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RESEARCH ARTICLE

An accurate finite difference formula for the numerical solution of delay-dependent fractional optimal control problems

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ABSTRACT

Time-delay fractional optimal control problems (OCPs) are an important research area for developing effective control and optimization strategies to address complex phenomena occurring in various natural sciences, such as physics, chemistry, biology, and engineering. By considering fractional OCPs with time delays, we can design control strategies that take into account the system's history and optimize its behavior over a given time horizon. However, applying the Pontryagin principle of maximization to solve these problems leads to a boundary value problem (BVP) that includes delay and advance terms, making analytical solutions difficult and demanding. To address this issue, this paper presents a precise finite difference formula to solve the aforementioned advance-delay BVP numerically. The suggested approximate method's error analysis and convergence properties are provided, and several illustrative examples demonstrate the applicability, validity, and accuracy of the proposed approach. Simulation results confirm the proposed technique's advantages for the optimal control of delay fractional dynamical equations.

1. Introduction

Over the past few years, fractional calculus (FC), as a generalization of classical calculus, has attracted the attention of scientists and engineers for describing various types of physical phenomena [1]. In fact, this calculus is known as a powerful tool for the modelling of complex dynamical systems related to memory effects and non-locality [2]. The FC has some applications in epidemic modelling [3], finance [4], diffusion equations [5], outbreak control [6], quasi-synchronization [7], image diagnosis [8], chaos control [9], etc. Due to the difficulty of analytical solution for fractional dynamical systems, some efficient approximation approaches have been proposed for the numerical solution of various problems containing fractional-order operators, *e.g.*, differential equations [10], delay-dependent systems [11], etc.

Optimal control problems (OCPs) play a crucial role in determining the best strategies for controlling dynamic systems over time, with applications ranging from engineering and economics to biology and robotics [12–14]. A delay fractional OCP tries to find a control law for a delay fractional dynamical system by minimizing a cost functional in terms of the corresponding state and control variables [15]. The study of time-delay fractional OCPs is critical to develop efficient control and optimization strategies for addressing complex phenomena in various natural sciences, such as physics, chemistry, biology, and engineering.

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However, due to the high complexity of fractional OCPs with time-delay, it is extremely difficult to obtain their analytical solution [16]. To solve this issue, in the past decade, some numerical techniques have been developed including finite difference method [17, 18], Bernstein polynomials [19], Legendre polynomials [20, 21], linear programming technique [22], Lagrange polynomials [23], neural network [24], Taylor expansions [25], Chelyshkov wavelets [26], embedding process [27], and fractional orthogonal basis functions [28]. More recently, the paper [29] presented a collocation method for solving nonlinear delay fractional optimal control systems with constraints on the state and control variables. Another study [30] focused on time-optimal feedback control of nonlocal Hilfer fractional state-dependent delay inclusion with Clarke's subdifferential. The new work [31] also introduced Mittag-Leffler wavelets and their applications for solving fractional OCPs with and without delay.

The field of fractional OCPs with time delays presents a significant challenge due to the complexity introduced by considering both FC and time-delay terms simultaneously. While there is existing research on fractional OCPs and time-delay systems independently, the intersection of these two areas remains relatively unexplored. Current methods for solving delay-dependent fractional OCPs often face difficulties in providing accurate and efficient solutions due to the intricate nature of the boundary value problem (BVP) resulting from applying the Pontryagin maximum principle. Analytical solutions for such advance-delay BVPs are scarce, leading to a gap in the literature regarding effective numerical solution techniques tailored specifically for this challenging class of problems. Therefore, there is a pressing need for innovative approaches that can accurately and reliably address the unique characteristics of delay-dependent fractional OCPs, providing researchers and practitioners with appropriate tools for optimizing complex dynamical systems subjected to FC and time delays.

This research article addresses the above-mentioned critical research gap in the field of fractional OCPs with time delays. The study's significance lies in its focus on developing effective control and optimization strategies for complex phenomena present in various natural sciences and engineering, where FC and time delays play crucial roles. By introducing a precise finite difference formula to numerically solve advance-delay BVPs arising from applying the Pontryagin maximum principle, this research offers an innovative approach tailored specifically for this challenging class of problems. The study's novelty is evident in its unique contributions, including the development of a novel numerical solution technique for delay-dependent fractional OCPs, comprehensive error analysis and convergence properties of the proposed method, as well as illustrative examples demonstrating its applicability and accuracy. This research's potential impact is substantial, as it provides researchers and practitioners with appropriate tools for optimizing complex dynamical systems subjected to FC and time delays, ultimately advancing the state-of-the-art in this underexplored intersection of FC and time-delay systems.

