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RESEARCH ARTICLE

A comparative view to $\mathcal{H}_\infty$-norm of transfer functions of linear DAEs

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ABSTRACT

In this paper, bisection and extended-balanced singular perturbation methods are used to calculate the $\mathcal{H}_\infty$-norm of the transfer function of a linear DAEs system for the particular case $D = 0$. In the beginning the approaches' algorithms and error analysis are provided separately. Next, the methods are employed to calculate the $\mathcal{H}_\infty$-norms of a numerical example pertaining to an automotive gas turbine model, and the error limits are used to check the norms in the suitable range, respectively. Ultimately, every solution is compared individually with the problem's $\mathcal{H}_\infty$-norm values, which are retrieved from MATLAB.

1. Introduction

The phenomenon of control means "making a system capable of acting as desired". It dates back to Ancient Egypt. The first known control tools are water clocks, Although it is not known when and by whom water clocks were first invented, the first example was found in tomb of Pharaoh Amenhotep I, in 1500s BC. These mechanisms, known to have been designed by Vitrivius and Ktesibos in 325 BC and called clepsydra (water thief) were used by the Greeks to adjust speaking times in assemblies and courts.

Control, in the modern sense, begins with Watt's steam engine, in 1789. Until the 1870s, hundreds of regulators (governors) were patented worldwide using Watt's principles. From then until today, major seminal works in the field of control were carried out by many famous scientists from various areas such that; Maxwell, Vyshnegradski, Routh, Lyapunov, Hurwitz, Sickels, McFarlane, Farcot, Minorsky, Nyquist, Bode, Bellman, Pontryagyn, Kalman etc [1]. For more detail about historical development of control theory see [2].

One of the most powerful techniques of modern control is $\mathcal{H}_\infty$ control. $\mathcal{H}_\infty$ control is a very useful tool for large-scale multivariable problems to numerically measure the performance, sensitivity and durability of closed loop (feedback) system controllers. Its primary aim is to reduce modeling inaccuracies and account for unquantified disturbances, such as environmental factors, inner uncertainties, and noise, by transforming an optimization problem into a sensitivity problem involving the $\mathcal{H}_\infty$-norm. Here, $\mathcal{H}_\infty$ refers to "the space encompassing all bounded analytic matrix-valued functions within the open right-half complex plane." This concept was initially introduced by Zames in 1981 [3] and has since found applications across numerous works utilizing various control theory techniques [4–9]. For more comprehensive information, please refer to [10].

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Boyd, Balakrishnan and Kamamba (1988) presented the Bisection algorithms, which is an important tool in control theory [11, 12]. In this field, the energy associated with each state of the system is characterized by Hankel singular values, whereas eigenvalues indicate the stability of a system. Making a link between the stability of the system and the energy of its states is the basic idea underlying the bisection method. This entails connecting the imaginary eigenvalues of the associated Hamiltonian matrix, designated as M_γ in Eq.(12), with the singular values of the transfer matrix evaluated along the imaginary axis. The technique is applied in numerous works [13, 14].

In high-level control problems which contain large number of variables and parameters, researches confront many difficulties and complexity. To cope with these adverse conditions, researchers try to create some alternative methods to convert these high-level problems into far smaller dimensional models which can be solved more easily, without losing structural characteristic of the original problems. These kinds of methods are called model order reductions [15–18]. One of the methods is balanced truncation approach. Balanced truncation approach means, to find appropriate balanced realization and truncate this realization preserving the structural characteristic of the original problems.

Let $\mu > 0$ be a parameter, a dynamical system which contains some state component derivatives with μ coefficients is called a singular perturbation model. Singular perturbation models are represented by following set of equations,

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u \quad (1)$$

$$\mu\dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u \quad (2)$$

$$y = C_1x_1 + C_2x_2 + Du \quad (3)$$

here x_1, x_2 are called slow and fast variables, respectively, Eq.(1), Eq.(2) are called slow (powerful) and fast(weak) subsystems, respectively and μ is called perturbation parameter.

Analysis of these system types is done by singular perturbation theory. Singular perturbation theory means to investigate behavior of solutions of the system Eq.(2) for an interval $0 \leq t \leq T$ (or $0 \leq t < +\infty$). The basic idea of singular perturbation method is to protect the slow(low-frequency) part (Eq.(1)) while neglecting the fast(high-frequency) (Eq.(1)). When considered from this point of view the method can be associated with a dominant mode state. In other words, it is process of examining solutions of the given system for $\mu = 0$ [18, 19]. μ-parameter may correspond to different concepts depending on the structure of the system. For example, it represents machine reactance or transients in voltage regulators in power systems, actuators in industrial control, enzymes in biochemical models and fast neutrons in nuclear reactor models.

