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RESEARCH ARTICLE

Intuitionistic fuzzy eigenvalue problem

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ABSTRACT

The purpose of this paper is the study of the eigenvalues of the second order fuzzy boundary value problem (FBVP). By using the (α-β)-level set of intuitionistic fuzzy numbers and Zadeh's extension principle, the FBVP is solved with the proposed method. Furthermore, a numerical example is illustrated and the advantages of the proposed approach are compared with other well-known methods such as the solutions based on the generalized Hukuhara derivative.

1. Introduction

Consider the following FBVP

$$u'' = -\lambda u, \quad t \in [a,b] \tag{1}$$

which satisfy the conditions

$$\widehat{a^i}_1 u(a) = \widehat{a^i}_2 u'(a) \tag{2}$$

$$\widehat{b^i}_1 u(b) = \widehat{b^i}_2 u'(b) \tag{3}$$

where $\widehat{a^i}_1, \widehat{a^i}_2, \widehat{b^i}_1, \widehat{b^i}_2$ intuitionistic fuzzy numbers, $\lambda > 0$, at least one of the numbers $\widehat{a^i}_1$ and $\widehat{a^i}_2$ and at least one of the numbers $\widehat{b^i}_1$ and $\widehat{b^i}_2$ are nonzero.

The subject of fuzzy differential equations (FDEs) was first introduced by Kaleva [1] and Seikkala [2] and has been expended and studied by many researchers for the purpose of modeling problems in science and engineering [3–6]. Most practical problems require the solution of an FDE satisfying fuzzy initial or boundary conditions., so a fuzzy initial value problem (IVP) or boundary value problem (BVP) should be solved. There are several approaches to solve fuzzy problems such as the Hukuhara derivative or Seikkala derivative, the differential inclusion and the derivative based on the Zadeh's extension principle which is widely used for FDEs [7–16].

Puri and Dan introduced the H-derivative [17], and later it was further explored by Kaleva [1] and Seikkala [2]. But in some cases the H-derivative method has a disadvantage that a fuzzy differential equation may have only solutions with nondecreasing lengths of the diameter of the level sets [1, 18]. This disadvantage was solved by Hüllermeier [19], who interpreted a FDE as a family of differential inclusions. Another approach to solve fuzzy problem has been proposed, including Zadeh's extension principle expanding the ordinary differential equations to the fuzzy cases [20]. Then the arithmetic operations are considered to be operations on fuzzy numbers [21].

An effective concept of the differentiability of fuzzy-valued functions is given as the strongly generalized differentiability concept (gh-differentiability) which was first introduced by Bede et al [22]. The fuzzy solutions with gh-differentiability have some not an interval solutions which are associated with the existence of switch points [23]. In addition, Gasilov et al. argued that the solutions obtained by the method of

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Khastan and Nieto [7] are difficult to evaluate, because the solutions to the four different problems may not reflect the nature of the phenomenon being studied [9].

Recently, intutionistic fuzzy set theory (IFST) has become very popular. It is used in various industries, robotics, in audiovisual systems etc. Therefore, many researchers have dedicated their time to the development of IFST. Atanassov [24] generalized the concept of fuzzy set theory by intuitionistic fuzzy set (IFS) which is an extension of fuzzy set introduced by Zadeh [25]. The degree of acceptance in fuzzy sets is only considered, otherwise IFS is characterized by a membership function and a non-membership function so that the sum of both values is less than one [26, 27]. The concept of intuitionistic fuzzy metric space has been introduced Melliani et al. [28] and differential and partial differential equations have been discussed under intuitionistic fuzzy environment.

On the other hand, Melliani et al. [10] gave the the existence and uniqueness theorem of a solution to the intuitionistic FDE. Numerical solution of intuitionistic FDE by Runge-Kutta method has been introduced with intuitionistic treatment in [29] and by Euler method has been discussed by Nirmala and Chenthur Pandian based on the $\alpha-$level [30].

In literature, although there are many approaches to solve the FDEs, there are only few papers such as [11–14, 31] in which the eigenvalues and the eigenfunctions of the FBVP are examined by using different methods such as H-differentiabilty, gH-differentiability and the Zadeh's extension principle.

The main aim of this research is to find eigenvalues of FBVP under the intuitionistic Zadeh's extension principle [32].

In this work, the solutions of the intuitionistic fuzzy eigenvalue problem are studied. The rest of this study is organized as follows, In Section 2, consists of basic definitions related to intuitionistic fuzzy set theory. In Section 3, intuitionistic fuzzy problem and a numerical example is given. Conclusion of the paper is in section 4.