2. Problem Statement

Consider the following fractional dynamical system with time-delay

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = A_1(\tau)z(\tau) + A_d(\tau)z(\tau - m) \\ \qquad + B_1(\tau)v(\tau), \quad \tau_0 \le \tau \le \tau_f, & \text{(1a)} \\ z(\tau) = \psi(\tau), \quad \tau_0 - m \le \tau \le \tau_0, & \text{(1b)} \end{cases}$$

in which $z \in \mathbb{R}^q$ is the state vector, and the symbol ${}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau)$ signifies the left Caputo fractional derivative [32]

$${}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = \frac{1}{\Gamma(1-\gamma)} \int_{\tau_0}^{\tau} (\tau - \xi)^{-\gamma} \frac{dz(\xi)}{d\xi} d\xi, \quad (2)$$

in which the derivative order is denoted by γ $(0 < \gamma \le 1)$. Also, the parameter m is the state time-delay, $v \in \mathbb{R}^r$ is the control variable, and the coefficients $A_1(\tau)$, $A_d(\tau)$, and $B_1(\tau)$ are continuous-time matrix functions. Following the optimal control concept, it is desired to determine the control $v(\tau)$ minimizing the following performance index

$$J = \frac{1}{2} \int_{\tau_0}^{\tau_f} \left(z^T(\tau) Q z(\tau) + v^T(\tau) R v(\tau) \right) d\tau, \quad (3)$$

where the matrices $R \in \mathbb{R}^{r \times r}$ and $Q \in \mathbb{R}^{q \times q}$ are, respectively, assumed to be positive definite and positive semi-definite.

Theorem 1. *(Pontryagin conditions of optimality) Under the constraint given by the dynamical system* (1)*, if* $(z(\tau), v(\tau))$ *is a minimizer of* (3)*, then the costate vector* $y(\tau)$

exists such that the following conditions are satisfied:

- *the Hamiltonian system, for* $\tau_0 \le \tau \le \tau_f$,

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = \dfrac{\partial \mathcal{H}}{\partial y(\tau)}, & \text{(4a)} \\ {}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau) = \dfrac{\partial \mathcal{H}}{\partial z(\tau)} + A_2(\tau) y(\tau+m), & \text{(4b)} \end{cases}$$

- *the stationary condition, for* $\tau_0 \le \tau \le \tau_f$,

$$\frac{\partial \mathcal{H}}{\partial v(\tau)} = 0, \tag{5}$$

- *and the transversality condition*

$$y(\tau)|_{\tau=\tau_f} = 0, \tag{6}$$

where ${}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau)$ $(0 < \gamma \le 1)$ *is the* γ*-th order right Riemann-Liouville fractional derivative of* $y(\tau)$ *defined by [32]*

$${}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\tau} \int_{\tau}^{\tau_f} (\xi-\tau)^{-\gamma} y(\xi) d\xi, \tag{7}$$

$A_2(\tau) = A_d(\tau+m)\chi_{[\tau_0,\tau_f-m]}(\tau)$*, and* $\chi_{[a,b]}$ *represents the characteristic function on the interval* $[a,b]$*. The function* $\mathcal{H}$*, called the Hamiltonian, has also the following form*

$$\begin{aligned} \mathcal{H} := 0.5 &\left(z^T(\tau) Q z(\tau) + v^T(\tau) R v(\tau)\right) \\ &+ y^T(\tau)\left(A_1(\tau) z(\tau) + A_d(\tau) z(\tau-m)\right. \\ &\left. + B_1(\tau) v(\tau)\right). \end{aligned} \tag{8}$$

Proof. First, we adjoin the dynamical constraint (1) to the performance index (3) by introducing the Lagrange multiplier $y(\tau) \in \mathbb{R}^q$, so the following augmented functional can be formed

$$J_a(v) = \int_{\tau_0}^{\tau_f} \left[\mathcal{H} - y^T(\tau)\, {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau)\right] d\tau. \tag{9}$$

Let $\delta f(\tau)$ denote the variation of the function $f(\tau)$; then we take the variation of $J_a(v)$ as

$$\begin{aligned} \delta J_a(v) = \int_{\tau_0}^{\tau_f} &\left\{ \left[\frac{\partial \mathcal{H}}{\partial z(\tau)}\right]^T \delta z(\tau) \right. \\ &+ \left[\frac{\partial \mathcal{H}}{\partial z(\tau-m)}\right]^T \delta z(\tau-m) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial y(\tau)} - {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau)\right]^T \delta y(\tau) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial v(\tau)}\right]^T \delta v(\tau) \\ &\left. - y^T(\tau)\, {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} \delta z(\tau) \right\} d\tau. \end{aligned} \tag{10}$$

Next, it is easily derived that

$$\begin{aligned} &\int_{\tau_0}^{\tau_f} \left\{ \left[\frac{\partial \mathcal{H}}{\partial z(\tau-m)}\right]^T \delta z(\tau-m) \right\} dt \\ &= \int_{\tau_0}^{\tau_f} \left\{ y^T(\tau) A_d^T(\tau) \delta z(\tau-m) \right\} d\tau \\ &= \int_{m}^{\tau_f} (A_d(\tau) y(\tau))^T \delta z(\tau-m) d\tau \\ &= \int_{\tau_0}^{\tau_f} (A_2(\tau) y(\tau+m))^T \delta z(\tau) d\tau, \end{aligned} \tag{11}$$

where $A_2(\tau) = A_d(\tau+m)\chi_{[\tau_0,\tau_f-m]}(\tau)$, and $\chi_{[a,b]}$ denotes the characteristic function on the interval $[a,b]$. Furthermore, by using the fractional integration by parts [32] and taking into account the transversality condition (6), we have

$$\begin{aligned} &\int_{\tau_0}^{\tau_f} y^T(\tau)\, {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} \delta z(\tau) d\tau \\ &= \int_{\tau_0}^{\tau_f} \left({}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau)\right)^T \delta z(\tau) d\tau. \end{aligned} \tag{12}$$