The extended-balanced singular perturbation method represents a hybrid approach that combines the principles of both balanced truncation and singular perturbation methods. It begins by reducing the model order through the application of balanced truncation. Subsequently, the norm of the transfer function for the reduced model is determined using the singular perturbation method.

This paper organized into six sections. A number of basic definitions and notations which will be used next chapters are given in Section 2. In Section 3, general information about bisection method is told and algorithm of the method is given. In Section 4, extended-balanced singular perturbation method is told and its algorithm summarize as a table with the error bounds. A numerical example is solved by both methods and tolerances are computed in Section 5. Finally, in Section 6, the results are compared and discussed.

2. Preliminaries

Let's examine the linear dynamic system;

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (4)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}, D \in \mathbb{R}^{p\times m}$. Transfer matrix (or function) of the system Eq. (4) is defined as;

$$G(s) = C(sI - A)^{-1}B + D \quad (5)$$

Let $\lambda_j(M), \sigma_j(M)$ denote the j^{th} eigenvalue and j^{th} singular value of a matrix M respectively, where $\sigma_j(M) = \sqrt{\lambda_j\left(MM^T\right)}$.$A$ is stable if $Re\left(\lambda_j(A)\right) < 0$ for all j. If A is stable H_∞-norm of the transfer matrix $G(s)$ is given as follows;

$$\|G\|_\infty = \sup_{\text{Re } s>0} \sigma_{\max}(G(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(G(i\omega)) \quad (6)$$

where $\sup_{\omega\in\mathbb{R}}$ denotes least upper bound for all real frequencies ω.

Let $J_{2n\times 2n} = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}$ be a skew-symmetric matrix where $0_n, I_n$ are n-dimensional zero and identity matrices, respectively. $H_{2n\times 2n}$ is called a Hamiltonian matrix, if HJ is symmetric, such that $(HJ)^T = HJ$. The definition confirms that the distinctive block structure form of Hamiltonian matrices is as follows;

$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & -H_{11}^T \end{bmatrix}$, where H_{12} and H_{21} are symmetric. For the system Eq.(4) the matrices $W_{\mathcal{C}}(t)$ and $W_{\mathcal{O}}(t)$ are called controllable and observable Grammians, respectively, defined as follows;

$$\begin{aligned} W_{\mathcal{C}}(t) &= \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \\ W_{\mathcal{O}}(t) &= \int_0^t e^{A^T\tau}C^TCe^{A\tau}d\tau \end{aligned} \tag{7}$$

which satisfy the Lyapunov equations as follows;

$$\begin{aligned} A^TW_{\mathcal{O}} + W_{\mathcal{O}}A + C^TC &= 0 \\ AW_{\mathcal{C}} + W_{\mathcal{C}}A^T + BB^T &= 0 \end{aligned} \tag{8}$$

and singular values of $W_{\mathcal{C}}(t)W_{\mathcal{O}}(t)$ are called Hankel singular values of the system Eq.(4) which describes the energy of each state of the system Eq.(4) and are denoted as σH_j for $j = 1, 2, \ldots$

Any positive definite matrix M can be expressed in the form of

$$M = LL^T \tag{9}$$

where L is a lower triangular matrix. The expression Eq.(9) and the matrix L are called Cholesky factorization and Cholesky factor of M, respectively. Let $M \in \mathbb{R}^{m\times n}$ and $rank(M) = r = \min(m, n)$, the expression

$$M = U\Sigma V^T \tag{10}$$

is called singular value decomposition of the matrix M. Here U and V are orthogonal matrices of type of $m \times m$ and $n \times n$, respectively, that is, $U^TU = I_m, V^TV = I_n$ and Σ is a half-diagonal matrix which contains singular values $(\sigma_1, \ldots, \sigma_r)$ of the matrix M. Singular value decomposition can be formulated clearly as follows for a matrix M,

$$M = U\Sigma V^T = \underbrace{\begin{bmatrix} u_1 & | & u_2 & | & \cdots & | & u_m \end{bmatrix}}_{u(m\times m)} \times \underbrace{\begin{bmatrix} \sigma_1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ddots & 0 & 0 & 0 & \ldots & 0 \\ \vdots & 0 & \sigma_r & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \ldots & 0 & \ldots & 0 \end{bmatrix}}_{\Sigma(m\times n)} \underbrace{\begin{bmatrix} \frac{v_1^T}{v_2^T} \\ \frac{\vdots}{v_n^T} \end{bmatrix}}_{V^T(n\times n)}. \tag{11}$$