2. Preliminaries

Before proceeding to the solution method, the notations and definitions that will be used throughout the paper are given. To denote an intuitionistic fuzzy number, a bar of the form $\widehat{\ }^i$ is placed over a letter. Also, $\widehat{u^i}(t)$ is written for intuitionistic fuzzy-valued functions defined over the real numbers.

Definition 1. *[26] Let $A \subseteq X$ and let $\mu_A(t) : X \to [0,1]$, $\zeta_A(t) : X \to [0,1]$ be two functions such that $0 \le \mu_A(t) + \zeta_A(t) \le 1$. The set*

$$\begin{aligned}\widehat{A^i} &= \{(t, \mu_A(t), \zeta_A(t)) : t \in X, \\ &\quad \mu_A(t), \zeta_A(t) : X \to [0,1]\}\end{aligned}$$

is called an intuitionistic fuzzy set of X.

Here $\mu_A(t)$ is called membership function and $\zeta_A(t)$ is called non-membership function and the set of all intuitionistic fuzzy sets of X will be denoted by $IF(X)$.

Definition 2. *[26] Let $\widehat{A^i} \in IF(X)$. The set*

$$\begin{aligned}A(\alpha,\beta) &= \{t \in X : \alpha,\ \beta \in [0,1]\ ; \\ &\quad \mu_A(t) \ge \alpha,\ \zeta_A(t) \le \beta,\ 0 \le \alpha+\beta \le 1\}\end{aligned}$$

is called the (α,β)-level of the intuitionistic fuzzy set $\widehat{A^i}$.

Theorem 1. *[26] Let $\widehat{A^i} \in IF(X)$. Then $A(\alpha,\beta) = A(\alpha) \cap A^*(\beta)$ holds.Here $A(\alpha)$ is $\alpha-$level set and $A^*(\beta)$ is $\beta-$level set.*

Definition 3. *[26] An intuitionistic fuzzy set $\widehat{A^i} \in IF(R^n)$ satisfying the following properties is called an intuitionistic fuzzy number in R^n*

1) $\widehat{A^i}$ is a normal set, i.e., $\exists\ t_0 \in R^n$ such that $\mu_A(t_0) = 1$ and $v_A(t_0) = 0$,

2) $A(0)$ and $A^*(1)$ are bounded sets in R^n,

3) $\mu_A : R^n \to [0,1]$ is an upper semi-continuous function, i.e.,

$\forall\ k \in [0,1]$, $(\{t \in A : \mu_A(t) < k\})$ is an open set.

4) $\zeta_A : R^n \to [0,1]$ is a lower semi-continuous function, i.e.,

$\forall\ k \in [0,1](\{t \in A : \zeta_A(t) > k\})$ is an open set.

5) The membership function $\mu_A(t)$ is quasi-concave, i.e.,

$\forall\ n \in [0,1]$, $\forall\ x, y \in R^n$

$$\mu_A(nt + (1-n)x) \ge min(\mu_A(t), \mu_A(x)),$$

6) The non-membership function $\zeta_A(t)$ is quasi-convex; i.e.,

$\forall\ n \in [0,1]$, $\forall x, y \in R^n$

$$\zeta_A(nt + (1-n)x) \le max(\zeta_A(t), \zeta_A(x)).$$

The set of all intuitionistic fuzzy numbers of R^n will be denoted by $IFN(R^n)$.

Definition 4. *[10] A triangular intuitionistic fuzzy number (TIFN) $\widehat{A^i} \in IF(R^n)$ is defined with the following membership and non-membership functions:*

$$\mu_A(t) = \begin{cases} \frac{t-a_1}{a_2-a_1};\ a_1 \le t \le a_2 \\ \frac{a_2-t}{a_3-a_2};\ a_2 \le t \le a_3 \\ 0;\ otherwise \end{cases}$$

and

$$\zeta_A(t)=\begin{cases}\frac{a_2-t}{a_2-a_1^*};\ a_1^*\le t\le a_2\\ \frac{t-a_2}{a_3^*-a_2};\ a_2\le t\le a_3^*\\ 1;\ otherwise\end{cases}.$$

Here $a_1^*\le a_1\le a_2\le a_3\le a_3^*$ *and it is denoted by* $\widehat{A^i}=(a_1,a_2,a_3;a_1^*,a_2,a_3^*)$.