From Eqs. (10), (11) and (12), we deduce

$$\begin{aligned} \delta J_a(v) = \int_{\tau_0}^{\tau_f} &\left\{ \left[\frac{\partial \mathcal{H}}{\partial z(\tau)} + A_2(\tau) y(\tau+m) \right.\right. \\ &\left. - {}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau)\right]^T \delta z(\tau) \\ &+ \left[\frac{\partial \mathcal{H}}{\partial y(\tau)} - {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau)\right]^T \delta y(\tau) \\ &\left. + \left[\frac{\partial \mathcal{H}}{\partial v(\tau)}\right]^T \delta v(\tau) \right\} d\tau. \end{aligned} \tag{13}$$

On an extremal v^*, we require that $\delta J_a(v^*) = 0$. Thus, in Eq. (13), each factor multiplying a variation has to be vanished. Since $z(\tau_0)$ is specified, it is concluded $\delta z(\tau_0) = 0$, but $\delta z(\tau_f)$ is not equal to 0; thus, it is required that $y(\tau_f) = 0$. Furthermore, the necessary conditions given by Eqs. (4) and (5) are achieved by setting to 0 the coefficients of $\delta z(\tau)$, $\delta y(\tau)$, and $\delta v(\tau)$ in Eq. (13). □

Applying the Pontryagin's optimality conditions given by Theorem 1 for the time-delay fractional OCP (1)-(3) leads to the following fractional advance-delay BVP

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = A_1(\tau)z(\tau) \\ +A_d(\tau)z(\tau-m) - S(\tau)y(\tau), \quad (14a) \\ \tau_0 \le \tau \le \tau_f, \\ \\ {}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} y(\tau) = Qz(\tau) + A_1^T(\tau)y(\tau) \\ +A_2(\tau)y(\tau+m), \quad (14b) \\ \tau_0 \le \tau \le \tau_f, \end{cases}$$

with the following conditions

$$\begin{cases} z(\tau) = \psi_1(\tau), \quad \tau_0 - m \le \tau \le \tau_0, & (15a) \\ y(t_f) = 0, & (15b) \end{cases}$$

where $y(\tau+m)$ is the advance term in time, $z(\tau-m)$ is the time-delay argument, and $S(\tau) = B_1(\tau)R^{-1}B_1^T(\tau)$. Moreover, the optimal control law has the following form

$$v^*(\tau) = -R^{-1}B_1^T(\tau)y(\tau), \quad \tau_0 \le \tau \le \tau_f. \quad (16)$$

The analytical solution of the fractional BVP (14)-(15), including the advance and the delay arguments, is not accessible. Thus, our main objective is to develop an effective approximate procedure to solve the above-mentioned BVP numerically.

3. Some Notations and Lemmas

The fractional derivatives in the senses of left Caputo and right Riemann-Liouville have previously been defined in (2) and (7), respectively. In the following, we give some more definitions and properties of Caputo and Riemann-Liouville fractional operators.

The left Riemann-Liouville fractional derivative of $z(\tau)$ is defined by [32]

$${}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{d\tau} \int_{\tau_0}^{\tau} (\tau-\xi)^{-\gamma} z(\xi) d\xi, \quad (17)$$

where $0 < \gamma \le 1$ denotes the fractional order.

Regarding the left and right fractional derivatives in the senses of Riemann-Liouville and Caputo, the following properties hold [32]

$$\begin{cases} {}^{C}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = {}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) - \dfrac{z(\tau_0)}{\Gamma(1-\gamma)}(\tau-\tau_0)^{-\gamma}, \\ \\ {}^{C}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} z(\tau) = {}^{R}_{\tau}\mathbb{D}^{\gamma}_{\tau_f} z(\tau) - \dfrac{z(\tau_f)}{\Gamma(1-\gamma)}(\tau_f-\tau)^{-\gamma}. \end{cases} \quad (18)$$