3. Bisection method

Let $\gamma > 0$ related Hamiltonian matrix M_γ for system Eq. (4) is given as follows;

$$\begin{aligned} M_\gamma &= \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -C^T \end{bmatrix} \times \\ &\quad \begin{bmatrix} -D & \gamma I \\ \gamma I & -D^T \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & B^T \end{bmatrix} \\ &= \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^TS^{-1}C & -A^T + C^TDR^{-1}B^T \end{bmatrix} \end{aligned} \tag{12}$$

where $R = D^TD - \gamma^2 I$ and $S = DD^T - \gamma^2 I$.

For special case

$$D = 0, M_\gamma = \begin{bmatrix} A & \frac{1}{\gamma}BB^T \\ -\frac{1}{\gamma}C^TC & -A^T \end{bmatrix}.$$

Prior to initiating the bisection algorithm, it is essential to establish clear lower (γ_{lb}) and upper (γ_b) bounds. While one option is to set $\gamma_{lb} = 0$ and γ_{ub} to a sufficiently large value before proceeding with the bisection protocol, this approach can be time-consuming and inefficient. To streamline this process and determine suitable bounds, we can leverage Hankel singular values, as derived by Enns [20] and Glover [21], which are outlined below:

$$\begin{aligned} \gamma_{lb} &= \max\left\{\sigma_{\max}(D), \sqrt{Tr\left(W_{\mathcal{C}}W_{\mathcal{O}}\right)/n}\right\} \\ \gamma_{ub} &= \sigma_{\max}(D) + 2\sqrt{nTr\left(W_{\mathcal{C}}W_{\mathcal{O}}\right)} \end{aligned} \tag{13}$$

or alternative formulas;

$$\begin{aligned} \gamma_{lb} &= \max\left\{\sigma_{\max}(D), \sigma H_1\right\} \\ \gamma_{ub} &= \sigma_{\max}(D) + 2\sum_{j=1}^{n}\sigma H_i \end{aligned} \tag{14}$$

here, σH_i s represents the Hankel singular values, while $W_{\mathcal{O}}$ and $W_{\mathcal{C}}$ stand for the observability and controllability Grammians of the system Eq.(4)

Assuming A is stable and $\varepsilon > 0$ is the error margin for system Eq.(4), the bisection algorithm is outlined as follows:

Step 1. Determine the lower and upper bounds for the bisection algorithm, where

$$\gamma_{lb} = \max\{\sigma_{\max}(D), \sigma H_1\}$$
$$\gamma_{ub} = \sigma_{\max}(D) + 2\sum_{j=1}^{n} \sigma H_i$$

Step 2. Set $\gamma = (\gamma_{lb} + \gamma_{ub})/2$
If $\gamma_{ub} - \gamma_{lb} < \frac{\varepsilon}{2}$, end.

Step 3. Calculate M_γ.

Step 4. Check eigenvalues of M_γ. If there exists a purely imaginary eigenvalue set $\gamma_{lb} = \gamma$. Else set $\gamma_{ub} = \gamma$.

4. Extended balanced singular perturbation method

The extended balanced singular perturbation method, as introduced in the Introduction, combines the principles of both balanced truncation and singular perturbation methods, as described below.

Suppose we have an asymptotically stable, minimal realization of the system Eq.(4) as defined in equation Eq.(5). The algorithm for the balanced truncation approach is implemented using the following MATLAB commands:

Step 1. Find controllable and observable Grammians $W_\mathcal{C}$ and $W_\mathcal{O}$ of the given system through the Lyapunov equtions with the MATLAB commands

Wc=gram(sys,'c')

Wo=gram(sys, 'o').

