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RESEARCH ARTICLE

Some results regarding observability and initial state reconstruction for time-fractional systems

Hamza Ben Brahim[a], Fatima-Zahrae El Alaoui[a], Khalid Zguaid[b*]

[a] *TSI Team, Faculty of Sciences, Moulay Ismail University, Meknes, Morocco*
[b] *LRST Laboratory, The Higher School of Education and Training of Agadir (ESEFA), Ibn Zohr University, Agadir, Morocco*
brahim.hamzaben@edu.umi.ac.ma, f.elalaoui@umi.ac.ma, zguaid.khalid@gmail.com

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ABSTRACT

The aim of this study is to present the notion of observability for a specific class of linear time-fractional systems of Riemann-Liouville type with a differentiation order between 1 and 2. To accomplish this goal, we first define the concept of observability and its features, then we extend the Hilbert Uniqueness Method (HUM) to determine the system's initial state. This method converts the reconstruction problem into a solvability one, leading to an algorithm that calculates the initial state. The effectiveness of the proposed algorithm is demonstrated through numerical simulations, which are presented in the final section.

1. Introduction

Over the past two decades, fractional differential systems have been widely used in the mathematical modeling of real phenomena such as diffusion, fluid mechanics, and viscoelasticity [1, 2]. These applications have motivated many researchers in the field of differential systems to study fractional differential systems with different fractional derivatives. In many processes or phenomena with long-range temporal cumulative memory effects and/or long-range spatial interactions, numerical and theoretical results have also shown that fractional differential systems offer more advantages than integer-order systems. Recently, the theory of fractional differential systems has become an important research topic in the field of evolutionary systems [3–5].

In this paper, we consider Ω as a bounded region in $\mathbb{R}^n$ whose boundary is sufficiently smooth $\partial\Omega$, and $\varepsilon \in]1,2[$. From now on, we denote $Q = \Omega\times]0,T]$ and $\Sigma = \partial\Omega\times]0,T]$ and we consider the following fractional system on the finite interval $]0,T]$:

$$
\begin{cases}
{}^{RL}\mathcal{D}^{\varepsilon}_{0^+}\Theta(x,t) = A\Theta(x,t) & in\ Q,\\
\lim\limits_{t\to 0^+} \mathcal{I}^{2-\varepsilon}_{0^+}\Theta(x,t) = \Theta_0(x) & in\ \Omega,\\
\lim\limits_{t\to 0^+} \dfrac{\partial}{\partial t}\mathcal{I}^{2-\varepsilon}_{0^+}\Theta(x,t) = \Theta_1(x) & in\ \Omega,\\
\Theta(\xi,t) = 0 & on\ \Sigma,
\end{cases}
\tag{1}
$$

where ${}^{RL}\mathcal{D}^{\varepsilon}_{0^+}$ is the Riemann-Liouville fractional order derivative, A is a second order, linear, differential operator, and $\mathcal{I}^{\varepsilon}_{0^+}$ is the Riemann–Liouville fractional integral of order ε. The Riemann-Liouville fractional derivative is one of the most important extensions of ordinary integer order derivatives. Differential equations with this type of fractional derivative require special forms of

*Corresponding Author

initial conditions. The study of distributed systems involves analyzing and designing these systems based on their mathematical models, including tasks such as determining system stability and observability, as well as designing control and observer algorithms to ensure the system behaves as desired. Fractional calculus provides a powerful tool for analyzing and designing fractional distributed systems, as it allows for a more complete and accurate description of the system's behavior.