Definition 1. *In order to approximate the left and right Riemann-Liouville fractional derivatives, the shifted Grünwald-Letnikov (SGL) difference operators are defined as below [33]*

$$\Lambda^{\gamma}_{h,p} z(\tau) = \frac{1}{h^{\gamma}} \sum_{k=0}^{[\frac{\tau-\tau_0}{h}]+p} w_k^{(\gamma)} z(\tau-(k-p)h), \quad (19)$$

$$\Upsilon^{\gamma}_{h,p} z(\tau) = \frac{1}{h^{\gamma}} \sum_{k=0}^{[\frac{\tau_f-\tau}{h}]+p} w_k^{(\gamma)} z(\tau+(k-p)h), \quad (20)$$

where h is the time step size, p is an integer, and $w_k^{(\gamma)} = (-1)^k \begin{pmatrix} \gamma \\ k \end{pmatrix}$. Also, within the following power series, the coefficients $w_k^{(\gamma)}$ are satisfied

$$(1-x)^{\gamma} = \sum_{k=0}^{\infty} w_k^{(\gamma)} x^k, \quad (21)$$

so the following recursive formula computes them

$$w_0^{(\gamma)} = 1, \quad w_k^{(\gamma)} = (1 - \frac{\gamma+1}{k}) w_{k-1}^{(\gamma)}, \quad k \ge 1. \quad (22)$$

From (21) and (22), some important properties of the coefficients $w_k^{(\gamma)}$ can easily be deduced, as stated in the following lemma.

Lemma 1. *Let $0 < \gamma < 1$; then the coefficients $w_k^{(\gamma)}$, given by Eq. (22), satisfy the properties*

(1) $w_0^{(\gamma)} = 1, \quad w_1^{(\gamma)} = -\gamma, \quad w_k^{(\gamma)} < 0, \quad k \ge 2,$
(2) $-\sum_{k=1}^{n} w_k^{(\gamma)} < 1, \quad \forall\ n \ge 1,$
(3) $\sum_{k=0}^{\infty} w_k^{(\gamma)} = 0.$

Now, the space function $\mathcal{L}^j(\mathbb{R})$ is defined as

$$\mathcal{L}^j(\mathbb{R}) = \Big\{ z : \int_{-\infty}^{\infty} (1+|\omega|)^j |\hat{z}(\omega)| d\omega < \infty; \ \hat{z} \text{ is the Fourier transform of } z \Big\}. \quad (23)$$

It is easy to show that for $0 < \gamma \le 1$, if $z \in \mathcal{L}^2(\mathbb{R})$, then $z \in \mathcal{L}^{1+\gamma}(\mathbb{R})$.

Lemma 2. *Let $z(\tau) \in C^j(\mathbb{R})$, $\frac{d^{j+1}z(\tau)}{d\tau^{j+1}} \in \mathcal{L}^1(\mathbb{R})$, $\frac{d^k z(\tau)}{d\tau^k}|_{\tau=\tau_0} = 0$ for $k = 0, 1, 2, \ldots, j$, and $0 < \gamma \le 1$; then*

$$\Lambda^{\gamma}_{h,p} z(\tau) = {}^{R}_{\tau_0}\mathbb{D}^{\gamma}_{\tau} z(\tau) + \sum_{l=1}^{j-1} \omega_l(p)\, {}^{R}_{\tau_0}\mathbb{D}^{\gamma+l}_{\tau} z(\tau) h^l + \mathcal{O}(h^j), \quad (24)$$

in which $\omega_l(p)$ is the coefficient of the power series $\left(\frac{1-e^{-x}}{x}\right)^{\gamma} e^{px} - 1$; in particular,

$$\omega_1(p) = p - \frac{\gamma}{2}, \quad \omega_2(p) = \frac{\gamma}{24} + \frac{1}{2}(p - \frac{\gamma}{2})^2. \quad (25)$$

Proof. The proof of this lemma is easily followed from Theorem 1 in [34]. □

Using Lemma 2, we can formulate a third-order difference operator for the Riemann-Liouville fractional derivative (17), as given by the following definition.

Definition 2. *We define a weighted SGL difference operator for the Riemann-Liouville fractional derivative* (17) *as follows*

$$ {}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau) = \frac{2+\gamma}{2}\Lambda_{h,0}^{\gamma} z(\tau) - \frac{\gamma}{2}\Lambda_{h,-1}^{\gamma} z(\tau), \quad (26) $$

where the operator $\Lambda_{h,p}^{\gamma}$ *has been given by* (19).

Lemma 3. *Let* $0 < \gamma \le 1$, *and* $z(\tau)$, *its Fourier transform, and* ${}_{\tau_0}^{R}\mathbb{D}_{\tau}^{\gamma+2} z(\tau)$ *belong to* $\mathcal{L}^1(\mathbb{R})$. *Then for* $\tau \in \mathbb{R}$

$$ {}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau) = {}_{\tau_0}^{R}\mathbb{D}_{\tau}^{\gamma} z(\tau) + \mathcal{O}(h^2), \quad (27) $$

uniformly as $h \to 0$, *where the operator* ${}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau)$ *has been defined in* (26).

