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RESEARCH ARTICLE

Modeling the dependency structure between quality characteristics in multi-stage manufacturing processes with copula functions

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ABSTRACT

This study is about multi-stage manufacturing processes and their control by statistical process control modeling. There are two kinds of dependence structures in a multi-stage manufacturing process: one is the dependence between the stages of the process, and the other is the dependence between the concerned quality characteristics. This study employs state-space models to demonstrate the dependency structure between the process stages and uses the Kalman filter method to estimate the states of the processes. In this setup, copula modeling is proposed to determine the dependence structure between the quality characteristics of interest. A simulation study is conducted to assess the model's accuracy. As a result, it was found that the model gives highly accurate predictions according to the mean absolute percentage error (MAPE) criteria (<10%).

1. Introduction

Today, production and service processes generally consist of many serial or parallel stages in which products are completed by passing from one to the other. In a multi-stage manufacturing process, it is not clear from which stage and which variables characterizing the process arise the variability in quality characteristics. The key to reducing quality variability in a product is understanding how, much of this variability occurs at each stage of the process and how much is transmitted to other stages.

The most important problem in the multi-stage manufacturing process is how to define the process in the context of interactions within and between stages and time dynamics. In past research, multistage processes have been described with statistical models such as the linear regression model. Conversely, for more effective monitoring and control of the process, engineering knowledge must also be combined with statistics in modeling and analysis of the multi-stage process. In this context; Many articles can be found in the sources that describe the multi-stage manufacturing process in a linear state-space model structure based on production engineering knowledge. A complex system, such as a multistage manufacturing process, may have many inputs and outputs. These inputs and outputs can be complexly interrelated. The hierarchical structure of the data obtained can be explained by multi-level dynamic models. An example of this is a two-level linear state-space model.

In this study, in addition to a dynamic modeling approach such as the state-space model of the dependency between stages in multi-stage manufacturing processes, it is proposed to use copula modeling to reveal the internal dependencies of the quality characteristics of interest at each stage. In order to present the practical implications of the proposed model, the process was simulated and the applicability of the model was discussed.

The following sections of the study are organized as follows: In the second section, studies on statistical process control (SPC) methods used for modeling multi-stage manufacturing processes and monitoring these processes will be discussed. In the third section, modeling of multi-stage manufacturing processes with state-space models will be explained. Additionally, this chapter will include the proposal of the Kalman filter method for the statistical estimation of the state variables of the process equations put forward by state-space models. In the fourth chapter, the statistical dependence of quality characteristics and the explanation of dependence with copula functions will be highlighted, and multi-stage manufacturing process modeling under dependence will be presented. Multi-stage manufacturing processes under dependency The example and process simulation of SPC approaches

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will be presented in the fifth chapter. In the sixth chapter, the results of the study and some determinations about future studies as extensions of these will be stated.

2. Literature review

In order to identify out-of-control situations in multi-stage manufacturing processes, SPC methods have been applied to the quality measurements of the product in the final stage of the process. Generally, Shewart, CUSUM, EWMA control charts for univariate quality measurements of the final product; It has been suggested to use Hotelling's T^2 control chart for multivariate quality measurements [1]. Since these control charts were applied to a single stage of the process, they were insufficient to determine the stage that was the source of variability. In another study, quality measurements obtained from each stage of the process were evaluated separately [2]. In this study, where simultaneous confidence intervals were established for the average of each of the quality variables, it was examined whether the quality measurements of interest were within the confidence intervals in terms of the defined quality levels, and it was stated that the explanatory power of the method decreased as the size of the problem increased.

Statistical process control tools used to monitor multi-stage manufacturing processes have a wide place in the literature. These tools can be examined under three headings: multivariate control charts, control charts based on regression modeling, and methods based on engineering-based models.

In many production processes, it may be necessary to simultaneously monitor and control one or more interrelated quality characteristics. Independent examination of quality characteristics causes loss of information to be obtained from the process. The concept of multivariate quality control originated in Hotelling's work in 1947 [3]. In this study, he applied his proposed method to bombardment viewfinder data used in World War II. The most well-known multivariate process monitoring and control method used to monitor the mean vector of the process is Hotelling's T^2 control chart, which is similar to the univariate Shewhart's $\bar{x}$ chart. Applied to multi-stage manufacturing processes, Hotelling's T^2 chart indicates when the entire process is out of control, but does not indicate which stage is out of control. Alternatively, quality metrics at each stage can be tracked with T^2 cards. In this case, the effect of the quality output of the previous stage on the quality measurements at a certain stage will be ignored. As a result, it is difficult to interpret an out-of-control situation in a multi-stage manufacturing process with a T^2 chart [1]. Following this pioneering work by Hotelling, control methods for many related variables have been proposed [4]. Nowadays, the issue of multivariate quality control (or process monitoring) has maintained its importance as many quality characteristics of products manufactured with automatic inspection methods can be measured at the same time. For example; Chemical and semiconductor manufacturers try to keep the process under control by constantly updating their databases for hundreds of important variables in their manufacturing processes.

It was thought that quality measurements in multi-stage manufacturing processes are affected by the output of the previous stage and the regression analysis technique was introduced [5]. This method is based on establishing univariate control cards for the residuals obtained from the multivariate regression line established on other variables for each quality variable [6]. Regression models can give misleading results when quality measurements from different stages are strongly correlated with each other. This problem in regression analysis can be partially reduced by the cause-selection method and is effective in identifying out-of-control stages [7]. A compilation of cause-selection method studies was compiled by Wade and Woodall [8]. Nowadays, the use and applications of cause-selection schemes for multi-stage processes are also found in Shu and Tsung's article [9].

The hierarchical structure of data obtained from the multistage manufacturing process suggests a two-level model: At the first level, quality measurements are fitted to the system input and quality information. At the second level, the change in quality measurements is modeled as a function of measurements obtained from earlier stages of the process. An example of this situation is the state-space model.

Quality measurements for the kth stage of a production process consisting of N stages are formulated as a linear state-space model as in Eq. (1) and Eq. (2) [10].

$$x_k = A_{k-1}x_{k-1} + v_k \quad (1)$$

$$y_k = C_k x_k + w_k \quad \{k\} \subset \{1,2,\dots,N\} \quad (2)$$

In Eq. (1), x_k shows unobservable product quality information such as dimensional deviations of products at the kth stage. v_k indicates the cause of variability and unmodelable errors (process noise). $A_{k-1}x_{k-1}$ shows the transformation of quality information from the $(k-1)$th stage to the kth stage. In Eq. (2), w_k is the measurement error of the product, and C_k is the matrix used to relate x_k with quality measurements (y_k).

A_{k-1} and C_k are constant matrices obtained from engineering knowledge, laws of physics and process/product design information and known at the kth stage of the process. For univariate cases, $v_k \sim N(0, \sigma_{v_k}^2)$ and $w_k \sim N(0, \sigma_{w_k}^2)$ with its variance depending on the stage index k and the initial state $x_0 \sim N(a_0, \tau^2)$. Various methods have been researched to monitor whether the process is out of control, and fixture errors, machine errors and thermal errors in the process are seen as process out of control or process errors.

A multistage manufacturing process can have many inputs and outputs. These inputs and outputs can be intricately related to each other. There are many articles explaining multi-stage manufacturing processes with state-space models based on process management expertise. Lawless et al. [11] and Agrawal et al. [12] revealed quality variability in multi-stage manufacturing processes with AR(1) type models in the form of state-space models. Part assembly process [12] and sheet metal assembly [13] are examples of modeling proposals in the form of state-space model. Detailed descriptions of state-space models can be found in [10] and [14]. There are many studies in the literature on error detection, error prevention and corrective methods in multi-stage manufacturing operations. Tsung et al.'s study compiled past studies on multi-stage manufacturing and service operations and provided ideas for future research [15].

Today, modeling for monitoring and control of multi-stage manufacturing processes, which have become more complex with developing technology, is a complex issue that still maintains its importance. State-space models are a modeling method that has a wide place in the literature and includes the physics rules surrounding engineering and production structures suitable for the structure of multi-stage manufacturing processes. In a dynamic system represented by a state-space model, the state of the system can be predicted from the input and output information together with the previous information of the model. Estimation of the state of the system from a series of noisy measurements obtained from a dynamic system can be made with the Kalman filter.

In this study, it is suggested to model the dependency structure between quality characteristics with copula and combine it with Kalman filter. Some studies in which copulas, Kalman filter and/or state space models are used together are given in Table 1.