Step 2. Find the Cholesky factors $L_\mathcal{C}$ and $L_\mathcal{O}$ of $W_\mathcal{C}$ and $W_\mathcal{O}$, respectively, such that

$$W_\mathcal{C} = L_\mathcal{C} L_\mathcal{C}^T$$
$$W_\mathcal{O} = L_\mathcal{O} L_\mathcal{O}^T$$

with the MATLAB commands

$$Lc = chol(Wc,' lower')$$
$$Lo = chol(Wo,' lower').$$

Step 3. Find the singular value decomposition of $L_\mathcal{O}^T L_\mathcal{C}$ such that

$$L_\mathcal{O}^T L_\mathcal{C} = U\Sigma V^T$$

with the MATLAB command

$[U, S, V] = svd(Lo$ ' *Lc).

Step 4. Make the transformation $T = L_\mathcal{C} V \Sigma^{-1/2}$ and obtain coefficient matrices of balanced system by similarity transformation as follows,

$$\tilde{A} = T^{-1}AT, \quad \tilde{B} = T^{-1}B, \quad \tilde{C} = CT, \quad \tilde{D} = D$$

where $\tilde{G}(s) = \begin{bmatrix} \tilde{A} & | & \tilde{B} \\ -\tilde{C} & | & \tilde{D} \end{bmatrix}$ and find controllable and observable Grammians of the balanced system $\widetilde{W}_C$ and $\widetilde{W}_0$ respectively which are given as below,

$$\widetilde{W}_\mathcal{C} = T^{-1} W_\mathcal{C} T^{-T}$$
$$\widetilde{W}_\mathcal{O} = T^T W_\mathcal{O} T$$

here $\widetilde{W}_\mathcal{C} = \widetilde{W}_\mathcal{O} = \Sigma = diag(\sigma_1, \sigma_1, \ldots, \sigma_n)$.

Let $\tilde{G}(s) = \begin{bmatrix} \tilde{A} & | & \tilde{B} \\ - & - & - \\ \tilde{C} & | & \tilde{D} \end{bmatrix}$ be the balanced system obtained by balanced truncation approach, the algorithm of singular perturbation method is given as follows;

Step 1. Separate the balanced system $\tilde{G}(s) = \begin{bmatrix} \tilde{A} & | & \tilde{B} \\ - & - & - \\ \tilde{C} & | & \tilde{D} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$ into two subsystem as slow(powerful) and fast(weak). Choose A_{11} as coefficient matrix of the slow part where $A_{11}, \Sigma_1 \in \mathbb{R}^{r\times r}$, for $r \ll n$. Rearrange the matrices $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ in block matrix form as seen below,

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tilde{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$
$$\tilde{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tilde{D} = D$$

add perturbation parameter μ and rewrite $\tilde{G}(s)$ as the followings,

$$\begin{bmatrix} \dot{x}_1 \\ \mu\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$
$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Du.$$

Step 2. Eliminate the fast(weak) part $\mu = 0$ and find the system as;

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1 u$$
$$0 = A_{21}x_1 + A_{22}x_2 + B_2 u$$
$$y = C_1 x_1 + C_2 x_2 + Du$$

and weak variable as,

$x_2 = -A_{22}{}^{-1} A_{21} x_1 - A_{22}{}^{-1} B_2 u.$

Step 3. Substitute x_2 to the other equations to get the final version of the system which denoted by $G_f(s)$ as is below

$$G_f(s) = \begin{bmatrix} A_f & | & B_f \\ - & - & - \\ C_f & | & D_f \end{bmatrix} =$$

Table 1. Algorithm of extended balanced singular perturbation method step by step.

	Balanced Truncation Approach	Singular Perturbation Method
Step 1.	Find Grammians of the original system $(W_{\mathcal{C}}, W_{O})$	Separate the balanced system $\tilde{G}(s)$ into two parts as; strong and weak
Step 2.	Find Cholesky factors of Grammians $(L_{\mathcal{C}}, L_{\mathcal{O}})$	Eliminate the weak part taking $\mu = 0$ and find weak variable x_2
Step 3.	Find singular value decomposition of $L_O^T L_{\mathcal{C}} = U\Sigma V^T$	Substitute x_2 in other equations, get the final version of the system $G_f(s)$
Step 4.	Make the transformation $T = L_{\mathcal{C}} V \Sigma^{-1/2}$ and find the balanced system $\tilde{G}(s)$	Obtain the H_∞-norm of $\|G_f(s)\|_\infty$
Error Analysis	Compute actual and theoretical infinity error bounds and apply the error tolerance criterion which says actual bound must be less than or equal to theoretical bound	

$$\left[\begin{array}{ccc} A_{11} - A_{12}A_{22}{}^{-1}A_{21} & | & B_1 - A_{12}A_{22}{}^{-1}B_2 \\ -------- & - & ---------- \\ C_1 - C_2A_{22}^{-1}A_{21} & | & D - C_2A_{22}{}^{-1}B_2 \end{array}\right]$$

Step 4. Obtain the H_∞-norm of $\|G_f(s)\|_\infty$ in MATLAB.