Proof. Let $\mathcal{F}[z(\tau)](\omega) = \hat{z}(\omega) = \int e^{-i\omega\xi} z(\xi) d\xi$ be the Fourier transform of $z(\tau)$, where $i = \sqrt{-1}$; thus, we have $\mathcal{F}[z(\tau - kh)](\omega) = e^{-ik\omega h}\hat{z}(\omega)$. For each $\tau \in \mathbb{R}$, we also have $\mathcal{F}[{}_{\tau_0}^{R}\mathbb{D}_{\tau}^{\gamma} z(\tau)](\omega) = (i\omega)^{\gamma}\hat{z}(\omega)$. Applying the Fourier transform to the both sides of Eq. (26), for each $\tau \in \mathbb{R}$ we obtain

$$ \begin{aligned} &\mathcal{F}[{}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau)](\omega) \\ &= \frac{1}{h^{\gamma}}(1 - e^{-i\omega h})^{\gamma}\left(\frac{2+\gamma}{2} - \frac{\gamma}{2}e^{-i\omega h}\right)\hat{z}(\omega) \\ &= \sigma_2(i\omega h)(i\omega)^{\gamma}\hat{z}(\omega), \end{aligned} \quad (28) $$

where

$$ \begin{aligned} \sigma_2(x) &= \left(\frac{1 - e^{-x}}{x}\right)^{\gamma}\left(\frac{2+\gamma}{2} - \frac{\gamma}{2}e^{-x}\right) \\ &= 1 - \frac{\gamma}{24}(5 + 3\gamma)x^2 + \mathcal{O}(x^3). \end{aligned} \quad (29) $$

There exists a positive constant C_2 such that $|1 - \sigma_2(-ix)| \le C_2|x|^2$. Now, we apply the inverse Fourier transform; since $z(\tau) \in \mathcal{L}^{\gamma+2}(\mathbb{R})$, we derive

$$ \begin{aligned} &\left|{}_{\tau_0}^{R}\mathbb{D}_{\tau}^{\gamma} z(\tau) - {}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau)\right| \\ &= \left|\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\omega\tau} \times \right. \\ &\qquad \left.(\mathcal{F}[{}_{\tau_0}^{R}\mathbb{D}_{\tau}^{\gamma} z(\tau) - {}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau)](\omega)) d\omega\right| \\ &= \left|\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-i\omega\tau} \times \right. \\ &\qquad \left.(1 - \sigma_2(i\omega h))(i\omega)^{\gamma}\hat{z}(\omega) d\omega\right| \\ &\le |h|^2 \frac{1}{2\pi i}\int_{-\infty}^{\infty} |\omega|^{2+\gamma}|\hat{z}(\omega)| d\omega \\ &\le C_2|h|^2 \frac{1}{2\pi i}\int_{-\infty}^{\infty} |1 + \omega|^{2+\gamma}|\hat{z}(\omega)| d\omega \\ &\le \tilde{C}|h|^2, \end{aligned} \quad (30) $$

where $\tilde{C} = \frac{C_2}{2\pi i}\int_{-\infty}^{\infty} |1 + \omega|^{2+\gamma}|\hat{z}(\omega)| d\omega$. □

Definition 3. *From* (26), *we can formally define the second-order weighted SGL difference (SGL2) operators as follows for the left and right Riemann-Liouville fractional derivatives*

$$ {}_{\tau_0}^{R}\Delta_h^{\gamma} z(\tau_n) = \frac{1}{h^{\gamma}}\sum_{k=0}^{n} g_k^{(\gamma)} z(\tau_n - kh), \quad (31) $$

$$ {}_{\tau_f}^{R}\Delta_h^{\gamma} z(\tau_n) = \frac{1}{h^{\gamma}}\sum_{k=0}^{n} g_k^{(\gamma)} z(\tau_n + kh), \quad (32) $$

where h *is the time step size and*

$$ \begin{cases} g_0^{(\gamma)} = \frac{2+\gamma}{2} w_0^{(\gamma)}, \\ g_k^{(\gamma)} = \frac{2+\gamma}{2} w_k^{(\gamma)} - \frac{\gamma}{2} w_{k-1}^{(\gamma)}, \quad k = 2, 3, \ldots. \end{cases} \quad (33) $$

Lemma 3 shows that the SGL2 operator (31) has the second-order of accuracy at every time level.

Remark 1. *Let* $z(\tau_0) = 0$ *and* $0 < \gamma \le 1$; *then by using integrating by parts, we have*

$$ \begin{aligned} {}_{\tau_0}^{R}\mathbb{D}_{\tau}^{\gamma} z(h) &= \frac{1}{\Gamma(1-\gamma)}\int_{\tau_0}^{h} \frac{z'(\xi)}{(h-\xi)^{\gamma}} d\xi \\ &= \frac{z'(\tau_0)h^{1-\gamma}}{\Gamma(2-\gamma)} + \frac{1}{\Gamma(2-\gamma)}\int_{\tau_0}^{h} \frac{z''(\xi)}{(h-\xi)^{\gamma-1}} d\xi. \end{aligned} \quad (34) $$

Therefore, if the function $z(\tau)$ *has no derivative at* $\tau = \tau_0$, *then the SGL2 formula* (31) *is of accuracy order* $1-\gamma$. *Moreover, the SGL2 formula is of accuracy order* $2 - \gamma$ *if* $z'(\tau_0) = 0$ *and the second derivative of* $z(\tau)$ *does not exist at* $\tau = \tau_0$.