Table 1. Some selected studies on copulas, Kalman filter and state space models.

Authors	Methods/Models	Examples/Application Area
Lindsey [16].	Kalman filter and copulas	The application to autoimmunity in multiple sclerosis data
Junker, Szimayer and Wagner [17	Kalman filter based on copula functions	Nonlinear cross-sectional dependence in the term structure of US-Treasury yields and points out risk management implications
Hafner and Manner [18]	A multivariate stochastic volatility models with Gaussian copula	The application to two bivariate stock index series
Goto [19]	State space model to describe the target system's behaviour	A simulation study conducted to show the effectiveness of the developed controller
Creal and Tsay [20]	Gaussian, Student's t, grouped Student's t, and generalized hyperbolic copulas with time-varying correlations matrices	Modeling an unbalanced, 200-dimensional panel consisting of credit default swaps and equities for 100 US corporations
Alpay and Hayat [21]	Copula and Data Envolopment Analysis (DEA)	The application to simulated and real hospital data
Zhang and Choudhry [22]	Four generalized autoregressive conditional heteroscedasticity (GARCH) models and the Kalman filter method	Empirically forecasting the daily betas of a few European banks during the pre-global financial crisis period and the crisis period
Fernández, García and González-López [23]	Copula and the multivariate Markov chain	Spike prediction in neuronal data
Smith and Maneesoonthorn [24]	Construction of copulas from the inversion of nonlinear state space models	Forecasting of quarterly U.S. broad inflation and electricity inflation
Wang, Meng, Liui Fu and Cau [25]	The Unscented Kalman Filter (UKF), copula and the worst case analysis	A two-stage dynamic attack strategy using global network information
Xu, Liang, Li and Wang [26]	Characterization of the dependence among all components by a copula function	Investigation of the optimal condition-based maintenance policy under periodic inspection for a *K*-out-of-*N*: G system
Kreuzer, Dalla Valle and Czado [27]	Non-linear non-Gaussian state space model	Estimation of airborne pollutant concentrations
Ly, Sriboonchitta, Tang and Wong [28]	A hybrid of ARMA-GARCH, static and dynamic copulas and dynamic state space models	Investigation of dependence and integration among the European electricity markets
Wang, Xu, Trajcevski, Zhang, Zhong and Zhou [29]	A non-linear neural state space model based on copula-augmented mechanism	Electricity forecasting
Kreuzer, Dalla Valle and Czado [30]	Multivariate nonlinear non-Gaussian state space models	The application to atmospheric pollutant measurement data

The rest of the study is organized as follows. In the third section of the study, state-space models will be discussed. In the fourth section, the copulas proposed to model the dependency structure between quality characteristics will be explained in detail. Application of the proposed approach by a simulation study is given in the fifth section. The last section includes the conclusions of the study, and the future studies.

3. Multi-stage manufacturing processes and state-space models

Dynamic systems, such as multistage manufacturing processes, can be more generally represented in the form of state-space models by the equations shown in Eq. (3) and Eq. (4).

$$x_k = A_{k-1}x_{k-1} + B_k u_k + D_k \varepsilon_k \quad (3)$$

$$y_k = C_k x_k + H_k \eta_k \quad (4)$$

Similar to Eq. (1) and Eq. (2), x_k is the state and y_k is the measurement or observation vectors ($k = 1, \ldots, N$). The vectors ε_k and η_k express the noise in the state and the observations, and the vector u_k represents the effects of managerial inputs at the kth stage in the process in Eq. (3) and Eq. (4).

Estimation of the state vector $x_k, k = 1, \ldots, N$ in state-space models and other related analyzes can be done within the framework of three main approaches [31]. These are Bayesian, Fisher and unknown-bounded approaches. In the Bayesian approach, the error terms ε_k and η_k in the equations are stochastic, and the initial state vector x_0 is a random variable. In the Fisher approach, the measurement equation term η_k has a stochastic feature, ε_k can be stochastic or completely unknown, and x_0 can be random. Within the framework of the unknown – bounded approach, ε_k, η_k and x_0 are unknown but are limited from above to the values of the ellipsoids expressing the variance-covariance quantities [32].

When A_{k-1}, B_k and C_k matrices are accepted as known matrices in state-space models, the model estimation problem is solved by using the observation values $y_1, y_2, \ldots, y_{k_1}$ obtained up to time k_1 and estimating x_{k_2} at time k_2. When $k_1 = k_2$, the estimation problem becomes a filtering process, for which Kalman filter (KF) or weighted least squares (WLS) methods can be used. Estimation equations that can be applied within the framework of the Bayesian model approach are known as Kalman filters in the literature [33].

The Bayesian model approach is the most widely used state-space modeling approach and can offer flexible perspectives on the dependence and independence of the vectors ε_k, η_k and x_0 within and among themselves in the time dimension. In this sense, the issues of determining the prior and posterior probability distributions for the random variables in the state-space model and the expected value and covariance functions are needed in estimation process.

Control effects that can be applied in a dynamic stochastic process are represented by the sequence $\{u_k\}$ in state-space models. While control effects, state vectors should be a function of x_k's, in the absence of a complete and direct observation of the situations, measurement or observation values must be considered as a function of y_k's and determined by the opinion of system experts; $u_k = \omega_0\,(y_0, y_1, \ldots, y_k)$. In the literature, it is also recommended to impose a constraint such as $|u_k| \leq 1$ for u_k's [34].

The Kalman filter and its calculation equations are explained in detail in the next section. In the weighted least squares method, the aim is to estimate the state vector with the deviation of x_k, which minimizes the quantity in Eq. (5), where the covariance matrix of the variable η_k is $R_k > 0$.

$$J(x_k) = (y_k - C_k x_k)' R_k^{-1} (y_k - C_k x_k) \quad (5)$$

In Eq. (5), the R_k^{-1} matrix is a positive definite matrix and must be determined in the context of the inputs, states and outputs of the dynamic system of interest. For x_k estimation that gives the smallest value of $J(x_k)$. The solution in Eq. (6) is found for the x_k estimation that gives the smallest value of $J(x_k)$.

$$\hat{x}_k = (C_k R_k^{-1} C_k)^{-1} C_k R_k^{-1} y_k \quad (6)$$

Estimation of x_k in the context of the weighted least squares method for the state-space model in Eq. (5) and Eq. (6); P_0 is the covariance matrix for the initial state vector x_0, and Q_k is the covariance matrix for the vector ε_k, and Equation 7 is obtained by reaching its minimum value under the $x_k = A_{k-1}x_{k-1} + B_k u_k + D_k e_k$ constraint.

$$J(x_k, \varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{k-1}) = \sum_{k=1}^{N} (y_k - C_k x_k)' R_k^{-1} (y_k - C_k x_k) + \sum_{k=0}^{n-1} \varepsilon_k Q_k^{-1} \varepsilon_k + x_0' P_0^{-1} x_0 \quad (7)$$

In Eq. (7), R_k, Q_k and P_0 matrices are positive definite and determined based on expert knowledge about the dynamic system of interest [10].

3.1. State estimation with filtering: Kalman filter

The state of the system may not be directly measurable. In a dynamic system represented by a state-space model, the state of the system can be estimated by using the model's information obtained at previous times and its output information. Kalman filter, which was first introduced by Kalman in 1960, is an effective analysis algorithm that estimates the state of the system from a series of measurements obtained from a dynamic system that may contain error (noise), and updates the estimate as observations are made [35]. The Kalman filter combines measurement data, a priori information about the system, and indirectly measuring state values to make the desired predictions by minimizing the error statistically. Therefore, it gives better results than most other filters for statistical estimation purposes. Within the framework of the Bayesian approach, by conditioning the real data information provided by measuring devices, the spread of conditional probability densities for the features to be estimated can be filtered. Kalman filter helps the purpose of predictive analysis of a system that can be expressed with a linear model, where measurement errors are white noise and normally distributed, by providing conditional probability distribution [36].

For the dynamic and stochastic multi-stage production system represented by the state-space model equations Eq. (3) and Eq. (4), a series of prediction and filtering processes are required in line with the estimation of the state vector x_k at stage k. The difference equations needed for this purpose within the scope of the Bayesian approach are known as Kalman or Kalman-Bucy equations. There are various approaches and generalizations in determining the equations in question, and it is possible to consider equivalent criteria that form the basis for all of them. Minimizing the expected value of prediction error squares is one of these criteria [37].