In recent years, the observability of fractional equations has received considerable attention. Observability refers to the ability to determine an initial state on the basis of its inputs and outputs, and is a crucial concept in the analysis of control systems. In general, the observability of fractional equations in abstract spaces comprises two cases: exact observability and approximate observability. When studying the exact observability of fractional systems in abstract spaces, we assume that the observability operator has a bounded inverse operator. Approximate observability, as opposed to exact observability, is better suited to describing natural phenomena. Previous research has extensively studied observability in classical systems, including integer-order distributed parameter systems. For instance, Wang [6] discussed observability in such systems, while Goodson and Klein [7] established observability criteria for simple systems, such as the wave equation and heat equation. More recent studies have examined observability through regional analysis. For example, Bourray, Boutoulout, and El Alaoui [8, 9] studied the regional boundary observability and the regional gradient observability for distributed parameter systems of integer order, while [10] developed regional enlarged observability for integer-order linear parabolic systems. For time-fractional distributed parameter systems with Riemann-Liouville fractional derivative, [11] developed the regional gradient observability. Zguaid, El Alaoui, and Boutoulout [12] studied the observability of a class of linear time-fractional diffusion systems with Caputo derivative of order $0 < \varepsilon < 1$. Additionally, other definitions of observability have been proposed in the literature, such as the fractional observability Gramian and matrix, studied by the authors in [13]. They also derived controllability and observability conditions for fractional continuous-time linear systems based on Gramian matrices. Furthermore, [14] explored regional observability for Hadamard-Caputo time fractional distributed parameter systems, and in [15], the pseudo-state representation was used to construct Luenberger-like observers for estimating various variables. For a deeper understanding of observability for classical and fractional systems we refer the reader to the literature [16–28]. According to the academic literature, the observability of linear systems has been widely studied, with multiple methods proposed for calculation. One such method is the Hilbert Uniqueness Method (HUM). The HUM approach is based on the principles of Hilbert space theory and can be extended to time-fractional distributed systems. Using HUM, observability can be determined by converting the reconstruction problem into a solvency one [29]. There are many articles that discuss the applications of the HUM approach, and we refer the reader to [30–34].

Inspired by the above-mentioned articles, the aim of this work is to study the observability of Riemann-Liouville time-fractional system (1). Our contribution consists of giving several characterizations for the exact and approximate observability of the linear system under consideration. We present a method for reconstructing the initial state in the desired region. In addition, we provide some simple numerical simulations that support our theoretical results.

This work is structured as follows: The first section provides an overview of the mathematical and conceptual foundations that will be used throughout the work. The second section focuses on the definitions and observability characteristics of fractional linear-time distributed systems with a Riemann-Liouville type derivative of order $1 < \varepsilon < 2$, including exact and approximate observability. In the third section, we introduce the concept of fractional Green's formula with order $1 < \varepsilon < 2$, and apply the HUM approach to determine the initial state of the system. To validate the effectiveness of the HUM approach, we present some examples and conclude with a numerical simulation.

2. Considered system and preliminaries

In this section, we give some definitions of fractional derivatives and integrals of Riemann-Liouville and Caputo types with an order of differentiation between 1 and 2. In addition, we introduce the essential concepts related to the theory of the cosine family, which will be used in this work. Moreover, we define the two-parameter Mittag-Leffler function, which has many important applications in fractional calculus. Let $\mathfrak{X}$ be a Hilbert space with the norm $\|.\|_{\mathfrak{X}}$.

We begin by outlining the definitions and key properties of fractional integrals and derivatives of Riemann-Liouville and Caputo types with order ε in the interval $]1,2[$.

Definition 1. *[35] Let us consider $\mathcal{Q} \in L^1(]a,b[;\mathfrak{X})$. We define the left-sided fractional integral of ζ with order $\varepsilon \in]1,2[$ by the following from:*

$$\mathcal{I}^{\varepsilon}_{a+}\mathcal{Q}(t) = \frac{1}{\Gamma(\varepsilon)}\int_a^t (t-s)^{\varepsilon-1}\mathcal{Q}(s)ds, \quad a < t \leq b.$$

Where $\Gamma(\varepsilon) = \displaystyle\int_0^{+\infty} \theta^{\varepsilon-1}e^{-\theta}d\theta$, is the Gamma function.

We define $AC([a,b];\mathfrak{X}) := \{\varphi : [a,b] \to \mathfrak{X} \;, \varphi \textit{ is absolutely continuous}\}$.

Definition 2. *[35] Let us consider $\zeta \in AC([a,b];\mathfrak{X})$ then:*

(1) *The left-sided Riemann–Liouville fractional derivative of order $\varepsilon \in]1,2[$ of a function ζ is defined by:*

$$^{RL}\mathcal{D}^{\varepsilon}_{a+}\mathcal{Q}(t) = \frac{1}{\Gamma(2-\varepsilon)}\frac{d^2}{dt^2}\int_a^t (t-s)^{1-\varepsilon}\mathcal{Q}(s)ds,$$

for $a < t \leq b$.

(2) *The right-sided Caputo fractional derivative of order $\varepsilon \in]1,2[$ of a function ζ is defined by:*

$$^{C}\mathcal{D}^{\varepsilon}_{b-}\mathcal{Q}(t) = \frac{1}{\Gamma(2-\varepsilon)}\int_t^b (s-t)^{1-\varepsilon}\frac{d^2}{ds^2}\mathcal{Q}(s)ds,$$

for $a \leq t < b$.