Remark 1. *[33] Let* $\widehat{A^i}\in IFN(R)$*. Then* $\left[\widehat{A}\right]^\alpha$ *and* $\left[\widehat{A^*}\right]^\beta$ *are closed and bounded intervals such that*

$$\left[\widehat{A}\right]^\alpha=\left[A_\alpha^-,A_\alpha^+\right]=\left[(a_2-a_1)\alpha+a_1,a_3-(a_3-a_2)\alpha\right]$$

and

$$\left[\widehat{A^*}\right]^\alpha=\left[a_2-(a_2-a_1^*)\alpha,(a_3^*-a_2)\alpha+a_2\right].$$

Definition 5. *[32] Let* X *and* Y *be two sets and* $g:X\to Y$ *be a function. Let* $\widehat{A^i}$ *be an intuitionistic fuzzy set in* X*. Then* $f(\widehat{A^i})$ *is an intuitionistic fuzzy set in* Y *such that for every* $y\in Y$

$$\mu_{g\left(\widehat{A^i}\right)}(y)=\begin{cases}\sup\{\mu_A(x):g(x)=y\};y\in g(x)\\ 0;\qquad y\notin f(x),\end{cases}$$

and

$$\zeta_{g\left(\widehat{A^i}\right)}(y)=\begin{cases}\inf\{\zeta_A(x):g(x)=y\};y\in g(x)\\ 1;\qquad y\notin g(x),\end{cases}$$

Definition 6. *[33] The function*

$$\theta(x)=\begin{cases}1,& x\ge 0\\ 0,& x<0\end{cases}$$

is called the Heaviside step function.

3. Numerical Method for the FBVP

Here, the eigenvalues and the fuzzy eigenfunctions of the intuitionistic fuzzy problem (1)-(3) are investigated. Then, similar to the method applied by Titchmarsh [34], we will use the solutions of (1) that satisfy the fuzzy initial conditions instead of the fuzzy boundary conditions. To solve intuitionistic fuzzy IVPs, the method created by Akin and Bayeğ is used [33]. To do this, firstly the crisp IVP will be solved.

Then, the solution of intuitionistic FIVPs will be obtained from classical solutions using the intuitionistic Zadeh's extension principle. The fuzzy solutions do not require the analysis of existence of switching endpoints of α and β levels, because Heaviside (step) function will be applied during the interval operations on α and β levels.

Now, let the linear and homogeneous differential equation (1) be considered separately with intuitionistic fuzzy boundary conditions (2) and (3),respectively.

$$\begin{cases}\chi''+\lambda\chi=0\\ \chi(a)=\widehat{a^i}_2,\ \chi'(a)=\widehat{a^i}_1\end{cases}\tag{4}$$

and

$$\begin{cases}\Psi''+\lambda\Psi=0\\ \Psi(b)=\widehat{b^i}_2,\ \Psi'(b)=\widehat{b^i}_1.\end{cases}\tag{5}$$

where $\widehat{a^i}_1,\widehat{a^i}_2,\widehat{b^i}_1,\widehat{b^i}_2$ intuitionistic triangular fuzzy numbers, $\lambda>0$.

Theorem 2. *[33] Let* $\widehat{\chi^i}(t)$ *and* $\widehat{\Psi^i}(t)$ *be the solution of the intuitionistic IVP in (4) and (5) obtained by intuitionistic Zadeh's extension principle. Let* α *and* β *levels of* $\widehat{\chi^i}(t)$ *and* $\widehat{\Psi^i}(t)$*,* $\widehat{a_k^i}$ *and* $\widehat{b_k^i}$ $(k=1,2)$ *be given by* $[\chi_\alpha^-(t,\lambda),\chi_\alpha^+(t,\lambda)]$, $[\Psi_\alpha^-(t,\lambda),\Psi_\alpha^+(t,\lambda)]$ *and* $\left[(\chi^*)_\alpha^-(t,\lambda),(\chi^*)_\alpha^+(t,\lambda)\right]$, $\left[(\Psi^*)_\alpha^-(t,\lambda),(\Psi^*)_\alpha^+(t,\lambda)\right]$; $\left[(a_k)_\alpha^-,(a_k)_\alpha^+\right]$, $\left[(b_k)_\alpha^-,(b_k)_\alpha^+\right]$ *and* $\left[(a_k^*)_\alpha^-,(a_k^*)_\alpha^+\right]$, $\left[(b_k^*)_\alpha^-,(b_k^*)_\alpha^+\right]$*, respectively. Then the* α *and* β *levels of the solution can be determined as follows:*

$$\begin{cases}\chi_\alpha^-=\sum_{k=1}^{2}\left[(a_k)_\alpha^+-\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\\ \chi_\alpha^+=\sum_{k=1}^{2}\left[(a_k)_\alpha^-+\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\\ (\chi^*)_\beta^-=\sum_{k=1}^{2}\left[(a_k^*)_\alpha^+-\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\\ (\chi^*)_\beta^+=\sum_{k=1}^{2}\left[(a_k^*)_\alpha^-+\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)\end{cases}$$

and

$$\begin{cases}\Psi_\alpha^-=\sum_{k=1}^{2}\left[(a_k)_\alpha^+-\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\\ \Psi_\alpha^+=\sum_{k=1}^{2}\left[(a_k)_\alpha^-+\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\\ (\Psi^*)_\beta^-=\sum_{k=1}^{2}\left[(a_k^*)_\alpha^+-\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\\ (\Psi^*)_\beta^+=\sum_{k=1}^{2}\left[(a_k^*)_\alpha^-+\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{2k}(t))\right]K_{2k}(t)\end{cases}$$