3.1.1. Minimization of expected value of squared error criteria method

In order to make state estimation with the Kalman filter, explanations about the variables and coefficients in the state-space model equations Eq. (3) and Eq. (4) are given below:

$x_k \in R^n$: System state vector.

$y_k \in R^m$: System observation vector.

A_k: $n \times n$ dimensional system transition matrix.

B_k: $n \times n$ dimensional system input matrix.

C_k: $m \times n$ dimensional observation transition matrix.

u_k: Vector expressing the effect of managerial inputs at time (stage) k.

D_k: $n \times n$ dimensional system noise matrix.

H_k: $m \times n$ dimensional observation noise matrix.

It is assumed that the matrices A_k, B_k, C_k, D_k and H_k are known at all times $k = 0,1,2,....$ The zero-mean white noise processes $\varepsilon_k \in R^n$ and $\eta_k \in R^m$ are assumed to satisfy the following assumptions for each k, j value in Eqs. (8)-(17).

$$E[\varepsilon_k] = 0 \tag{8}$$

$$E[\eta_k] = 0 \tag{9}$$

$$E\left[\varepsilon_k \varepsilon_j'\right] = Q_k \delta_{kj} \tag{10}$$

$$E\left[\eta_k \eta_j'\right] = R_k \delta_{kj} \tag{11}$$

$$\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \tag{12}$$

$$E\left[\varepsilon_k \eta_j'\right] = 0 \tag{13}$$

$$E[x_0] = \bar{x}_0 \tag{14}$$

$$E[(x_0 - \bar{x}_0)(x_0 - \bar{x}_0)'] = P_0 \tag{15}$$

$$E\left[x_0 \varepsilon_j'\right] = 0 \tag{16}$$

$$E\left[x_0 \eta_j'\right] = 0 \tag{17}$$

Table 2. Discrete time Kalman filter equations based on minimization of mean squared errors [38].

System dynamic model: $x_k = A_{k-1}x_{k-1} + B_k u_k + \varepsilon_k$, $\varepsilon_k \sim N(0, Q_k)$
Measurement (observation) model: $y_k = C_k x_k + \eta_k$, $\eta_k \sim N(0, R_k)$
Starting conditions: $x_0 \sim N(\bar{x}_0, P_0)$, $\hat{x}_{0\|-1} = \bar{x}_0$, $P_{0\|-1} = P_0$
Independence conditions: $E\left[\varepsilon_k \eta_j'\right] = 0$, $E\left[x_0 \varepsilon_j'\right] = 0$, $E\left[x_0 \eta_j'\right] = 0$, $\forall k, j$
Estimation of prediction stage: State estimation: $\hat{x}_{k\|k-1} = A_{k-1}\hat{x}_{k-1\|k-1} + B_k u_k$ Measurement condition: $\hat{y}_{k\|k-1} = C_k \hat{x}_{k\|k-1}$ $= C_k\left[A_{k-1}\hat{x}_{k-1\|k-1} + B_k u_k\right]$
Errors of predition stage: State error: $x_k^* = x_k - \hat{x}_{k\|k-1}$ Measurement error: $w_k = y_k - \hat{y}_{k\|k-1} = C_k\left(x_k - \hat{x}_{k\|k-1}\right) + H_k \eta_k$
Covariance matrix of prediction stage: $P_k(w) = E[w_k w_k']$ $= C_k P_{k\|k-1} C_k' + H_k R_k H_k'$
Update of error covariance for prediction stage: For state: $P_{k\|k-1} = A_{k-1} P_{k-1\|k-1} A_{k-1}' + D_k Q_k D_k'$ For measurement: $P_k(y) = C_k[A_{k-1} P_{k-1} A_{k-1}' + D_k Q_k D_k'] C_k' + H_k R_k H_k'$
Observational update of the state estimate: $\hat{x}_{k\|k} = \hat{x}_{k\|k-1} + B_k u_k + P_{k\|k-1} C_k P_k^{-1}(w)\left[y_k - \hat{y}_{k\|k-1}\right]$ $= \hat{x}_{k\|k-1} + K_k\{y_k - C_k \hat{x}_{k\|k-1}\}$
Update of error covariance for filtering stage: $P_{k\|k} = [I - K_k C_k] P_{k\|k-1}$
Kalman gain matrix: $K_k = P_{k\|k-1} C_k' P_k^{-1}(w)$

In addition to all given assumptions, it is assumed that the matrices Q_k and R_k are known. It is aimed to obtain $\hat{x}_{k|m}$ by using observations $\{y_1, y_2, \ldots, y_m\}$ for the best estimation of the x_k vector. In this direction, It is possible to use the covariance matrix $\left(P_{k|m}\right)$ of the estimation error $(x_k - \hat{x}_{k|m})$. When $k = m$, the estimation is called as filtering. Considering that the observations are not error-free, the assumption of $R_k > 0$ will be a realistic and necessary assumption. Let the vector $Y_k = [y_1, \dots, y_k]'$ represent the observations obtained until time (stage) k. If $\vartheta_{k|m}$ is the estimation error of $\hat{x}_{k|m}$ using Y_k, the covariance matrix of this error is expressed as $P_{k|m} = E[\vartheta_{k|m}\vartheta'_{k|m}]$, with $E\left[\hat{x}_{k|m}\right] = E(x_k)$. The estimation of vector x_k is done in two stages with various calculation steps. In Table 2, discrete time Kalman filter equations are summarized according to the method of minimizing the expected value of error squares by showing the filter system relationship.

On the other hand, η_k in the system equation Eq. (4) may become unobtainable. The solution to this problem requires adding additional state equations to the system equation. Bryson and Johanson proposed the first general solution to the problem in question [39]. To solve the problem, Brown and Hwang suggest removing exactly known state variables from the system equations and estimating the remaining ones by filtering [34]. This recommendation requires the separation of system state variables from other exactly known system variables by linear transformation. Simon summarized adequate explanations and methods of Kalman filter application approaches by considering the dependence as linear dependence and correlation for the cases where the random vectors η_k and ε_k are dependent within and between themselves [40].

4. Modeling multi-stage manufacturing processes under the dependency between quality characteristics

In this section, a method is proposed by including copula functions in the approach of modeling and evaluating multi-stage manufacturing processes with state-space models under dependency. It has been suggested to use copula modeling to reveal the internal dependencies of the quality features within the state vector at each stage. With copula models, the stochastic relationship between quality characteristics can be determined by revealing the dependency structure without the need for common distributions of quality characteristics. In this context; Statistical properties such as marginal distributions, covariance, conditional probability distributions (and therefore regression function determination) of quality features that are random variables can be expressed.

4.1. Copula functions

Copula functions are statistical tools used to model dependency. Copulas are functions that combine multivariate distributions with their univariate marginal distributions. Let F be the m-dimensional cumulative distribution function and $F_1, F_2, \dots, F_m$ be the cumulative distribution functions of one-dimensional marginals. In this case, the m-dimensional copula function is defined as in Eq. (18).

$$F(y_1, y_2, \dots, y_m) = C(F_1(y_1), F_2(y_2), \dots, F_m(y_m); \theta) \quad (18)$$

θ in Eq. (18) is called the dependency parameter and the marginal distributions of each of the quality characteristics express the relationship. The most basic theoretical determination about copula functions is put forward by the Scalar theorem.

Theorem 1. *(Sklar's Theorem) The m-dimensional copula is a function C defined from the m-dimensional interval $[0,1]^m$ to the unit interval $[0,1]$ and satisfies the following conditions* [41].

- $C(1, \dots, 1, a_n, 1, \dots, 1) = a_n, \forall n \le m$ and $a_n \in [0,1]$.
- If $a_n = 0$ for any $n \le m$, $C(a_1, \dots, a_m) = 0$.
- C is m-increasing.

In other words, the m-copula is an m-dimensional distribution function with m univariate marginals, each of which is uniformly distributed in the range $(0,1)$.

There are many copula functions belonging to different copula families in the literature. When its application areas are investigated, it is seen that it has widespread use in finance, actuarial, time series and risk analysis. In this study, the focus is on the Gaussian (normal) copula, which belongs to the elliptic copula family and has many useful features.