Next, we will explore the key concepts of cosine family theory that are relevant to this work. For the rest of this paper, the adjoint of any operator P, is denoted by P^*.

Definition 3. *[36] A one parameter family $(\mathfrak{W}(t))_{t\in\mathbb{R}}$ of bounded linear operators mapping the Banach space $\mathfrak{X}$ into itself is called a strongly continuous cosine family if and only if:*

(1) $\mathfrak{W}(0) = \mathfrak{I}$, *where $\mathfrak{I}$ is the identity function of $\mathfrak{X}$.*
(2) $\mathfrak{W}(t+s)+\mathfrak{W}(t-s) = 2\mathfrak{W}(t)\mathfrak{W}(s), \forall (t,s) \in \mathbb{R}^2$.
(3) *The map $\eta \mapsto \mathfrak{W}(t)\eta$, is continuous in t for each fixed point $\eta \in \mathfrak{X}$.*

The sine family $(\mathfrak{U}(t))_{t\in\mathbb{R}}$ associated with the strongly continuous cosine family $(\mathfrak{W}(t))_{t\in\mathbb{R}}$ is defined by:

$$\mathfrak{U}(t)\eta = \int_0^t \mathfrak{W}(s)\eta ds, \quad \eta \in \mathfrak{X}, \quad t \in \mathbb{R}.$$

The infinitesimal generator of the cosine family $(\mathfrak{W}(t))_{t\in\mathbb{R}}$, which we denote A, is defined by:

$$A\eta = \frac{d^2}{dt^2}\mathfrak{W}(0)\eta, \quad \forall \eta \in \mathcal{D}(A).$$

Where $\mathcal{D}(A) = \left\{\eta \in \mathfrak{X} \; : \mathfrak{W}(t)\eta \in C^2(R,\mathfrak{X})\right\}$. This infinitesimal generator A is a closed, densely-defined operator in $\mathfrak{X}$.
We have the following proposition

Proposition 1. *[37] Let A be the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(\mathfrak{W}(t))_{t\in\mathbb{R}}$ on $\mathfrak{X}$, we have:*

$$\mathfrak{W}(t) = \sum_{n=0}^{+\infty}\frac{A^n t^{2n}}{2n!}, \quad \forall t \in \mathbb{R}.$$

For more details about strongly continuous cosine and sine families, we refer the reader to [38, 39].
The two-parameter Mittag-Leffler function, which plays an important role in this work, is given in the coming definition.

Definition 4. *[40] The two parameter Mittag-Leffler function is defined as:*

$$E_{p,\zeta}(u) = \sum_{j=0}^{+\infty}\frac{u^j}{\Gamma(jp+\zeta)}; \quad p,\ \zeta > 0; \quad u \in \mathbb{C}.$$

For more details, we refer to [41] .
Throughout this paper, we maintain the assumption that $\mathfrak{X} := L^2(\Omega)$, and that the family of linear operators $(\mathfrak{W}(t))_{t\in\mathbb{R}}$ is uniformly bounded. This means that there exists a constant $\mathbf{G} \geq 1$, such that for any t in the real numbers ($t \in \mathbb{R}$), the norm of the operator $\mathfrak{W}(t)$ in the space of all linear and bounded operators from $\mathfrak{X}$ to itself is bounded by $\mathbf{G}$.
In addition, we define the operator $A : \mathcal{D}(A) \subset \mathfrak{X} \longrightarrow \mathfrak{X}$ as the infinitesimal generator of the cosine family of uniformly bounded linear operators $(\mathfrak{W}(t))_{t\in\mathbb{R}}$ on the space $\mathfrak{X}$ is defined as follows:

$$A\Theta(x,t) = \sum_{h,k=1}^{n}\frac{\partial}{\partial x_d}\left[\sigma_{hk}(x)\frac{\partial\Theta(x,t)}{\partial x_k}\right]+\sigma_0\Theta(x,t), \tag{2}$$

$\forall x \in \Omega, \forall t \in]0,T]$, where the coefficients σ_{hk} are in $C^1(\overline{\Omega})$, with $1 \leq h \leq n$ and $1 \leq k \leq n$, and σ_0 is in $C^1(\overline{\Omega})$, such that:

$$\begin{cases} \sigma_{hk} = \sigma_{kh}, \quad 1 \leq h \leq n,\ 1 \leq k \leq n, \\ \exists B > 0,\ \forall\, \varsigma \in \mathbb{R}^n,\ \displaystyle\sum_{h,k=1}^{n}\sigma_{dk}(x)\varsigma_h\varsigma_k \geq B\|\varsigma\|^2, \end{cases} \tag{3}$$

for $\varsigma = (\varsigma_1, \varsigma_2, \ldots, \varsigma_n) \in \mathbb{R}^n$ and $\|\varsigma\|^2 = \displaystyle\sum_{i=1}^{n}\varsigma_i^2$.