The algorithm of extended balanced singular perturbation method is summarized in Table 1 as follows.

To analyze the error tolerance first we define modelling error transfer function as follows;

$$E_r = [G(s) - G_f(s)] \tag{15}$$

then, we have a criterion about sufficiency of error tolerance which is based on comparison of two error bounds called actual infinity error bound and theoretical infinity error bound defined in [22,23] given as below;

- Actual infinity error bound: $\|E_r\|_\infty = \|[G(s) - G_f(s)]\|_\infty$
- Theoretical infinity error bound: $2\sum_{i=r+1}^{n} \sigma_i$
- The criterion:

$$\|E_r\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i$$

5. Application to a numerical example

Example 1. *The system two-input, twelve-state, two-output model of an automobile gas turbine [24].*

For more detail and examples see [25]. Consider the system Eq.(4) with the coefficient matrices given as follows:

When employing the bisection method for this problem, we obtain the values presented in Table 2. The first and the last columns in the table pertain to number of iteration that denoted as Itr briefly and verifying the presence of purely imaginary eigenvalues that denoted as Eig briefly, respectively.

Table 2. Related values of Example 2.

Itr	γ_{lb}	γ_{ub}	γ	Eig
1	3.0368	36.4417	19.7397	no
2	3.0368	19.7397	11.3881	yes
3	11.3881	19.7397	15.5637	no
4	11.3881	15.5637	13.4759	yes
5	13.4759	15.5637	14.5198	no
6	13.4759	14.5198	13.9979	no
7	13.4759	13.9979	13.7369	yes
8	13.7369	13.9979	13.8674	no
9	13.7369	13.8674	13.8022	no
10	13.7369	13.8022	13.7695	no
11	13.7369	13.7695	13.7532	no
12	13.7369	13.7532	13.7450	no
13	13.7369	13.7450	13.7410	no
14	13.7369	13.7410	13.7389	no
15	13.7369	13.7389	13.7379	yes
16	13.7379	13.7389	13.7384	yes
17	13.7384	13.7389	13.7387	yes
18	13.7387	13.7389	13.7388	no
19	13.7387	13.7388	13.7388	yes
20	13.7388	13.7388	13.7388	

After 20 iterations γ_{ub} and γ_{lb} are so close, the all next iterations will be automatically assigned the same value by MATLAB and error margin ε is also will be satisfied. Thus, $\mathcal{H}_\infty$ norm of transfer function of the given problem found as $\|G(s)\|_\infty \approx 13.7388$. Now, if we apply balanced truncation approach algorithm step by step finally we get

$$\tilde{G}(s) = \left[\begin{array}{ccc} \tilde{A} & | & \tilde{B} \\ --- & & --- \\ \tilde{C} & | & \tilde{D} \end{array}\right] \text{ where;}$$

and Hankel singular values of the original system as,

$$\sigma(G) = (7.1833 \quad 1.4904 \quad 0.9279 \quad 0.5876 \quad 0.4633 \quad 0.2368 \quad 0.1613 \quad 0.0936 \quad 0.0006 \quad 0 \quad 0 \quad 0).$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.202 & -1.15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2.36 & -13.6 & -12.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.62 & -9.4 & -9.15 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -188 & -111.6 & -116.4 & -20.8 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1.0439 & 0 & 0 & -1.794 & 0 & 0 & 1.0439 & 0 & 0 & 0 & -1.794 \\ 0 & 4.1486 & 0 & 0 & 2.6775 & 0 & 0 & 4.1486 & 0 & 0 & 0 & 2.6775 \end{bmatrix}^T$$

$$C = \begin{bmatrix} 0.264 & 0.806 & -1.42 & -15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4.9 & 2.12 & 1.95 & 9.35 & 25.8 & 7.14 & 0 \end{bmatrix}$$