Definition 1. *(Gaussian Copula) Consider random variables $Z_1, Z_2, \dots, Z_k$ with correlation coefficients $\rho_{ij} = \rho(Z_i, Z_j)$ with multivariate normal probability distribution. Let the joint cumulative distribution function of the random variables $Z_1, Z_2, \dots, Z_k$ be $\Phi_G(z_1, z_2, \dots, z_k)$. In this case, the multivariate Gaussian (Normal) copula is defined in Eq. (19)* [42].

$$(u_1, \dots, u_k) = \Phi_G\left(\Phi^{-1}(u_1), \dots, \Phi^{-1}(u_k)\right) \quad (19)$$

The two-variable Gaussian (Normal) copula is in the form of Eq. (20).

$$C(u_1, u_2; \theta) = \Phi_G(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta) = \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\theta^2)^{1/2}} \times \left\{\frac{-(s^2 - 2\theta st + t^2)}{2(1-\theta^2)}\right\} ds dt \quad (20)$$

In Eq. (20), Φ denotes the cumulative distribution function for the standard normal random variable and $\Phi_G(u_1, u_2)$ denotes the standard bivariate normal distribution with the correlation parameter θ, which takes values in the range of $(-1,1)$. This copula function was proposed by Lee in 1983 [43]. *The density function of the two-variable Gaussian copula is also in the form in Eq. (21).*

$$c(u_1, u_2; \theta) = \frac{1}{\sqrt{1-\theta^2}} \exp\left[\frac{-(u_1^2 - 2\theta u_1 u_2 + u_2^2)}{2(1-\theta^2)}\right] \times \exp\left(\frac{u_1^2 + u_2^2}{2}\right) \quad (21)$$

According to the scalar theorem, the bivariate probability distribution of the random vector $X = (X_1, X_2)'$ can be determined by the non-normal (any distribution) marginal distributions of the vector and the Gaussian copula [44].

In order to determine the probability distribution of a random vector $X = (X_1, X_2)'$, it is necessary to determine the marginal distribution of each X_j and find the dependency structure between X_j. In order to determine the dependency structure between random variables, it is necessary to mention the measures and some special dependency structures included in the copula functions. There is a relationship between copula functions expressing dependence and dependence measurements, especially for two-variable cases. Dependency can be measured by many methods. The Pearson correlation coefficient is one of them; it is sensitive to outliers and does not change under strictly increasing linear transformations. The expression of the Pearson correlation coefficient in terms of copulas is shown in Eq. (22) [45].

$$\rho_P(X, Y) = \frac{1}{\sigma_X \sigma_Y} \int_0^1 \int_0^1 [C(u_1, u_2) - u_1 u_2] dF_X^{-1}(u_1) dF_Y^{-1}(u_2), \qquad u_i \in [0,1] \quad (22)$$

4.2. Integration of state-space model with copula modeling

In this section, the state vector of quality characteristics under dependency is estimated by combining the state-space model, Kalman filtering and copula functions for multi-stage manufacturing processes. Therefore, a unique approach has been introduced to monitor quality in a multi-stage manufacturing process.

4.2.1. Prediction error

Considering the general state-space model representation of a multi-stage manufacturing process with Eq. (3) and Eq. (4), the Kalman filter method for estimating the state vector x_k is introduced in Section 3.1. In the prediction phase of the estimation, it was seen that the uncertainty in the state vector $\hat{x}_{k|k-1}$ is a function of the estimation of $\hat{x}_{k-1|k-1}$ and the covariance Q_k of ε_k. In the next step, the prediction error components for vector x_k are; The statistical inference prediction error for x_k is $x_k - \hat{x}_{k|k-1}$ and the prediction error for the observation vector y_k is η_k. Therefore, as expressed in Table 2, the conditional variance of the prediction error given in Eq. (23) should be evaluated as a function of the uncertainties or errors related to $\hat{x}_{k|k-1}$ and R_k.

$$P_k(w) = C_k P_{k|k-1} C_k' + H_k R_k H_k' \quad (23)$$

According to the information obtained up to stage or time $k-1$, based on the conditional probability distribution of x_k to $y_k - \hat{y}_{k|k-1}$, the final estimation $\hat{x}_{k|k}$ and its covariance $P_{k|k-1}$ are obtained. Assuming that the joint probability distribution $X = x_k$ and $Y = y_k - \hat{y}_{k|k-1}$ is a normal distribution given in Eq. (24), the conditional probability distribution X given that Y is $N(\mu_{X|y}, \Sigma_{XX|y})$ with parameters $\mu_{X|y} = \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(Y - \mu_Y)$ and $\Sigma_{XX|y} = \Sigma_{XX} - \Sigma_{XY}\Sigma_{YY}^{-1}\Sigma_{YX}$.

$$\left[\begin{pmatrix}\mu_X \\ \mu_Y\end{pmatrix}, \begin{pmatrix}\Sigma_{XX} & \Sigma_{XY} \\ \Sigma_{YX} & \Sigma_{YY}\end{pmatrix}\right] \quad (24)$$

In Eq. (24), $\mu_X = \hat{x}_{k|k-1}$, $\Sigma_{XX} = P_{k|k-1}$, $\Sigma_{XY} = P_{k|k-1}C_k'$ and $\Sigma_{YY} = P_k(w)$ (as in Eq. (23)) are some definitions. It is seen that by using these definitions, the expressions $\hat{x}_{k|k}$ ve $P_{k|k}$ which are the final estimates in the second stage of the Kalman filter, will be reached (see Table 2). The importance of the conditional variance $P_k(w)$ of the prediction error in the estimation of x_k can be revealed from another perspective. For example; assuming that x_0, ε_k and η_k are random variables whose joint distribution is the normal distribution, the probability distribution of the random vector y_k conditional on the information $\{y_1, y_2, \dots, y_{k-1}\}$ is normal distribution $N[y_{k|k-1}, P_k(w)]$, the estimates of x_k depend on the parameters of the distribution, conditional expected value and conditional variance-covariance $y_{k|k-1}$ and $P_k(w)$, respectively. The estimation of the model parameters of interest can be achieved by maximizing the function given in Eq. (25), which expresses the log-likelihood in the context of all observation values.

$$L_C^{(k)}(whole) = \ln\mathcal{L} = -\frac{1}{2}\sum_k \ln[2\pi|P_k(w)|] - \frac{1}{2}\sum_k w_k'[P_k(w)]^{-1}w_k = \sum_{k=1}^{N} L_C^{(k)} \quad (25)$$

In the filtering stage, equations that express the estimates $\hat{x}_{k|k}$ and $P_{k|k}$ emerge. This was mentioned in Section 3.1, where the error quantities $y_k - \hat{y}_{k|k-1} = w_k$, and the variance expression of these error quantities $P_k(w)$ are highlighted and discussed in state vector estimation. It has been emphasized more clearly that it constitutes the necessary essential element. Maximazing the likelihood function specified in Eq. (25) is equivalent to minimizing the quantity in Eq. (26), especially under the assumption of a normal distribution in general.

$$J = E\left[y_k - C_k\hat{x}_{k|k-1}\right]\left[y_k - C_k\hat{x}_{k|k-1}\right]' \quad (26)$$

The assumptions and definitions for an illustrative example of this when the state vector x_k is a two-element random vector with two quality characteristics are as follows:

- x_k is a 2×1 dimensional state vector and y_k is a 2×1 dimensional measurement vector.
- The quality characteristics in the measurement vector have marginal distributions $y_{k1} \sim F_{k1}(.)$ and $y_{k2} \sim F_{k2}(.)$, respectively, and it is assumed that the internal dependency structure between them is modeled with an appropriate copula. The cumulative joint probability distribution of random variables y_{k1} and y_{k2} can be determined through copula functions as $F_{k12}(y_{k1}, y_{k2})$.
- $E(y_{ki}) = \mu_{ki}, \quad Var(y_{k1}) = \sigma_{ki}^2, \quad E(y_{ki}^2) = \sigma_{ki}^2 + \mu_{ki}^2, \quad k = 1,2,\ldots,N; \; i = 1,2.$
- $Cov(y_{k1}, y_{k2}) = \sigma_{k12}, \quad E(y_{k1}y_{k2}) = \sigma_{k12} + \mu_{k1}\mu_{k2}, \quad k = 1,2,\ldots,N.$
- It is assumed that the 2×2 dimensional matrices A_k, B_k and C_k are known.
- $y_k = \begin{bmatrix} y_{k1} \\ y_{k2} \end{bmatrix}, \quad \hat{x}_{k|k-1} = \begin{bmatrix} \hat{x}_{k1} \\ \hat{x}_{k2} \end{bmatrix}, \quad \hat{x}_{k-1|k-1} = \begin{bmatrix} \hat{x}_{(k-1)1} \\ \hat{x}_{(k-1)2} \end{bmatrix}, \quad u_k = \begin{bmatrix} u_{k1} \\ u_{k2} \end{bmatrix} \quad k = 1,2,\ldots,N.$
- $B_k = \begin{bmatrix} b_{11}^{(k)} & b_{12}^{(k)} \\ b_{21}^{(k)} & b_{22}^{(k)} \end{bmatrix}, \quad A_{k-1} = \begin{bmatrix} a_{11}^{(k-1)} & a_{12}^{(k-1)} \\ a_{21}^{(k-1)} & a_{22}^{(k-1)} \end{bmatrix}, \quad C_k = \begin{bmatrix} c_{11}^{(k)} & c_{12}^{(k)} \\ c_{21}^{(k)} & c_{22}^{(k)} \end{bmatrix}, \quad k = 1,2,\ldots,N.$