We consider the system (1) augmented by the following output functional:

$$\varpi(t) = \mathfrak{C}\Theta(.,t), \quad \forall t \in]0,T]. \tag{4}$$

Where $\mathfrak{C}$ is a linear, possibly unbounded, operator called the observation operator with dense domain in $\mathfrak{X}$ and range in $\mathfrak{O}$, where $\mathfrak{O}$ is a Hilbert space (observation space).

Next, we give the definition of the mild solution for the system (1).

Definition 5. *[42] For any $t \in]0,T]$ and $1 < \varepsilon < 2$, a function $\Theta(.,.) \in C(]0,T];\mathfrak{X})$, is said to be a mild solution of the system (1) if it satisfies:*

$$\Theta(x,t) = M_q(t)\Theta_0(x) + R_q(t)\Theta_1(x), q = \frac{\varepsilon}{2}. \tag{5}$$

for all $t \in]0,T], x \in \Omega$.

Where

$$M_q(t) = \frac{d}{dt}t^{q-1}P_q(t), \quad t \in]0,T],$$
$$R_q(t) = t^{q-1}P_q(t) = \int_0^t M_q(s)ds, \quad t \in]0,T],$$
$$P_q(t) = \int_0^\infty q\theta S_q(\theta)\mathfrak{W}(t^q\theta)d\theta, \quad t \in]0,T],$$

and

$$S_q(\rho) = \frac{1}{q}\rho^{-1-\frac{1}{q}}\xi_q(\rho^{\frac{-1}{q}}), \quad \rho \in]0,+\infty[,$$
$$\xi_q(\rho) = \frac{1}{\pi}\sum_{k=1}^{\infty}(-1)^{n-1}(\rho)^{-nq-1}\frac{\Gamma(nq+1)}{n!}sin(n\pi q),$$

for all $\rho \in]0,+\infty[$.

Note that $S_q(.)$ is the Mainardi's Wright-type function, which is defined on $]0,+\infty[$ and satisfies:

- $S_q(\rho) \geq 0, \rho \in]0,+\infty[$.
- $\displaystyle\int_0^{+\infty} \rho^n S_q(\rho)d\rho = \frac{\Gamma(n+1)}{\Gamma(1+nq)}$.

The measurement functional (4) can also be written as:

$$\varpi(t) = \mathfrak{I}_\varepsilon(t)\begin{pmatrix}\Theta_0\\ \Theta_1\end{pmatrix}, \quad \forall t \in]0,T], \tag{6}$$

where $\mathfrak{I}_\varepsilon : \mathfrak{X}^2 \longrightarrow L^2(]0,T];\mathfrak{O})$ is called the observability operator, and it's defined by:

$$\mathfrak{I}_\varepsilon(t)\begin{pmatrix}\Theta_0\\ \Theta_1\end{pmatrix} = \mathfrak{C}M_q(t)\Theta_0 + \mathfrak{C}R_q(t)\Theta_1.$$

Note that the operator $\mathfrak{I}_\varepsilon$ is bounded if $\mathfrak{C}$ is bounded, and it plays an important role in the characterization of observability.

The objective is to determine the initial state of a system from the output function (4). To achieve this, the adjoint of $\mathfrak{I}_\varepsilon$ must be calculated which is not always defined when $\mathfrak{C}$ is unbounded. This calculation will later aid in defining and understanding the characteristics of observability. Then, we consider the following definition.