Here $K_{1k}(t)$ *and* $K_{2k}(t)$ *are Heaviside function.*

$$\chi_\alpha^-=\sum_{k=1}^{2}\left[(a_k)_\alpha^+-\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

$$\chi_\alpha^+=\sum_{k=1}^{2}\left[(a_k)_\alpha^-+\left((a_k)_\alpha^+-(a_k)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

$$(\chi^*)_\alpha^-=\sum_{k=1}^{2}\left[(a_k^*)_\alpha^+-\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

$$(\chi^*)_\alpha^+=\sum_{k=1}^{2}\left[(a_k^*)_\alpha^-+\left((a_k^*)_\alpha^+-(a_k^*)_\alpha^-\right)\theta(K_{1k}(t))\right]K_{1k}(t)$$

First, let us look for the solution to the problem in Equation (4). Then, by performing similar operations, we find the solution to the problem (5). First of all we solve the following crisp IVP related to the fuzzy IVP in Eq. (4) and then apply intuitionistic Zadeh's Extension Principle to the solution [33]:

$$\begin{cases} \chi'' + \lambda\chi = 0 \\ \chi(a) = a_2,\ \chi'(a) = a_1 \end{cases} \quad (6)$$

where a_1, a_2 and λ are real numbers. The general solution of the differential equation (6) can be written as:

$$\chi_\lambda(t) = C_1\chi_1(t) + C_2\chi_2(t), \quad (7)$$

where C_1 and C_2 are arbitrary constants; $\chi_1(t)$ and $\chi_1(t)$ are linearly independent functions satisfying the Eq. (6).

Let us substitute the initial conditions to find the coefficients C_1 and C_2 in equation Eq. (7). Therefore, the following system of equations is obtained:

$$\begin{cases} C_1\chi_1(a) + C_2\chi_2(a) = a_2 \\ C_1\chi_1'(a) + C_2\chi_2'(a) = a_1 \end{cases} \quad (8)$$

In Eq.(8) C and B are unknown coefficients and the following notations are used for convenience.

$$\begin{aligned} W &= \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix}; \\ w_{11} &= \chi_1(a),\ w_{12} = \chi_2(a),\ w_{21} = \chi_1'(a),\ w_{22} = \chi_2'(a); \\ \overrightarrow{C} &= \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},\ \overrightarrow{a} = \begin{pmatrix} a_2 \\ a_1 \end{pmatrix}. \end{aligned}$$

According to these notations, (8) is written in the matrix form:

$$W\overrightarrow{C} = \overrightarrow{a}.$$

Using Cramer's method, C_1 and C_2 are obtained as follows:

$$C_J = \frac{|W_1|}{|W|} - \frac{|W_2|}{|W|}.$$

Here

$$\begin{aligned} |W| &= \begin{vmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{vmatrix} = w_{11}w_{22} - w_{21}w_{12}, \\ |W_1| &= \begin{vmatrix} a_2 & w_{12} \\ a_1 & w_{22} \end{vmatrix} = a_2w_{22} - a_1w_{12}, \\ |W_2| &= \begin{vmatrix} w_{11} & a_2 \\ w_{21} & a_1 \end{vmatrix} = a_1w_{11} - a_2w_{21}. \end{aligned}$$

Thus, C_1 and C_2 can be rewritten as

$$\begin{aligned} C_1 &= \frac{|W_1|}{|W|} = \frac{a_2w_{22} - a_1w_{12}}{|W|}, \\ C_2 &= \frac{|W_2|}{|W|} = \frac{a_1w_{11} - a_2w_{21}}{|W|}. \end{aligned}$$

C_1 and C_2 can be rewritten as follows to simplify the above results, respectively

$$\begin{aligned} C_1 &= a_2f_{22} - a_1f_{12}, \\ C_2 &= a_1f_{11} - a_2f_{21} \end{aligned}$$

where $f_{ij} = \frac{w_{ij}}{|W|}; i, j = 1, 2$.