If Eq. (26) is rewritten according to the definitions, the matrix in Eq. (27) is obtained.

$$J = E\left[y_k - C_k\hat{x}_{k|k-1}\right]\left[y_k - C_k\hat{x}_{k|k-1}\right]' = E\begin{bmatrix} N & L \\ L & M \end{bmatrix} \qquad (27)$$

The expansion of the matrix elements in Eq. (27) is given in Eqs. (28)-(30).

$$N = \left(y_{k1} - c_{11}^{(k)}\hat{x}_{k1} - c_{12}^{(k)}\hat{x}_{k2}\right)^2 \qquad (28)$$

$$M = \left(y_{k2} - c_{21}^{(k)}\hat{x}_{k1} - c_{22}^{(k)}\hat{x}_{k2}\right)^2 \qquad (29)$$

$$L = \left(y_{k1} - c_{11}^{(k)}\hat{x}_{k1} - c_{12}^{(k)}\hat{x}_{k2}\right) \times \left(y_{k2} - c_{21}^{(k)}\hat{x}_{k1} - c_{22}^{(k)}\hat{x}_{k2}\right) \qquad (30)$$

When the elements of matrix J are considered separately, the expected values in Eqs. (31)-(33) are obtained.

$$E(N) = \sigma_{k1}^2 + \mu_{k1}^2 - 2\left(c_{11}^{(k)}\hat{x}_{k1} + c_{12}^{(k)}\hat{x}_{k2}\right)\mu_{k1} + \left(c_{11}^{(k)}\hat{x}_{k1} + c_{12}^{(k)}\hat{x}_{k2}\right)^2 \qquad (31)$$

$$E(M) = \sigma_{k2}^2 + \mu_{k2}^2 - 2\left(c_{21}^{(k)}\hat{x}_{k1} + c_{22}^{(k)}\hat{x}_{k2}\right)\mu_{k1} + \left(c_{21}^{(k)}\hat{x}_{k1} + c_{22}^{(k)}\hat{x}_{k2}\right)^2 \qquad (32)$$

$$E(L) = \sigma_{k12} + \mu_{k1}\mu_{k2} - \left(c_{21}^{(k)}\hat{x}_{k1} + c_{22}^{(k)}\hat{x}_{k2}\right)\mu_{k1} - \left(c_{11}^{(k)}\hat{x}_{k1} + c_{12}^{(k)}\hat{x}_{k2}\right)\mu_{k2} + \left(c_{11}^{(k)}\hat{x}_{k1} + c_{12}^{(k)}\hat{x}_{k2}\right) \times \left(c_{21}^{(k)}\hat{x}_{k1} + c_{22}^{(k)}\hat{x}_{k2}\right) \qquad (33)$$

If the system transition matrix C_k is optimized (minimum), the partial derivatives of the expected values according to the elements of the C_k matrix are equal to zero. Then, the values in Eq. (34) and Eq. (35) for $c_{11}^{(k)}$and $c_{12}^{(k)}$ are obtained.

$$c_{11}^{(k)} = \frac{\mu_{k1} - c_{12}^{(k)}\hat{x}_{k2}}{\hat{x}_{k1}} \qquad (34)$$

$$c_{12}^{(k)} = \frac{\mu_{k1} - c_{11}^{(k)}\hat{x}_{k1}}{\hat{x}_{k2}} \qquad (35)$$

It is necessary to test that the expressions in Eq. (34) and Eq. (35) are the values that minimize $E(N)$. The values found for this are the values that make the second derivatives of $E(N)$ with respect to $c_{11}^{(k)}$ and $c_{12}^{(k)}$ greater than zero, ($\partial^2 E(N)/\partial\left(c_{11}^{(k)}\right)^2 = 2\hat{x}_{k1}^2 > 0$ and $\partial^2 E(N)/\partial\left(c_{12}^{(k)}\right)^2 = 2\hat{x}_{k2}^2 > 0$, will be the values that minimize $E(N)$. Similarly, if the partial derivatives according to $c_{21}^{(k)}$ and $c_{22}^{(k)}$ in $E(M)$ are taken and set equal to zero, the expressions in Eq. (36) and Eq. (37) are obtained.

$$c_{21}^{(k)} = \frac{\mu_{k2} - c_{22}^{(k)}\hat{x}_{k2}}{\hat{x}_{k1}} \qquad (36)$$

$$c_{22}^{(k)} = \frac{\mu_{k2} - c_{21}^{(k)}\hat{x}_{k1}}{\hat{x}_{k2}} \qquad (37)$$

Since, $\partial^2 E(M)/\partial\left(c_{21}^{(k)}\right)^2 = 2\hat{x}_{k1}^2 > 0$ and $\partial^2 E(M)/\partial\left(c_{22}^{(k)}\right)^2 = 2\hat{x}_{k2}^2 > 0$, the values in Eq. (36) and Eq. (37) are the values that minimize $E(M)$. If these values are substituted in $E(L)$, the result will be as in Eq. (38) for k=1,2,…,N.

$$E(L) = E(Y_{k1}Y_{k2}) - \mu_{k1}\mu_{k2} = Cov(Y_{k1}, Y_{k2}) = \sigma_{k12}, \qquad k = 1,\ldots,N \qquad (38)$$

In conclusion, the dependency between quality characteristics at any stage k is a phenomenon that affects the quality values of the production process. In the derivation made above, it is seen that the variance and covariance values directly affect the values symbolizing the quality status of the system, under the assumptions about the moments of the Y variables, which express the observable values of the X variables, which are the quality characteristics. Considering that variance and covariance values are quantities that determine correlation values; The conclusion is that the dependence, which can be expressed in general and specifically in the context of Gaussian copulas, is effective in the Kalman filter state estimation equations.

To state this more clearly, let us consider the Pearson correlation measure $\rho_P = (y_{k1}, y_{k2}) = \sigma_{k12}(\sigma_{k1}\sigma_{k2})^{-1}$ in the context of observations for a two-element state vector. Pearson correlation measure can be expressed as a function of the copula function $C(.,.)$ and the marginal distributions F_1 and F_2, as shown in Eq. (22), in the form in Eq. (39).

$$\rho_P(y_{k1}, y_{k2}) = (\sigma_{k1}\sigma_{k2})^{-1} \int_0^1 \int_0^1 [C(u_1, u_2) - u_1u_2]\, dF_1^{-1}(u_1)F_2^{-1}(u_2), \quad u_i \in [0,1], \quad i = 1,2 \quad (39)$$

By expressing the covariance σ_{k12} given in Eq. (38) in terms of copula, using Eq. (22), the adequacy of combining copula functions in the estimation of state-space models through the Kalman filter is demonstrated with Eq. (40).

$$\sigma_{k12} = (\sigma_{k1}\sigma_{k2})\rho_P(y_{k1}, y_{k2}) = \int_0^1 \int_0^1 [C(u_1, u_2) - u_1u_2]\, dF_1^{-1}(u_1)F_2^{-1}(u_2) \quad (40)$$

4.2.2. Copula likelihood functions

Considering the copula functions and Sklar's Theorem, the joint probability distribution function for the elements of the observation vector y_k, which takes continuous values, will be in the form in Eq. (41) with the expression of the copula function.

$$F_{k12}(y_{k1}, y_{k2}; \gamma, \theta) = C(F_{k1}(y_{k1}; \gamma), F_{k2}(y_{k2}, \gamma); \theta) \quad (41)$$