Now, we present the following properties for $\{g_k^{(\gamma)}\}$ by using Lemmas 1 and 3.

Lemma 4. *For* $0 < \gamma \le 1$, *the following properties are satisfied by the coefficients in* (33)*:*

(1) $g_0^{(\gamma)} = 1 + \frac{\gamma}{2}$, $g_1^{(\gamma)} = -\frac{\gamma(\gamma+3)}{2}$, $g_2^{(\gamma)} = \frac{\gamma(\gamma+3\gamma-2)}{4}$, $g_k^{(\gamma)} < 0$, $k \ge 3$,

(2) $-\sum_{k=1}^{n} g_k^{(\gamma)} < g_0^{(\gamma)}$, $\forall\, n \ge 2$,

(3) $\sum_{k=0}^{\infty} g_k^{(\gamma)} = 0$.

4. Numerical Method Formulation

Following the theoretical parts given in the previous section, here we formulate an accurate finite difference method to solve the fractional advance-delay BVP (14)-(15). To this end, first consider that the approximate values of $z(\tau_n)$ and $y(\tau_n)$ are denoted by z_n and y_n, respectively. Applying the SGL2 formulas (31) and (32) on the uniform grid points $\tau_n = \tau_0 + nh$ $(n = 0, 1, \ldots, N)$ with $h = \frac{\tau_f - \tau_0}{N}$ as the time step size, a full discretization of the Pontryagin's conditions

(14)-(15) is formulated as follows

$$\begin{cases} {}^{R}_{\tau_0}\Delta^{\gamma}_{h} z_n = A_1(\tau_n)z_n + A_d(\tau_n)\hat{z}_n \\ \qquad -S(\tau_n)y_n,\ \tau_0 \le \tau \le \tau_f, & \text{(35a)} \\ {}^{R}_{\tau_f}\Delta^{\gamma}_{h} y_n = Qz_n + A_1^T(\tau_n)y_n \\ \qquad +A_2(\tau_n)\tilde{y}_n,\ \tau_0 \le \tau \le \tau_f, & \text{(35b)} \\ z_{-n} = \psi(\tau_{-n}),\ n = 0,1,2,\ldots, & \text{(35c)} \\ y_N = 0, & \text{(35d)} \end{cases}$$

where $\tau_{-n} = \tau_0 - nh$, and

$$\hat{z}_n = z(\tau_n - h) \approx \begin{cases} \psi(\tau_n - h), \\ \tau_n - h \le \tau_0, \\ \\ p_1(\tau_n - h; z_k, z_{k+1}), \\ \tau_0 < \tau_k \le \tau_n - h < \tau_{k+1}, \end{cases} \tag{36}$$

$$\tilde{y}_n = y(\tau_n + h) \approx \begin{cases} p_1(\tau_n + h; y_{i-1}, y_i), \\ \tau_{i-1} \le \tau_n + h < \tau_i, \\ \\ 0, \\ \tau_f \le \tau_n + h, \end{cases} \tag{37}$$

in which $0 \le i, k \le N-1$. Besides, the function p_1 is the linear interpolation polynomial

$$p_1(\xi; z_k, z_{k+1}) = \frac{\xi - \tau_k}{h} z_{k+1} + \frac{\tau_{k+1} - \xi}{h} z_k, \tag{38}$$

determined by the support points (τ_k, z_k) and (τ_{k+1}, z_{k+1}). Therefore, the value of the optimal control for $n = 0, 1, \ldots, N$ is approximated by

$$v_n^* = -R^{-1} B_1^T(\tau_n) y_n, \tag{39}$$

where v_n^* represents the numerical approximation of $v^*(\tau_n)$.

5. Numerical Examples

Here, we employ three numerical examples to show the effectiveness of the proposed finite difference technique. Comparative results are also given to verify the superiority of the suggested scheme over the other methodologies available in the literature.

Example 1. *As the first case, consider a delay fractional OCP in the form of minimizing*

$$J = \frac{1}{2}\int_0^2 \left(z^2(\tau) + v^2(\tau)\right) d\tau, \tag{40}$$

subject to

$$\begin{cases} {}^{C}_{0}\mathbb{D}^{\gamma}_{\tau} z(\tau) = \tau z(\tau - 1) + v(\tau), \quad 0 \le \tau \le 2, \\ z(\tau) = 1, \quad -1 \le \tau \le 0. \end{cases} \tag{41}$$

Solving the problem (40)-(41) for different values of γ, we portray, in Figure 1, the approximate state and control functions. Meanwhile, the performance index values $J = 1.0807, 1.0658, 1.0510$ were attained for $\gamma = 0.8, 0.9, 1$, respectively. As can be seen from Figure 1, the numerical approximation goes to the classic solution when γ tends to unity. Also, as depicted in Table 1, the cost functional values obtained by our proposed scheme is less than those previously achieved in [35] by using a linear programming (LP) control strategy. Thus, the given comparative discussion in this part verifies the efficiency of the suggested technique for solving the fractional OCP (40)-(41).