From the results for C_1 and C_2, the classical solution of the given crisp IVP can be derived as follows:

$$\begin{aligned} \chi_\lambda(t) &= C_1\chi_1(t) + C_2\chi_2(t), \\ &= (a_2f_{22} - a_1f_{12})\,\chi_1(t) \\ &\quad + (a_1f_{11} - a_2f_{21})\,\chi_2(t). \end{aligned}$$

This solution can also be written as:

$$\begin{aligned} \chi_\lambda(t) = a_2\left(f_{22}\chi_1(t) - f_{21}\chi_2(t)\right) \\ + a_1\left(f_{11}\chi_2(t) - f_{12}\chi_1(t)\right). \end{aligned}$$

Next the following notations are used for the sake of its comprehension:

$$\begin{aligned} K_{11}(t) &= f_{22}\chi_1(t) - f_{21}\chi_2(t), \\ K_{12}(t) &= f_{11}\chi_2(t) - f_{12}\chi_1(t). \end{aligned} \quad (9)$$

Thus the solution of the crisp IVP (6) can be written as:

$$\chi_\lambda(t) = a_2K_{11}(t) + a_1K_{12}(t). \quad (10)$$

It is easy to see that the solution in Eq. (10) is linearly dependent only on the initial values. Now, Zadeh's extension principle is applied to the intuitionistic fuzzy sets and the solution of the fuzzy IVP as follows:

$$\widehat{\chi^i}_\lambda(t) = \widehat{a^i}_2K_{11}(t) + \widehat{a^i}_1K_{12}(t) \quad (11)$$

In terms of α and β levels of the intuitionistic fuzzy numbers it is obtained that

$$\begin{cases} \left[\chi_\alpha^-(t,\lambda), \chi_\alpha^+(t,\lambda)\right] = \sum_{k=1}^{2}\left[(a_k)_\alpha^-, (a_k)_\alpha^+\right]K_{1k}(t) \\ \left[(\chi^*)_\alpha^-(t,\beta), (\chi^*)_\alpha^+(t,\beta)\right] = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^-, (a_k^*)_\alpha^+\right]K_{1k}(t) \end{cases}$$

where $\chi_\alpha^-(t,\lambda)$, $(a_k)_\alpha^-$, ; $(\chi^*)_\alpha^-(t,\lambda)$, $(a_k^*)_\alpha^-$ are lower bounds for $\alpha-$levels and $\beta-$levels, respectively and $\chi_\alpha^+(t,\lambda)$, $(a_k)_\alpha^+$, ; $(\chi^*)_\alpha^+(t,\lambda)$, $(a_k^*)_\alpha^+$ are upper bounds for $\alpha-$levels and $\beta-$levels, respectively

Using the Heaviside function and interval arithmetic the α and β levels of the solution $\widehat{\chi^i}_\lambda(t)$ can be written as follows:

$$\begin{cases} \chi_\alpha^- = \sum_{k=1}^{2}\left[(a_k)_\alpha^+ - \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta\left(K_{1k}(t)\right)\right]K_{1k}(t) \\ \chi_\alpha^+ = \sum_{k=1}^{2}\left[(a_k)_\alpha^- + \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta\left(K_{1k}(t)\right)\right]K_{1k}(t) \\ (\chi^*)_\alpha^- = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^+ - \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta\left(K_{1k}(t)\right)\right]K_{1k}(t) \\ (\chi^*)_\alpha^+ = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^- + \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta\left(K_{1k}(t)\right)\right]K_{1k}(t) \end{cases} \quad (12)$$

For $\left[\widehat{\Psi^i}_\lambda(t)\right]^\alpha$, a solution is found for the problem (5) by doing similar operations. So the solution of the crisp IVP $\Psi_\lambda(t)$ can be written as:

$$\Psi_\lambda(t) = b_2 K_{21}(t) + b_1 K_{22}(t). \quad (13)$$

Then Zadeh's extension principle is applied and the solution of the fuzzy IVP as follows:

$$\widehat{\Psi^i}_\lambda(t) = \widehat{a^i}_2 K_{21}(t) + \widehat{a^i}_1 K_{22}(t). \quad (14)$$

By taking $\alpha-$levels and $\beta-$levels, into account in the solution (5) and using the Heaviside function, the solution $\widehat{\Psi^i}_\lambda(t)$ can be written as follows:

$$\begin{cases} \Psi_\alpha^- = \sum_{k=1}^{2}\left[(a_k)_\alpha^+ - \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \\ \Psi_\alpha^+ = \sum_{k=1}^{2}\left[(a_k)_\alpha^- + \left((a_k)_\alpha^+ - (a_k)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \\ (\Psi^*)_\alpha^- = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^+ - \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \\ (\Psi^*)_\alpha^+ = \sum_{k=1}^{2}\left[(a_k^*)_\alpha^- + \left((a_k^*)_\alpha^+ - (a_k^*)_\alpha^-\right)\theta\left(K_{2k}(t)\right)\right]K_{2k}(t) \end{cases} \quad (15)$$

Because the eigenvalues of the problem (1)-(3) if and only if consist of the zeros of function $W(\chi,\Psi)(t,\lambda)$ in [34], Wronskian function is found from the classical solutions (10) and (13) for classic eigenvalue λ as follows :

$$W(\chi,\Psi)(t,\lambda) = \chi_\lambda(t)\Psi'_\lambda(t) - \chi'_\lambda(t)\Psi_\lambda(t). \quad (16)$$

Now we give the following numerical example to demonstrate the proposed method.