In Eq. (41), the vector γ represents the probability distribution parameters except the dependence parameter θ between y_{k1} and y_{k2}. It is not necessary for θ parameter to express only correlation. If the distribution function in Eq. (41) is differentiated according to (y_{k1}, y_{k2}), the joint probability density function in Eq. (42) is obtained $k = 1,2, \ldots, N$.

$$f_{k12}(y_{k1}, y_{k2}; \gamma, \theta) = c(F_{k1}(y_{k1}; \gamma_1), F_{k2}(y_{k2}; \gamma_2); \theta) \times f_{k1}(y_{k1}; \gamma_1) f_{k2}(y_{k2}; \gamma_2) \quad (42)$$

$c(.,.;\theta)$ in Eq. (42) is the copula density function corresponding to $C(.,.;\theta)$. Assuming that there are n observations that can be expressed as $(y_{k11}, \ldots, y_{k1n})$ and $(y_{k21}, \ldots, y_{k2n})$ for each of the y_k vector elements y_{k1} and y_{k2} at any stage or time k, the copula log-likelihood function, For k =1,2,…,N, the expression in Eq. (43) is obtained.

$$\mathcal{L}_c^{(k)} = \sum_{i=1}^{n} \ln f_{k12}(y_{1k}, y_{2k}; \gamma, \theta) = \sum_{i=1}^{n} \ln c(F_{k1}(y_{k1}; \gamma_1), F_{k2}(y_{k2}; \gamma_2); \theta) + \sum_{i=1}^{n} \ln f_{k1}(y_{k1}; \gamma_1) + \sum_{i=1}^{n} \ln f_{k2}(y_2; \gamma_2) \quad (43)$$

The difference between the expression in Eq. (43) and the ordinary log-likelihood function is that the sum of the log copula density functions is included in the equation. In the observation vector given in Eq. (3) where $H_k = I_k$, $\eta_{k1} = y_{k1} - c_{11}^{(k)}x_{k1} - c_{12}^{(k)}x_{k2}$ and $\eta_{k2} = y_{k2} - c_{21}^{(k)}x_{k1} - c_{22}^{(k)}x_{k2}$ are defined as in the covariance matrix R_k in Eq. (44) for $y_k = (y_{k1}, y_{k2})'$ with $\eta_k = (\eta_{k1}, \eta_{k2})' \sim N(0, R_k)$, $k = 1,2, \ldots, N$.

$$R_k = \begin{bmatrix} Var(\eta_{k1}) & Cov(\eta_{k1}, \eta_{k2}) \\ Cov(\eta_{k1}, \eta_{k2}) & Var(\eta_{k2}) \end{bmatrix} = \begin{bmatrix} \sigma_{\eta_{k1}}^2 & \rho_k\sigma_{\eta_{k1}}\sigma_{\eta_{k2}} \\ \rho_k\sigma_{\eta_{k1}}\sigma_{\eta_{k2}} & \sigma_{\eta_{k2}}^2 \end{bmatrix} \quad (44)$$

When the marginal density functions and joint probability density functions are given in Eq. (45), Eq. (46) and Eq. (47), respectively, the corresponding Gaussian (normal) copula density function expression can be calculated only with the help of the marginal distribution functions as in Eq. (48).

$$f_{k1}\left(y_{k1}; x_{k1}, x_{k2}, c_{11}^{(k)}, c_{12}^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_{\eta_{k1}}^2}} \exp\left\{-\frac{\left(y_{k1} - c_{11}^{(k)}x_{k1} - c_{12}^{(k)}x_{k2}\right)^2}{2\sigma_{\eta_{k1}}^2}\right\} \quad (45)$$

$$f_{k2}\left(y_{k2}; x_{k1}, x_{k2}, c_{21}^{(k)}, c_{22}^{(k)}\right) = \frac{1}{\sqrt{2\pi\sigma_{\eta_{k2}}^2}} \exp\left\{-\frac{\left(y_{k2} - c_{21}^{(k)}x_{k1} - c_{22}^{(k)}x_{k2}\right)^2}{2\sigma_{\eta_{k2}}^2}\right\} \quad (46)$$

$$f_{k12}(y_k; x_k, C_k, R_k) = \frac{1}{2\pi\sqrt{|R_k|}} \times \exp\left\{-\frac{1}{2}(y_k - C_kx_k)'R_k^{-1}(y_k - C_kx_k)\right\} \quad (47)$$

$$c(F_{k1}(y_{k1}; x_{k1}, \gamma_{1k}), F_{k2}(y_{k2}; x_{k2}, \gamma_{2k}); \theta) = \frac{1}{\sigma_{\eta_{k1}}^2\sigma_{\eta_{k2}}^2 - \rho_k^2} \exp\left\{-\frac{(\eta_{k1}^2 - 2\rho_k\eta_{k1}\eta_{k2} + \eta_{k2}^2)}{2(1-\rho_k^2)}\right\} \times \exp\left(\frac{\eta_{k1}^2 + \eta_{k2}^2}{2}\right) \quad (48)$$

As seen from the copula log-likelihood function obtained in Eq. (43), the value size of the copula log-likelihood function is determined by the dependency parameter values when other parameters are given. Under normal distribution, the dependence parameter θ is the parameter expressed in terms of moment factors and corresponding to the Pearson correlation.

4.2.3. Copula functions and Kalman filter

In order to define the stochastic dependency structure between the quality characteristics of a product in multi-stage manufacturing processes with copula functions, it is sufficient to know the marginal probability distributions of the characteristics. By analyzing the representation of multi-stage manufacturing processes with the state-space models approach under dependency, it is possible to make Kalman filter estimations better trackable and interpretable on the basis of copula likelihood functions. To better express this, it would be useful to express the combination of Kalman filter estimation steps with copula functions, as shown in Table A1 in Appendix.

For the explicit expression of the copula log-likelihood functions $L_c(prediction)$ and $L_c(whole)$ in Table A1, it is necessary to know or predict the likelihood distribution models for the state-space model state vector x_k and therefore the observation vector y_k. In the predictions made for the Kalman filter method state-space model, normal distribution is assumed for the relevant model variables, and it is stated by many researchers that the predictions are efficient under these conditions [42].

For this reason, it is necessary to determine copula density functions and copula log-likelihood functions under certain distributions by using Eq. (43) to express the $L_c(prediction)$ and $L_c(whole)$ functions in Table A1.

For example; when the joint probability distribution of x_0, e_k and η_k is a normal distribution, the $L_c(prediction)$ and $L_c(whole)$ functions for the y_k observation vector will be as in Eq. (49) and Eq. (50).

$$L_c(prediction) = \sum_{i=1}^{n} \ln f_{k12}(y_{1k}, y_{2k}; \gamma, \theta)$$
$$= \sum_{i=1}^{n} \ln c(F_{k1}(y_{k1}; \gamma_1), F_{k2}(y_{k2}; \gamma_2); \theta)$$
$$+ \sum_{i=1}^{n} \ln f_{k1}(y_{k1}; \gamma_1) + \sum_{i=1}^{n} \ln f_{k2}(y_2; \gamma_2) \quad (49)$$

$$L_c(whole) = \sum_{i=1}^{N} L_c(prediction) \quad (50)$$

The copula density function in Eq. (49) is defined in Eq. (48). On the other hand, γ_1 and γ_2 in the expressions $f_{k1}(y_{k1}; \gamma_1)$ and $f_{k2}(y_{k2}; \gamma_2)$ defined in Eq. (45) and Eq. (46) show the distribution parameters.

It has been stated in the previous sections that in determining the x_k and $P_{k|k}$ expressions in the filtering stage of the Kalman filter equations, the likelihood function should be maximized or, equivalently, the sum of squares of the errors $w_k = y_k - \hat{y}_{k|k-1}$ should be minimized. .In this regard, copula log-likelihood functions must be determined to write the $L_c(prediction)$ and $L_c(whole)$ expressions shown in Table A1, the joint probability function of the random variables x_k and $w_k = y_k - \hat{y}_{k|k-1}$ with normal distribution and the copula function were used.