Table 1. Comparison of the approximate values for J (Example 1).

γ	Method	
	LP strategy [35]	Proposed technique
0.8	1.0807	1.0658
0.9	1.0658	1.0658
0.1	1.0514	1.0510

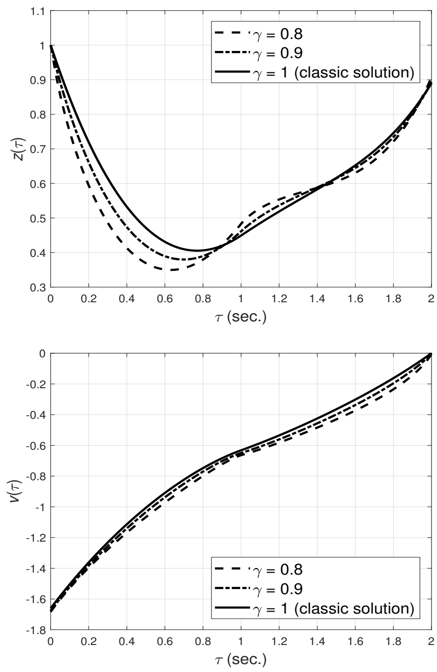

Figure 1. Simulation curves of $z(\tau)$ and $v(\tau)$ for Example 1.

Example 2. *Let us take into account, as the second example, the performance index*

$$J = \frac{1}{2}\int_0^1 \left\{ \left(z_1(\tau) + z_2(\tau)\right)^2 + v^2(\tau) \right\} d\tau, \tag{42}$$

together with the delay fractional dynamical equations

$$\begin{cases} {}_0^C\mathbb{D}_\tau^\gamma z_1(\tau) = \tau z_1(\tau) + z_2(\tau - \frac{1}{4}),\ 0 \le \tau \le 1, \\ {}_0^C\mathbb{D}_\tau^\gamma z_2(\tau) = \tau^2 z_2(\tau) - 5 z_1(\tau - \frac{1}{4}) \\ \quad -z_2(\tau - \frac{1}{4}) + v(\tau),\ 0 \le \tau \le 1, \end{cases} \tag{43}$$

and the initial conditions

$$\begin{bmatrix} z_1(\tau) \\ z_2(\tau) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad -\frac{1}{4} \le \tau \le 0. \tag{44}$$

We plot the state and control variables in Figure 2 for some values of γ. Also, the performance index values $J = 2.7999, 2.2393, 1.7548$ were obtained for $\gamma = 0.8, 0.9, 1$, respectively. Comparing the results with those reported in [35] shows a good agreement, a fact which confirms the efficiency of our proposed scheme to solve the delay fractional OCP (42)-(44). In addition, the classic solution is recovered by the fractional response in Figure 2 when γ goes to 1, a fact which is in line with the correctness of our numerical implementation.

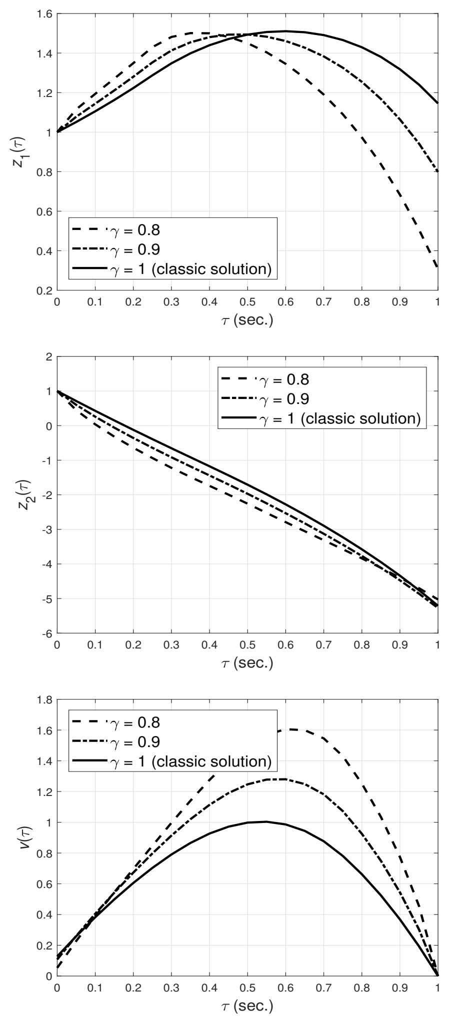

Figure 2. Simulation curves of $z_1(\tau)$, $z_2(\tau)$, and $v(\tau)$ for Example 2.

