\}$, $\tau (f)^0(x, y-x) < 0$.

That is, $\sum_{i=1}^{k} \tau_i (f_i)^0 (x, y-x) < 0.$

Since each $f_i, i = 1, 2, ..., k$ is regular we have

$$\sum_{i=1}^{k} \tau_i (f_i)' (x, y-x) < 0.$$

By continuity of f we have that there exist $\lambda^* > 0$ for each $\tau \in K^+ /\{0\}$, such that $(\tau f)(x + \lambda(y-x)) - (\tau f)(x) < 0$, for all $0 < \lambda < \lambda^*$.

Let $0 < \bar{\lambda} < \lambda^*$, then from above inequality it follows that for each $\tau \in K^+ /\{0\}$,

$(\tau f)(x + \bar{\lambda}(y-x)) - (\tau f)(x) < 0,$

which gives

$$f(x + \bar{\lambda}(y-x)) - f(x) \in -\operatorname{int} K,$$

which is contradiction to (3).
Hence x is a solution of (SVVIP). □

Theorem 3. If $x \in C$ is a solution of (SVVIP) and f is K-∂^c pseudoconvex then x is a weak minimum for (VOP).

Proof Suppose on contrary $x \in C$ is not a weak minimum for (VOP), then there exists $y \in C$ such that

$f(y) - f(x) \in -\operatorname{int} K.$

Since f is K-∂^c pseudoconvex we have

$\left\langle \partial^c f(x), y-x \right\rangle \subseteq -\operatorname{int} K.$

Hence for $x \in C$, there exists $y \in C$ such that

$\left\langle \partial^c f(x), y-x \right\rangle \subseteq -\operatorname{int} K,$

which is contradiction to the fact that x is a solution of (SVVIP).
Hence x is a weak minimum for (VOP). □

Theorem 4. Let $\partial^c f$ be strict K-naturally quasimonotone. Then $x \in C$ is a solution of (SVVIP) if and only if it is a solution of (MVVIP).

Proof Let $x \in C$ be a solution of (SVVIP).
Then for all $y \in C$, $\left\langle \partial^c f(x), y-x \right\rangle \not\subseteq -\operatorname{int} K.$

Since $\partial^c f$ is strict K-naturally quasimonotone, we have

$\left\langle \partial^c f(y), y-x \right\rangle \not\subseteq -\operatorname{int} K.$

Hence x is a solution of (MVVIP).
Conversely suppose that $x \in C$ is a solution of (MVVIP).
Then for all $y \in C$, $\langle \partial^c f(y), y-x \rangle \not\subseteq -\operatorname{int} K.$ (6)

Let $x(\lambda) = x + \lambda(y-x),\ 0 < \lambda < 1.$
Since C is convex, $x(\lambda) \in C.$
Replacing y by $x(\lambda)$ in (6) we get
$\langle \partial^c f(x(\lambda)), x(\lambda) - x \rangle \not\subseteq -\operatorname{int} K,$
which implies that there exists at least one $\xi_\lambda \in \partial^c f(x(\lambda))$ such that
$\langle \xi_\lambda, x(\lambda) - x \rangle \notin -\operatorname{int} K.$
That is,
$\langle \xi_\lambda, y-x \rangle \notin -\operatorname{int} K.$

Then proceeding on the similar lines of Theorem 1 we get that for any $y \in C$, there exist $\xi \in \partial^c f(x)$ such that $\langle \xi, y-x \rangle \notin -\operatorname{int} K$ which gives $\left\langle \partial^c f(x), y-x \right\rangle \not\subseteq -\operatorname{int} K.$
Hence x is a solution of (SVVIP). □

Acknowledgements

The authors wish to thank the anonymous referees of this paper whose comments have improved the presentation of the paper.

References

[1] Hartman, P., Stampacchia, G., On Some Nonlinear Elliptic Differential Functional Equations, *Acta Mathematica*, 115, pp.153-188, (1966).

[2] Giannessi, F., Theorems of the Alternative, Quadratic Programs and Complementarity Problem, In: R. W. Cottle, F. Giannessi and J. L. Lions (Eds.), *Variational Inequalities and Complementarity Problems*, John Wiley and Sons, New York, pp. 151-186 (1980).

[3] Chen, G. Y., Yang, X. Q., Vector Complementary Problem and its Equivalence With Weak Minimal Element in Ordered Spaces, *Journal of Mathematical Analysis and Applications*, 153, pp. 136-158, (1990).

[4] Chen, G. Y., Existence of Solutions for A Vector Variational Inequality, An extension of Hartman-Stampacchia theorem, *Journal of Optimization Theory and Applications*,74, pp. 445-456, (1992).

[5] Yao, J. C., Multivalued Variational Inequalities With K -pseudomonotone Operators, *Journal of Optimization Theory and Applications*, 83, pp. 391-403, (1994).