$$D = 0$$

$$\tilde{A} = \begin{bmatrix} -0.1647 & 0.0185 & 0.1116 & -0.0528 & -0.4698 & 0.2334 & 0.0559 & -0.0336 & 0.0179 & -0.0049 & -0.0001 & 0 \\ -0.0071 & -0.8293 & -0.2913 & 0.3813 & 0.2261 & -0.0513 & -0.7095 & -1.6433 & 0.0079 & -0.0065 & 0.0008 & -0.0003 \\ -0.0605 & 0.6599 & -0.1368 & 0.1909 & 1.9093 & -0.6958 & 0.0074 & 0.3709 & -0.0453 & 0.0129 & 0.0001 & 0 \\ -0.0676 & -0.2951 & -0.1328 & -0.1111 & 0.0156 & 0.2976 & 0.3495 & 0.4641 & 0.0339 & -0.0075 & -0.0005 & 0.0001 \\ -0.4673 & -0.3296 & -1.3820 & -1.3904 & -5.8806 & 3.7574 & 1.4509 & -1.1650 & 0.4207 & -0.1141 & -0.0018 & 0.0004 \\ 0.2332 & 0.0786 & 0.4107 & 0.5352 & 3.7468 & -2.6771 & -2.2760 & 1.6069 & -0.3983 & 0.1083 & 0.0016 & -0.0004 \\ -0.0103 & 0.7485 & -0.3240 & 0.0251 & -0.4105 & 1.6010 & -1.2640 & -2.6186 & -0.0926 & 0.0100 & 0.0033 & -0.0010 \\ 0.0416 & 1.6502 & -0.7122 & 0.3698 & 1.9050 & -1.8933 & -4.2162 & -11.6604 & 0.3158 & -0.1526 & 0.0113 & -0.0035 \\ -0.0144 & -0.0545 & 0.0096 & -0.0499 & -0.3612 & 0.3313 & 0.1843 & 0.9343 & -6.2677 & 3.6203 & 0.0367 & -0.0080 \\ -0.0032 & -0.0197 & 0.0056 & -0.0128 & -0.0838 & 0.0753 & 0.0693 & 0.3210 & -3.5719 & -13.7547 & -0.0809 & 0.0128 \\ -0.0002 & 0 & -0.0002 & -0.0005 & -0.0042 & 0.0040 & -0.0004 & 0.0010 & -0.0948 & -0.5578 & -0.2477 & 0.1440 \\ 0 & -0.0001 & 0.0001 & 0.0001 & 0.0011 & -0.0011 & 0.0005 & 0.0011 & 0.0177 & 0.0725 & 0.1343 & -0.9059 \end{bmatrix}$$

$$\tilde{B} = \begin{bmatrix} -0.4823 & 1.4882 & -0.5001 & 0.0220 & -0.5164 & 0.2995 & -0.6184 & -1.3307 & 0.0274 & 0.0129 & -0.0003 & 0.0002 \\ -1.4609 & -0.5073 & -0.0613 & -0.3607 & -2.2765 & 1.0855 & 0.1595 & 0.6417 & -0.0796 & -0.0200 & -0.0008 & 0.0002 \end{bmatrix}^T$$

$$\tilde{C} = \begin{bmatrix} -0.5368 & 1.4993 & 0.0399 & -0.3178 & -0.8139 & 0.3740 & 0.6374 & 1.3230 & 0.0181 & -0.0010 & -0.0008 & 0.0002 \\ -1.4417 & -0.4735 & 0.5023 & -0.1721 & -2.1879 & 1.0621 & 0.0399 & -0.6574 & 0.0823 & -0.0238 & -0.0001 & 0 \end{bmatrix}$$

$$\tilde{D} = 0$$

It is seen clearly in the Figure 1 the first three Hankel singular values are much greater than the others so we choose $r = 6$ and apply extended balanced singular perturbation method. First separate the balanced system $\tilde{G}(s)$ into two parts as slow(powerful) and fast(weak) and rewrite the system for perturbation parameter $\mu = 0$ as is given below;