Example 3. *As a practical case, here we consider the minimization of*

$$J = \int_0^{t_f} \left(10^4 z_1^2(\tau) + v^2(\tau)\right) d\tau, \tag{45}$$

subject to the simplified fractional model

$$\begin{aligned} {}_0^C\mathbb{D}_\tau^\gamma z(\tau) &= \begin{bmatrix} -a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & -2\xi\omega \end{bmatrix} z(\tau) \\ &+ \begin{bmatrix} 0 & ka & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} z(\tau - 0.33) \\ &+ \begin{bmatrix} 0 \\ 0 \\ \omega^2 \end{bmatrix} v(\tau), \quad \tau \ge 0, \end{aligned} \tag{46}$$

which is connected to a wind tunnel at the NASA Langley Research Center. The vector $z(\tau)$ *represents* $z(\tau) = (z_1(\tau), z_2(\tau), z_3(\tau))$, *the parameters in the model* (46) *take the values* $\frac{1}{a} = 1.964$, $\xi = 0.8$, $\omega = 6$, *and* $k = -0.0117$, *and the initial conditions are considered as*

$$z(\tau) = \begin{bmatrix} -0.1 \\ 8.547 \\ 0 \end{bmatrix}, \quad -0.33 \le \tau \le 0. \tag{47}$$

Simulation curves of $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$, and $v(\tau)$ for $\tau_f = 20$ and $\gamma = 0.8, 0.9, 1$ are shown in Figure 3. This figure confirms the convergence of the fractional response to the classic solution, given in [36], as γ goes to 1. Comparison of our numerical findings with those reported in [35] also shows that the new scheme is accurate and efficient to solve the delay fractional OCP (45)-(47).

6. Conclusion

In this study, we presented an approximate numerical solution for time-delay fractional OCPs using a novel finite difference formula. We began by formulating the optimality conditions as a system of fractional advance-delay BVPs and then applied our accurate finite difference method to solve these complex problems. The error analysis and convergence properties of the proposed method were discussed in detail, demonstrating its reliability and effectiveness. Through several illustrative examples and associated simulation results, we showed the accuracy, validity, and correctness of our approach. In particular, our third example, which is connected to a wind tunnel at the NASA Langley Research Center, served as a practical case demonstrating the applicability of our method to real-world problems in engineering

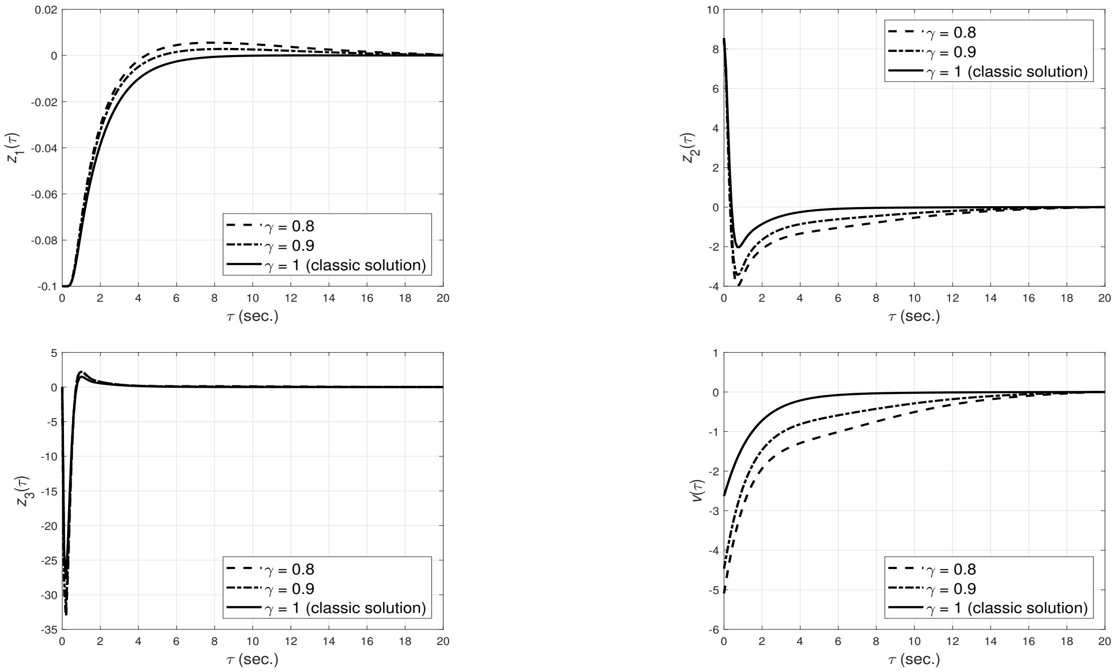

Figure 3. Simulation curves of $z_1(\tau)$, $z_2(\tau)$, $z_3(\tau)$, and $v(\tau)$ for Example 3.

and aerodynamics. Furthermore, comparative experiments highlighted the superiority of our new method over other approximation schemes developed in previous studies. These results not only validate the effectiveness of our approach but also emphasize its potential for addressing challenging problems in various natural sciences and engineering disciplines. Looking ahead, future perspectives of our work include exploring extensions of the proposed method to more complex systems and further practical applications. Future research directions may also involve further refining the algorithm, exploring additional applications across diverse scientific disciplines, and potentially integrating advanced computational techniques to enhance the method's efficiency.

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