Let $y_k = (y_{k1}, y_{k2})'$ be the values observed about the quality characteristics of the production process at the kth stage of the multi-stage manufacturing process. In the case of the existence of an observation set of size n, considering the equations $x_{k|k-1}, P_{k|k-1}$, $E(w_k w_k') = P_k(w)$ in Table A1, the probability density functions and the copula density function are given in Eqs.(51)-(54) where $w_{ki} = y_{ki} - y_{k|k-1}^{(i)}$, $i = 1,2$.

$$f_{k1}(w_{k1}; x_{k1}, \gamma_1)$$
$$= \frac{1}{\sqrt{2\pi\sigma_{k1}^2}} \exp\left\{-\frac{\left(y_{k1} - y_{k|k-1}^{(1)}\right)^2}{2\sigma_{w_{k1}}^2}\right\}, \quad (51)$$

$$f_{k2}(w_{k2}; x_{k2}, \gamma_2)$$
$$= \frac{1}{\sqrt{2\pi\sigma_{k2}^2}} \exp\left\{-\frac{\left(y_{k2} - y_{k|k-1}^{(2)}\right)^2}{2\sigma_{w_{k2}}^2}\right\}, \quad (52)$$

$$f_{k12}(w_{k1}, w_{k2}; x_{k1}, x_{k2}, \gamma_1, \gamma_2, \theta)$$
$$= \frac{1}{2\pi\sqrt{|P_k(w)|}} \exp\left\{-\frac{1}{2}\left(y_k - y_{k|k-1}\right)' P_k(w)\left(y_k - y_{k|k-1}\right)\right\}, \quad (53)$$

$$c(F_{k1}(w_{k1}; x_{k1}, \gamma_1), F_{k2}(w_{k2}; x_{k2}, \gamma_2); \theta)$$
$$= \frac{1}{\sigma_{w_{k1}}^2 \sigma_{w_{k2}}^2 - \rho_k^2} \exp\left\{-\frac{(w_{k1}^2 - 2\rho_k w_{k1} w_{k2} + w_{k2}^2)}{2(1 - \rho_k^2)}\right\}$$
$$\times \exp\left(\frac{w_{k1}^2 + w_{k2}^2}{2}\right) \quad (54)$$

$y_{k|k-1}^{(i)}, i = 1,2$ in Eq. (51) and Eq. (52) shows the prediction made for the ith element of vector y_k based on the values observed until the kth stage. $P_k(w)$ in Eq. (53) shows the error covariance matrix in the Kalman filter prediction stage and is in the form in Eq. (55). ρ_k, which shows the correlation between w_{k1} and w_{k2} in the sense of Pearson correlation, is a copula correlation parameter for y_{k1} and y_{k2} since $w_k = y_k - \hat{y}_{k|k-1}$ and $y_{k|k-1}$ are calculated values.

$$P_k(w) = \begin{bmatrix} \sigma_{w_{k1}}^2 & \sigma_{w_{k1}}\sigma_{w_{k2}}\rho_k \\ \sigma_{w_{k1}}\sigma_{w_{k2}}\rho_k & \sigma_{w_{k2}}^2 \end{bmatrix} \quad (55)$$

Then, the copula log-likelihood function derived by the log-likelihood expression in Eq. (43) is given Eq. (56).

$$L_c(w_k | y_1, \dots, y_{k-1})$$
$$= \sum_{j=1}^{n} \ln c\left(F_{k1}\left(w_{k1j}; x_{k1}, \gamma_1\right), F_{k2}\left(w_{k2j}; x_{k2}, \gamma_2\right); \theta\right)$$
$$+ \sum_{j=1}^{n} \ln f_{k1}\left(w_{k1j}; x_{k1}, \gamma_1\right)$$
$$+ \sum_{j=1}^{n} \ln f_{k2}\left(w_{k2j}; x_{k2}, \gamma_2\right) \quad (56)$$

As seen in Eq. (56), when the copula density function dependence parameter is different from zero, the copula log-likelihood function creates an increasing or decreasing effect on its value. The equations $x_{k|k}$ and $P_{k|k}$, which are the expressions of the Kalman filter filtering stage estimates, emerge as a result of the optimization of the copula log-likelihood functions according to w_{k1} and w_{k2} are obtained by using maximum likelihood (MLE) or minimization of the mean squared error (MMSE) approaches, effective estimates for x_k are obtained. When the marginal probability distributions of observation vectors y_k and error vectors w_k are distributions other than the normal distribution, the effectiveness of Kalman filter estimations may decrease.

5. Application of the proposed approach

In this section, a multi-stage manufacturing process is simulated with the modeling method presented and the results are discussed. The production process in the simulation study is based on assumption. Assume that parts are processed in the wood workshop of a factory. The wooden pieces, which are processed through a three-stage manufacturing process, are expected to weigh 150 grams (g) and be 30 centimeters (cm) long at the end of the production process. When unprocessed wood pieces arrive at the factory, they are weighed and their lengths are measured in the input quality control department. Based on past measurements, it will be assumed that the lengths of untreated wood pieces have a normal distribution with a mean of 32 cm and a standard deviation of 0.5 cm. Similarly, the weights of the raw parts will be assumed to have a normal distribution with a mean of 152g and a standard deviation of 1.1g. It will be assumed that the parts entering the processing process are cut in the first stage, sanded in the second stage and polished in the last stage. Fig.(1) shows a representative version of this process.

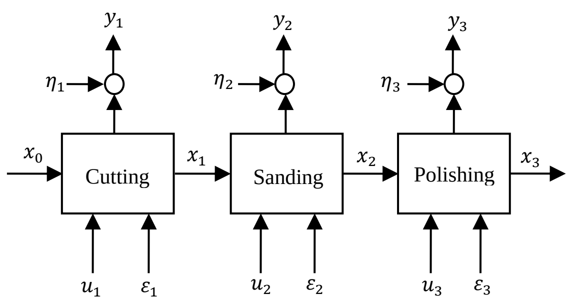

Figure 1. Three-stage processing scheme

The wooden parts, first checked in the input quality control department, are cut to the desired size during the cutting stage. After the parts are sanded, they move on to the polishing stage. Then, the products are left to dry to take their final form. It is assumed that the parts are weighed at the end of each stage and their length measured. Under these assumptions, the state equation will be as in Eq. (57), and the output (measurement) equation will be as in Eq. (58) where $D_k = I_k$ and $H_k = I_k$.

$$x_k = A_{k-1}x_{k-1} + B_k u_k + \varepsilon_k, \qquad k = 1,2,3 \quad (57)$$

$$y_k = C_k x_k + \eta_k\ , \qquad k = 1,2,3 \quad (58)$$

In a multi-stage processing process revealed by the linear state-space model, x_k is the directly unobservable quality characteristic of the product being inspected. x_k is a vector that contains all the information about the current state of the process at the kth stage. In this simulation study, the state vector x_k consists of two quality characteristics: x_{k1} represents the actual value of the part size at the kth stage; x_{k2} shows the actual value of the part weight at the kth stage. The vectors $x_k = [x_{k1} \quad x_{k2}]', k = 1,2,3$ in size (2×1) are known positive definite matrices that show the deviation during the transition from the kth stage to the $(k+1)$th stage of the process given as
$A_0 = [-0.645 \ \ 0.343\ ; -3.165 \ \ 1.660]$,
$A_1 = [-0.639 \ \ 0.329; -3.170 \ \ 1.670]$ and
$A_2 = [-0.761 \ \ 0.352; -3.660 \ \ 1.730]$
with MATLAB notation.

B_k is defined as the input matrix at the kth stage and $u_k = \left[u_1^{(k)} \quad u_2^{(k)}\right]', k = 1,2,3$ is defined as a (2×1) dimensional vector showing the contribution of the kth stage in the state equation. Here $u_i^{(k)}, i = 1,2; k = 1,2,3$ is the contribution of the kth stage to the ith quality characteristic. This contribution is provided by the multiplication of the known matrix B_k and the vector u_k. In this application, $B_1 u_1 = [-0.01 \ -0.025]'$, $B_2\, u_2 = [-0.01 \ -0.01]'$ and $B_3 u_3 = [0.01 \ \ 0.01]'$. Let the unobservable process noise be defined as $\varepsilon_k = \left[e_1^{(k)} \quad e_2^{(k)}\right]', k = 1,2,3$. In the simulation study, it is assumed that $e_i^{(k)} \sim N(\mu = 0, \sigma^2), i = 1,2; k = 1,2,3$. It is also assumed that the measurements can be taken from every stage. In this case, the measurement vector $Y\_k = [y_{k1} \quad y_{k2}]'\,, k = 1,2,3$ can be observed for every value of the phase index k. Let the C_k matrices, which provide the transition between the actual values of the quality characteristics and the measurement values, be determined as $C_1 = [1.30 \ -0.01; 0.01 \ \ 0.99], C_2 = [1.01 \ -0.01; 0.01 \ \ 0.99]$, and $C_3 = [1.01 \ -0.001; 0.01 \ \ 0.99]$.