[6] Giannessi, F., On Minty Variational Principle, *New Trends in Mathematical Programming*, Kluwer Academic Publishers, Dordrecht, Netherlands, (1997).

[7] Giannessi, F., On Minty Variational Principle. In: F. Giannessi, S. Komlosi and T. Rapesak (Eds.), *New Trends in Mathematical Programming*, Dordrecht, Netherlands: Kluwer Academic Publishers, pp. 93-99 (1998).

[8] Giannessi, F., *Vector Variational Inequalities and Vector Equilibria*, Dordrecht, Netherlands: Kluwer Academic Publishers, (2000).

[9] Komlosi, S., On the Stampacchia and Minty Variational Inequalities. In: G. Giorgi and F. Rossi (Eds), *Generalized Convexity and Optimization for Economic and Financial Decisions*, Bologna: Pitagora Editrice, pp. 231-260, (1999).

[10] Yang, X. M., Yang, X. Q., Teo, K. L., Some Remarks On the Minty Vector Variational Inequality, *Journal of Optimization Theory and Applications*, 121, pp. 193- 201, (2004).

[11] Lee, G. M., Lee, K. B., Vector Variational Inequalities for Nondifferentiable Convex Vector Optimization Problems, *Journal of Global Optimization*, 32, pp. 597-612 (2005).

[12] Mishra, S. K., Wang, S. Y., Vector Variational–like Inequalities and Nonsmooth Vector Optimization Problems, *Nonlinear Analysis*, 64, pp. 1939-1945, (2005).

[13] Chinaie, M., Jabarootian, T., Rezaie, M., Zafarani, J., Minty's Lemma and Vector Variational –like Inequalities, *Journal of Global Optimization*, 40, pp. 463- 473, (2008).

[14] Rezaie, M., Zafarani, J., Vector Optimization and Variational like Inequalities, *Journal of Global Optimization*, 43, pp. 47-66, (2009).

[15] Karamardian, S., Schaible, S., Seven Kinds of Monotone Maps, *Journal of Optimization Theory and Applications*, 66, pp. 37-46, (1990).

[16] Hadjisavvas, N., Schaible, S., On Strong Pseudomonotonicity and (Semi) Strict Quasimonotonicity, *Journal of Optimization Theory and Applications*, 79, 1, pp. 139-155, (1993).

[17] Schaible, S., Generalized Monotonicity, A Survey, *Generalized Convexity*, Edited by S. K. Komlosi, T. Rapesak and S. Schaible, Springer Verlag, Berlin, Germany, pp. 229-249, (1994).

[18] Cambini, R., Some New Classes of Generalized Concave Vector Valued Functions, *Optimization*, 36, pp.11-24, (1996).

[19] Cambini, A., Martein, L., Generalized Convexity and Optimality Conditions in Scalar and Vector Optimization, *Handbook of Generalized Convexity and Generalized Monotonicity*. Edited by N. Hadjisavvas, Komlosi and S. Schaible, Springer Netherlands, 76, pp. 151-193(2005).

[20] Vani, *The Study of Optimality and Duality in Vector Optimization Problems*, Thesis (PhD), University of Delhi, (2007).

[21] Clarke, F. H., *Optimization and Nonsmooth Analysis*, Wiley Interscience, (1983).

[22] Bector, C. R., Chandra, S., Dutta, J., *Principles of Optimization Theory*, India: Narosa Publishing House, (2005).

[23] Swaragi, Y. H., Nakayama, H., Tanino, T., *Theory of Multiobjective Optimization*, Academic Press, New York, (1985).

[24] Jahn, J., *Vector Optimization: theory, applications, and extensions*. Springer–Verlag Berlin Heidelberg, (2004).

[25] Aggarwal, S., *Optimality and Duality in Mathematical Programming Involving Generalized Convex Functions*, Thesis (PhD), University of Delhi, (1998).

Surjeet Kaur Suneja *is a (Retd.) Associate Professor from the Department of Mathematics, Miranda House, University of Delhi. She completed her M.Sc. in 1969 and Ph.D. in 1984 from the University of Delhi. She has published 60 research papers in various well reputed journals. Nine Students have completed their Ph.D. and 21 students have done their Mphil*

under her supervision. At present she is supervising 2 M.Phil. and 4 Ph.D. students. Her areas of interest include generalized convexity, vector optimization, nonsmooth optimization, set valued optimization and variational inequality problems.

Bhawna Kohli *completed her M.Sc. in 2001 and M.Phil. in 2004 from the Department of Mathematics, University of Delhi. At present she has submitted her Ph.D. thesis in the area of Mathematical programming in the Department of Mathematics, University of Delhi. She has published 5 research papers in various well known journals. Her areas of interest include Nonsmooth Optimization, vector optimization, bilevel programming and variational inequality problems.*