Example 1. *Consider the intuitionistic fuzzy boundary value problem*

$$-u'' = \lambda u \quad (17)$$

$$\widehat{2^i}u(0) = \widehat{1^i}u'(0) \quad (18)$$

$$\widehat{4^i}u(1) = \widehat{3^i}u'(1) \quad (19)$$

where $\widehat{1^i} = (0,1,2;-1,1,3)$, $\widehat{2^i} = (1,2,3;0,2,4)$, $\widehat{3^i} = (2,3,4;1,3,5)$, $\widehat{4^i} = (3,4,5;2,4,6)$ *intuitionistic triangular fuzzy numbers and* $\lambda = p^2$, $p > 0$. *From problem* (17)-(19) *, we get two intuitionistic FIVPs as follows:*

$$\chi'' + p^2\chi = 0, \quad \chi(0) = \widehat{1^i}, \quad \chi'(0) = \widehat{2^i} \quad (20)$$

and

$$\Psi'' + p^2\Psi = 0, \quad \Psi(1) = \widehat{3^i}, \quad \Psi'(1) = \widehat{4^i}. \quad (21)$$

Let us first solve the crisp IVP:

$$\chi'' + p^2\chi = 0, \quad \chi(0) = 1, \quad \chi'(0) = 2.$$

By solving the differential equation in the crisp IVP, the general crisp solution is obtained as:

$$\chi(t,\lambda) = C_1\cos(pt) + C_2\sin(pt).$$

The functions $K_{11}(t)$ *and* $K_{12}(t)$ *are obtained as follows:*

$$\begin{aligned} K_{11}(t) &= \cos(pt) \\ K_{12}(t) &= \frac{1}{p}\sin(pt). \end{aligned} \quad (22)$$

Thus the solution of the crisp IVP can be written using (22) *as:*

$$\begin{aligned} \chi(t,\lambda) &= a_2 K_{11}(t) + a_1 K_{12}(t) \\ &= \frac{2}{p}\sin(pt) + \cos(pt) \end{aligned} \quad (23)$$

Similarly, the solution $\Psi(t,\lambda)$ *is written as:*

$$\Psi(t,\lambda) = \frac{4}{p}\sin(pt-p) + 3\cos(pt-p). \quad (24)$$

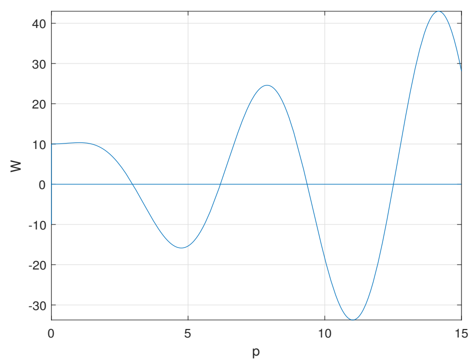

Figure 1. The function $W(\lambda) = \left(3p + \frac{8}{p}\right)\sin(p) + (4-6)\cos(p)$.

Then, Wronskian functions can be gotten from Eq. (16) *as:*

$$\begin{aligned} W(\lambda) &= W(\chi,\Psi)(t,\lambda) \\ &= \left(3p + \frac{8}{p}\right)\sin(p) + (-2)\cos(p). \end{aligned}$$

The classic eigenvalues of problem (17)-(19) *consist of the zeros of the* $W(\lambda)$ *functions. For this reason, an infinite number of eigenvalues satisfying the equation* $W(\lambda) = 0$ *can be obtained by calculating p values in Matlab programme in Figure 1.*

Table 1. Eigenvalues of the fuzzy problem.

	p_n	λ_n
$n = 1$	*3.30241*	*10.90581*
$n = 2$	*6.38091*	*40.71581*
$n = 3$	*9.49291*	*90.11511*
$n = 4$	*12.61831*	*159.22151*
$n = 5$	*15.74981*	*248.05621*
$n \approx$	$n\pi$	$(n\pi)^2$

The first five eigenvalues are found numerically and then the approximation of the remaining eigenvalues is written in tabl e 1.