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + B_1 u \\ 0 &= A_{21}x_1 + A_{22}x_2 + B_2 \\ y &= C_1 x_1 + C_2 x_2 + Du \end{aligned}$$

where;

$$A_{11} = \begin{bmatrix} -0.1647 & 0.0185 & 0.1116 & -0.0528 & -0.4698 & 0.2334 \\ -0.0071 & -0.8293 & -0.2913 & 0.3813 & 0.2261 & -0.0513 \\ -0.0605 & 0.6599 & -0.1368 & 0.1909 & 1.9093 & -0.6958 \\ -0.0676 & -0.2951 & -0.1328 & -0.1111 & 0.0156 & 0.2976 \\ -0.4673 & -0.3296 & -1.3820 & -1.3904 & -5.8806 & 3.7574 \\ 0.2332 & 0.0786 & 0.4107 & 0.5352 & 3.7468 & -2.6771 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0.0559 & -0.0336 & 0.0179 & -0.0049 & -0.0001 & 0 \\ -0.7095 & -1.6433 & 0.0079 & -0.0065 & 0.0008 & -0.0003 \\ 0.0074 & 0.3709 & -0.0453 & 0.0129 & 0.0001 & 0 \\ 0.3495 & 0.4641 & 0.0339 & -0.0075 & -0.0005 & 0.0001 \\ 1.4509 & -1.1650 & 0.4207 & -0.1141 & -0.0018 & 0.0004 \\ -2.2760 & 1.6069 & -0.3983 & 0.1083 & 0.0016 & -0.0004 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} -0.0103 & 0.7485 & -0.3240 & 0.0251 & -0.4105 & 1.6010 \\ 0.0416 & 1.6502 & -0.7122 & 0.3698 & 1.9050 & -1.8933 \\ -0.0144 & -0.0545 & 0.0096 & -0.0499 & -0.3612 & 0.3313 \\ -0.0032 & -0.0197 & 0.0056 & -0.0128 & -0.0838 & 0.0753 \\ -0.0002 & 0 & -0.0002 & -0.0005 & -0.0042 & 0.0040 \\ 0 & -0.0001 & 0.0001 & 0.0001 & 0.0011 & -0.0011 \end{bmatrix}$$

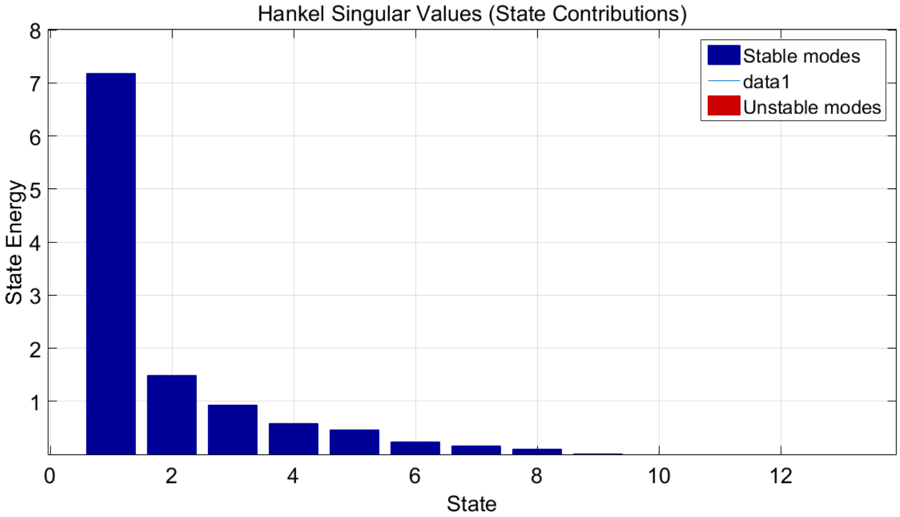

Figure 1. Hankel singular values of the original system.

$$A_{22} = \begin{bmatrix} -1.2640 & -2.6186 & -0.0926 & 0.0100 & 0.0033 & -0.0010 \\ -4.2162 & -11.6604 & 0.3158 & -0.1526 & 0.0113 & -0.0035 \\ 0.1843 & 0.9343 & -6.2677 & 3.6203 & 0.0367 & -0.0080 \\ 0.0693 & 0.3210 & -3.5719 & -13.7547 & -0.0809 & 0.0128 \\ -0.0004 & 0.0010 & -0.0948 & -0.5578 & -0.2477 & 0.1440 \\ 0.0005 & 0.0011 & 0.0177 & 0.0725 & 0.1343 & -0.9059 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.4823 & -1.4609 \\ 1.4882 & -0.5073 \\ -0.5001 & -0.0613 \\ 0.0220 & -0.3607 \\ -0.5164 & -2.2765 \\ 0.2995 & 1.0855 \end{bmatrix} \quad B_2 = \begin{bmatrix} -0.6184 & 0.1595 \\ -1.3307 & 0.6417 \\ 0.0274 & -0.0796 \\ 0.0129 & -0.0200 \\ -0.0003 & -0.0008 \\ 0.0002 & 0.0002 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -0.5368 & 1.4993 & 0.0399 & -0.3178 & -0.8139 & 0.3740 \\ -1.4417 & -0.4735 & 0.5023 & -0.1721 & -2.1879 & 1.0621 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.6374 & 1.3230 & 0.0181 & -0.0010 & -0.0008 & 0.0002 \\ 0.0399 & -0.6574 & 0.0823 & -0.0238 & -0.0001 & 0 \end{bmatrix} \quad D = 0$$