The elements of the vector $\eta_k = \left[\xi_1^{(k)} \xi_2^{(k)}\right]', k = 1,2,3$, which show the measurement error are distributed normally given as $\xi_i^{(k)} \sim N(\mu = 0\,, \sigma^2), i = 1,2; k = 1,2,3$. In this study, different correlation coefficient values (0.99, 0.90, 0.70, 0.50, 0.30 and 0.1) were tested for 0.1, 0.5 and 1 values of the σ^2 parameter, which indicates the noise level. Additionally, it is assumed that the dependency structure between the length and the weight measurement values can be determined with the Gaussian copula.

In this case, the Kalman filter equations in Table 2 will be taken into consideration for the observed values of quality characteristics $y_k = (y_{k1}, y_{k2})'$ in the kth stage of the multi-stage manufacturing process. Pearson correlation (ρ) between w_{k1} and w_{k2} is a copula correlation parameter for y_{k1} and y_{k2} since $w_k = y_k - \hat{y}_{k|k-1}$ and $y_{k|k-1}$ are calculated values. The values of the Gaussian copula dependence parameter which models the dependency structure between y_{k1} and y_{k2} are calculated for each stage by a MATLAB code. As previously shown in Eq. (22) and Eq. (23), by expressing the σ_{k12} covariance in terms of copula, the adequacy of combining copula functions in the estimation of state-space models through the Kalman filter was demonstrated. The obtained Gaussian copula dependence parameter values were used instead of ρ_k in the matrix given in Eq. (55). The prediction stage covariance matrix $P_k(w)$ is included in the Kalman gain matrix as in $K_k = P_{k|k-1}C_k' P_k^{-1}(w)$. As a result, the Kalman filter equations and copula functions have been integrated.

Assume that 100 wooden parts go through this machining process under the defined conditions. Due to the structure of the multi-stage manufacturing process, the output of the previous stage will be the input for any stage. For example, the outputs from the cutting stage will be the input for the sanding stage. The outputs obtained in the sanding stage will be the input for the polishing stage (see in Figure 1).

Table 3. MAPE values (%) for the simulation study

		Cutting		Sanding		Polishing	
ρ	σ^2	L	W	L	W	L	W
0.99	0.1	4.95	9.28	0.85	0.84	8.85	1.07
0.99	0.5	4.80	9.27	2.26	1.14	8.91	1.62
0.99	1.0	5.03	9.25	2.57	1.28	8.49	1.96
0.90	0.1	4.97	9.27	1.21	0.93	8.51	1.11
0.90	0.5	4.92	9.24	2.06	1.17	8.93	1.66
0.90	1.0	4.64	9.27	3.55	1.68	7.90	2.05
0.70	0.1	4.95	9.26	1.12	0.89	8.64	1.08
0.70	0.5	4.87	9.26	2.09	1.15	8.51	1.77
0.70	1.0	4.74	9.26	3.09	1.37	8.52	2.02
0.50	0.1	4.96	9.25	1.02	0.88	8.76	1.28
0.50	0.5	4.84	9.26	2.23	1.14	8.64	1.44
0.50	1.0	4.75	9.30	2.48	1.18	9.05	2.06
0.30	0.1	4.90	9.27	1.01	0.90	8.71	1.15
0.30	0.5	4.99	9.25	2.18	1.19	8.78	1.55
0.30	1.0	4.91	9.29	4.02	2.11	8.94	3.04
0.10	0.1	4.88	9.24	1.29	0.95	8.76	1.09
0.10	0.5	4.97	9.26	2.32	1.21	8.50	1.96
0.10	1.0	4.95	9.23	2.90	1.19	8.81	1.71

A MATLAB code was written to obtain simulation values for the quality characteristics, weight (W) and length (L), for each production stage under the assumptions. The mean absolute percentage error (MAPE) criterion was used to measure the performance of the Kalman filter model under the copula dependency. Table 3 displays the MAPE values that are obtained for various noise levels (σ^2) and correlation coefficients (ρ). Since every MAPE value is less than 10%, it is evident that the proposed model, which provides remarkably accurate predictions, allows for the examination of the dependencies between quality characteristics at every stage [46].

6. Conclusions

In this study, the state-space model established for the dependency between the stages in the multi-stage manufacturing process is integrated with the copula modeling used to reveal the internal dependency structure between the quality characteristics in a stage. The importance of the conditional variance of the prediction error in the Kalman filter equations in the estimation of x_k has been revealed and it has been emphasized that it constitutes the main element that needs to be addressed. In the application part of the study, first, the dependency structure between the quality variables of interest in a hypothetical production process was expressed with copulas, system state predictions were made with the Kalman filter, and evaluated under the mean absolute percentage error (MAPE) criterion. The resulting model has shown that it is a model that allows examining the dependency between quality characteristics at every stage and gives extremely accurate predictions.

The original contribution of this study to the theory, method and practice on the subject is as follows: The Kalman filter estimation method, based on state-space models of multi-stage manufacturing processes, has been presented in a broad perspective, with a solution algorithm proposed and subject-specific comments. In order to take into account the statistical dependence between the quality characteristics of interest at any stage of the process, the dependence was expressed with copula functions and integrated with the Kalman filter method.

The innovations and improvements that the specified original contributions brought to the modeling and analysis of multi-stage manufacturing processes are as follows: The fact that quality characteristics are essentially interdependent is reflected in the models and internalized in the analyses. Model components, structure and calculation steps that are dependent on modeling and analysis are clearly stated. The internal dependency structure that can exist between the quality characteristics of interest at any production stage is integrated with the dependency structure between the stages.

In order to further the results put forward in the study, future studies that are deemed useful are as follows: Prediction methods for various copulas that can be used in modeling the internal dependency between the quality characteristics of interest at any stage of multi-stage manufacturing process structures can be investigated and implemented. State-space modeling generalizations involving dependency can be made with multivariate copula models for more than two quality characteristics. In the presence of models containing noise terms and observation errors with distributions other than normal distribution, the robustness of Kalman filter estimates can be addressed.

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Appendix

Table A1. Kalman filter and copula functions [47]

Stages	Related Equations
Initial:	
x_0, P_0	
Prediction stage:	
$\hat{x}_{k\|k-1}$	(Table 2) $\hat{x}_{k\|k-1} = A_{k-1}\hat{x}_{k-1\|k-1} + B_k u_k$
$P_k(y)$	(Table 2) $P_k(y) = C_k\left[A_{k-1}P_{k-1\|k-1}A'_{k-1} + D_k Q_k D'_k\right]C'_k + H_k R_k H'_k$
$w_k = y_k - \hat{y}_{k\|k-1}$	(Table 2) $w_k = C_k\left(x_k - \hat{x}_{k\|k-1}\right) + H_k \eta_k$
$P_k(w) = E(w_k w'_k)$	(Table 2) $P_k(w) = C_k P_{k\|k-1} C'_k + H_k R_k H'_k$
Estimation prediction stage – copula likelihood function $L_c(prediction)$	General: Eq. (43) $\mathcal{L}_c^{(k)} = \sum_{i=1}^{n} \ln f_{k12}(y_{1k}, y_{2k}; \gamma, \theta) = \sum_{i=1}^{n} \ln c(F_{k1}(y_{k1};\gamma_1), F_{k2}(y_{k2};\gamma_2);\theta) + \sum_{i=1}^{n} \ln f_{k1}(y_{k1};\gamma_1) + \sum_{i=1}^{n} \ln f_{k2}(y_2;\gamma_2)$ Specific: Copula likelihood function for $\hat{x}_{k\|k-1}$ and w_k
Estimation-filtering stage	
$\hat{x}_{k\|k}$	(Table 2) $\hat{x}_{k\|k} = \hat{x}_{k\|k-1} + K_k\{y_k - C_k\hat{x}_{k\|k-1}\}$
$P_{k\|k}$	(Table 2) $P_{k\|k} = [I - K_k C_k]P_{k\|k-1}$
Copula likelihood function for all process stages $L_c(whole)$	Generalization of Eq. (43)