From (12) *and* (15) *$\alpha-$levels and $\beta-$levels of the solutions* $\widehat{\chi^i}_\lambda(t)$ *and* $\widehat{\Psi^i}_\lambda(t)$*, respectively can be found as follows:*

$$\begin{aligned}
\chi_\alpha^-(t,\lambda) &= [2-\alpha-2(1-\alpha)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[3-\alpha-2(1-\alpha)\theta(K_{12}(t))]K_{12}(t),\\
\chi_\alpha^+(t,\lambda) &= [\alpha+2(1-\alpha)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[\alpha+1+2(1-\alpha)\theta(K_{12}(t))]K_{12}(t),\\
(\chi^*)_\alpha^-(t,\beta) &= [2\beta+1-(4\beta)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[2+2\beta-(4\beta)\theta(K_{12}(t))]K_{12}(t),\\
(\chi^*)_\alpha^+(t,\beta) &= [1-2\beta+(4\beta)\theta(K_{11}(t))]K_{11}(t)\\
&\quad+[2-2\beta+(4\beta)\theta(K_{12}(t))]K_{12}(t).
\end{aligned}$$

and

$$\begin{aligned}
\Psi_\alpha^- &= [4-\alpha-2(1-\alpha)\theta(K_{21}(t))]K_{21}(t)\\
&\quad+[5-\alpha-2(1-\alpha)\theta(K_{22}(t))]K_{22}(t),\\
\Psi_\alpha^+ &= [2+\alpha+2(1-\alpha)\theta(K_{21}(t))]K_{21}(t)\\
&\quad+[3+\alpha+2(1-\alpha)\theta(K_{22}(t))]K_{22}(t),\\
(\Psi^*)_\alpha^- &= [3+2\beta-(4\beta)\theta(K_{21}(t))]K_{21}(t)\\
&\quad+[4+2\beta-(4\beta)\theta(K_{22}(t))]K_{22}(t),\\
(\Psi^*)_\alpha^+ &= [3-2\beta+(4\beta)\theta(K_{11}(t))]K_{21}(t)\\
&\quad+[4-2\beta+(4\beta)\theta(K_{22}(t))]K_{22}(t).
\end{aligned}$$

where $\theta(t)$ *is the Heaviside function,* $K_{11}(t)=\cos(pt)$, $K_{12}(t)=\frac{1}{p}\sin(pt)$, $K_{21}(t)=\cos(pt-p)$ *and* $K_{22}(t)=\frac{1}{p}\sin(pt-p)$.

In particular, $p_1=3.30241$ *in Table 1 and substitute* (25) *and* (25) *are selected. The α and β levels of the solutions* $\widehat{\chi^i}_{p_1}(t)$ *and* $\widehat{\Psi^i}_{p_1}(t)$ *are given in Figures 2, 3 and Figures 4, 5.*

Consider the FBVP given as in (17)-(19), using gh-differentiabilty by converting the FDE into a family of systems of classical differential equation [35]. Now we have that the graphical representation of the endpoint functions χ_α^-, χ_α^+ *in Figure 6 and* Ψ_α^-, Ψ_α^+ *in Figure 7 obtained of (1,1)-system for every* $\alpha\in[0,1]$*. In Figure 6 and 7, it is seen that the* $\widehat{\chi}$ *and* $\widehat{\Psi}$ *functions do not fulfil the fuzzy solution properties duo to the existence of switching points in the entire interval* $[0,3.5]$.

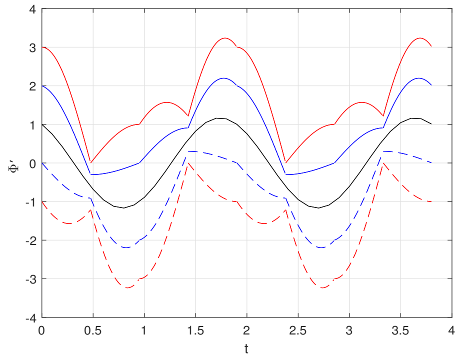

Figure 2. The $\widehat{\chi^i}_\lambda(t)$ solution in Example 1. The black line represents the reel solution. The red and blue lines represent upper solution for $\beta=1$ and $\alpha=0$, respectively and the dashed red and blue lines represent lower solution for $\beta=1$ and $\alpha=0$, respectively