and from the second equation find weak variable as, $x_2 = -{A_{22}}^{-1} A_{21} x_1 - {A_{22}}^{-1} B_2 u$. Continue from Step3 make necessary algebraic matrix operations and finally get, $G_f(s) = \begin{bmatrix} A_f & | & B_f \\ -- & -- & -- \\ C_f & | & D_f \end{bmatrix}$ where;

$$A_f = \begin{bmatrix} -0.1690 & 0.0951 & 0.0784 & -0.0663 & -0.6557 & 0.6759 \\ -0.0058 & -1.1996 & -0.1312 & 0.3503 & 0.2631 & -0.5240 \\ -0.0513 & 0.5609 & -0.0938 & 0.2259 & 2.3058 & -1.5693 \\ -0.0772 & -0.0122 & -0.2554 & -0.1296 & -0.3903 & 1.3889 \\ -0.5873 & 1.7458 & -2.2822 & -1.7701 & -11.0375 & 15.9797 \\ 0.4156 & -3.1167 & 1.7961 & 1.1096 & 11.5963 & -21.3316 \end{bmatrix}$$

$$B_f = \begin{bmatrix} -0.5472 & -1.4593 \\ 1.7922 & -0.6034 \\ -0.4144 & -0.0472 \\ -0.2146 & -0.3261 \\ -2.2766 & -2.2466 \\ 3.0081 & 1.0291 \end{bmatrix} \quad C_f = \begin{bmatrix} -0.5419 & -1.4612 \\ 1.8765 & -0.2345 \\ -0.1233 & 0.3985 \\ -0.3049 & -0.2438 \\ -1.0190 & -3.0315 \\ 1.1786 & 2.9523 \end{bmatrix}^T$$

$$D_f = \begin{bmatrix} -0.3115 & 0.0807 \\ -0.2056 & -0.0223 \end{bmatrix}$$

Obtain the $\mathcal{H}_\infty$-norm in MATLAB as $\|G_f(s)\|_\infty = 13.7413$ which is so close to the $\mathcal{H}_\infty-$ norm of the original system $\|G(s)\|_\infty = 13.7388$. Let's now assess the error tolerance between the original system and the reduced-order balanced model using both actual and theoretical infinity error bounds, as outlined below.

$$\|E_r\|_\infty = \|[G(s) - G_f(s)]\|_\infty = 0.3774$$

and for $r = 6$ and $n = 12$,

$$2\sum_{i=r+1}^{n} \sigma_i = 2(0.1613 + 0.0936 + 0.0006 + 0 + 0 + 0) = 0.5110$$

It is obvious that $\|E_r\|_\infty \leq 2\sum_{i=r+1}^{n} \sigma_i$ thus we can say that error tolerance is in a satisfied level.

6. Conclusion

In this research, we applied both the bisection method and the extended-balanced singular perturbation method to analyze a linear dynamic system with the parameter D set to 0 . Our goal was to compute the $\mathcal{H}_\infty-$ norm of its transfer function. We conducted a numerical experiment using both methods and performed a detailed error analysis. The outcomes of our investigation revealed that the bisection method performed satisfactorily, with error tolerances falling within an acceptable range after a certain number of iterations. Similarly, the extended-balanced singular perturbation method demonstrated satisfactory performance, as the error tolerances met the criteria for investigating the accuracy of the reduced-order models. According to the $\mathcal{H}_\infty$-norms computed by methods, we conclude that bisection method is a slightly accurate than extended-balanced singular perturbation method. Utilizing bisection and extended balanced singular perturbation methods, the research not only provides detailed algorithms and error analysis but also demonstrates practical application through a numerical example involving an automotive gas turbine model, enhancing the precision and reliability of $\mathcal{H}_\infty$-norm computations in real-world systems.

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