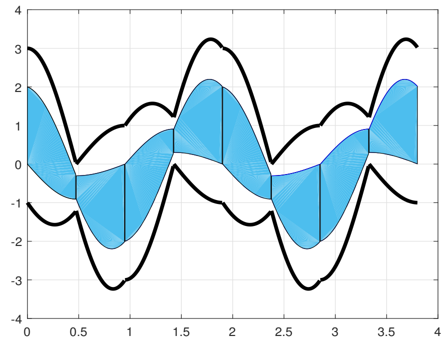

Figure 3. The blue region of the intersection of fuzzy solution $[\chi]^\alpha$ and $[\chi^*]^\alpha$ of the intuitionistic fuzzy solution in Example 1

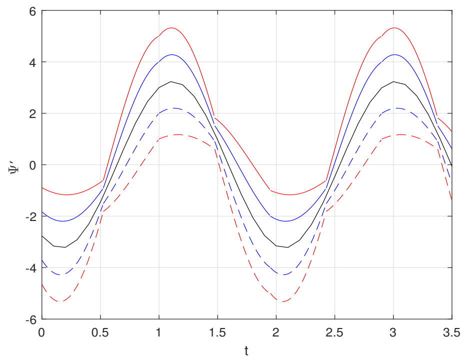

Figure 4. The $\widehat{\Psi^i}_\lambda(t)$ solution in Example 1. The black line represents the crisp solution. The red and blue lines represent upper solution for $\beta=1$ and $\alpha=0$, respectively and the dashed red and blue lines represent lower solution for $\beta=1$ and $\alpha=0$, respectively

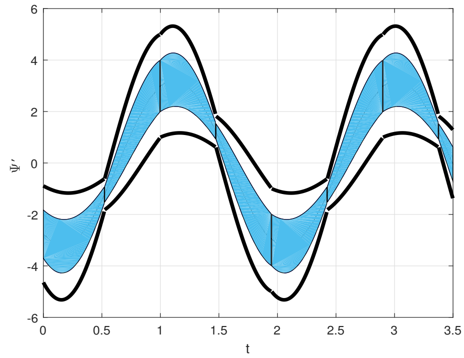

Figure 5. The blue region of the intersection of fuzzy solution $[\psi]^\alpha$ and $[\psi^*]^\alpha$ of the intuitionistic fuzzy solution in Example 1

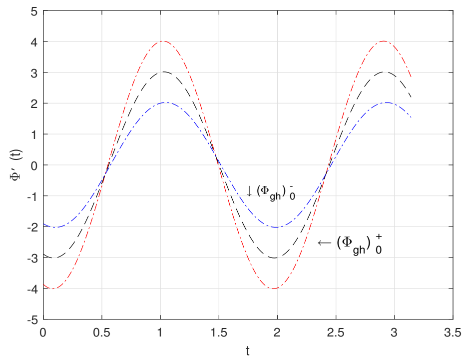

Figure 6. The χ solution of the (1,1)-system related to (17)-(19) in the sense of gH-derivative.The blue line and the red line represent respectively the left and right end-points of the 0-level of the solution the black line represent the reel solution in Example 1

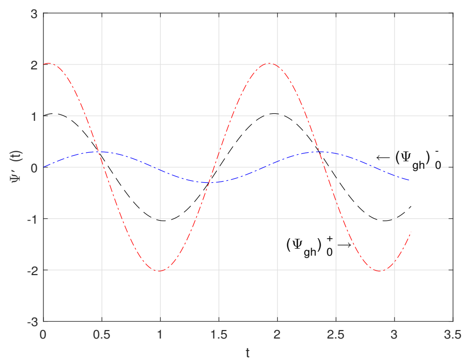

Figure 7. The ψ solution of (1,1)-system related to (17)-(19) in the sense of gH-derivative.The blue line and the red line represent respectively the left and right end-points of the 0-level of the solution the black line represent the reel solution for Example 1

4. Conclusion

The main contribution of this article is the study of intuitionistic fuzzy eigenvalue problem with boundary values given by intuitionistic fuzzy numbers. The eigenvalues of the fuzzy problem are found mainly on the idea of the intuitionistic Zadeh's extension principle. To do this the method proposed in Theorem 2 is used. Then one of the obtained eigenvalues is arbitrarily selected and substituted in the fuzzy solutions to obtain the intuitionistic fuzzy eigenfunctions $\widehat{\chi^i}_\lambda(t)$ and $\widehat{\Psi^i}_\lambda(t)$ which are shown in Figures 2, 3, 4 and 5. To prevent switch-points as illustrated in Figure 6 and in Figure 7, Heaviside function is used during the interval operations on α and β-levels.

The approach using the gH-derivative is equivalent to the study of some systems of classical differential equations, which can lead to an additional study of switching points as shown in Figures 6 and 7. Moreover from this approach, the sign of the solution is considered itself and the signs of its first and second derivatives.

By using the method in this paper, fuzzy eigenfunctions are obtained without dealing with these unfavourable situations.

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