Definition 6. *[43] We say that the operator $\mathfrak{C}$ is an admissible observation operator, respectively, for M_q and R_q, if:*

- *$\exists N_1$, such that:*
$$\forall z \in \mathcal{D}(A), \quad \int_0^T \|\mathfrak{C}M_q(t)z\|_{\mathfrak{O}}^2 dt \leq N_1\|z\|_{\mathfrak{X}}^2,$$
and
- *$\exists N_2$, such that:*
$$\forall z \in \mathcal{D}(A), \quad \int_0^T \|\mathfrak{C}R_q(t)z\|_{\mathfrak{O}}^2 dt \leq N_2\|z\|_{\mathfrak{X}}^2.$$

Remark 1. *The admissibility condition for M_q and R_q is always satisfied in the case where $\mathfrak{C}$ is bounded. Therefore, a bounded observation operator $\mathfrak{C}$ is an admissible one.*

Throughout this paper, we suppose that $\mathfrak{C}$ is an admissible observation operator. Then, the adjoint operator of $\mathfrak{I}_\varepsilon$ is given as follows:

$$\begin{aligned}\mathfrak{I}_\varepsilon{}^* : \quad & L^2(]0,T];\mathfrak{O}) \longrightarrow \mathfrak{X}^2\\ & \mathcal{Y} \longmapsto \begin{pmatrix}\int_0^T M_q^*(t)\mathfrak{C}^*\mathcal{Y}(t)dt\\ \int_0^T R_q^*(t)\mathfrak{C}^*\mathcal{Y}(t)dt\end{pmatrix}.\end{aligned} \tag{7}$$

Indeed:
Let us consider $(y_0, y_1) \in \mathfrak{X}^2$, and $\mathcal{Y} \in L^2(]0,T];\mathfrak{O})$, then:

$$\begin{aligned}&\left\langle \mathfrak{I}_\varepsilon(.)\begin{pmatrix}y_0\\ y_1\end{pmatrix}, \mathcal{Y}\right\rangle_{L^2(]0,T];\mathfrak{O})}\\ &= \int_0^T \left\langle \mathfrak{I}_\varepsilon(t)\begin{pmatrix}y_0\\ y_1\end{pmatrix}, \mathcal{Y}(t)\right\rangle_{\mathfrak{O}} dt\\ &= \int_0^T \langle \mathfrak{C}M_q(t)y_0 + \mathfrak{C}R_p(t)y_1, \mathcal{Y}(t)\rangle_{\mathfrak{O}} dt\\ &= \int_0^T \langle y_0, M_q^*(t)\mathfrak{C}^*\mathcal{Y}(t)\rangle_{\mathfrak{X}} dt\\ &\quad + \int_0^T \langle y_1, R_q^*(t)\mathfrak{C}^*\mathcal{Y}(t)\rangle_{\mathfrak{X}} dt\\ &= \left\langle \begin{pmatrix}y_0\\ y_1\end{pmatrix}, \begin{pmatrix}\int_0^T M_q^*(t)\mathfrak{C}^*\mathcal{Y}(t)dt\\ \int_0^T R_q^*(t)\mathfrak{C}^*\mathcal{Y}(t)dt\end{pmatrix}\right\rangle_{\mathfrak{X}^2}.\end{aligned}$$

We extend the definitions of observability for hyperbolic systems to the fractional case $\varepsilon \in]1,2[$. Consequently, we have the following definition:

Definition 7.
- *The system* (1) *with* (4) *is said to be exactly observable if:*
$$\mathrm{Im}\,(\mathfrak{J}_\varepsilon{}^*) = \mathfrak{X}^2.$$
- *The system* (1) *with* (4) *is said to be approximately observable if:*
$$\overline{\mathrm{Im}\,(\mathfrak{J}_\varepsilon{}^*)} = \mathfrak{X}^2.$$

We now present some useful properties regarding the exact and approximate observablity. This first proposition gives us many characterizations regarding the approximate observability of the system (1).

Proposition 2. *The following properties are equivalent:*

(1) *The system* (1) *with* (4) *is approximately observable .*
(2) $\mathrm{Ker}\,(\mathfrak{J}_\varepsilon) = \{0\}$.
(3) $\mathrm{Ker}\,(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon) = \{0\}$.
(4) $\overline{Im(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon)} = \mathfrak{X}^2$.
(5) $(\mathfrak{J}_\varepsilon\mathfrak{J}_\varepsilon{}^*)$ *is positive definite.*

Proof. We show that $1 \Leftrightarrow 2$, $2 \Leftrightarrow 3$, $3 \Leftrightarrow 4$, and $2 \Leftrightarrow 5$.

- (1)$\Leftrightarrow$(2) We know that the space $\mathfrak{X}^2$ is a reflexive Banach space, then:
$$Im\,(\mathfrak{J}_\varepsilon{}^*) = \mathfrak{X}^2 \Leftrightarrow \left[\overline{Im\,(\mathfrak{J}_\varepsilon{}^*)}\right]^\perp = \left[\mathfrak{X}^2\right]^\perp \Leftrightarrow \mathrm{Ker}\,(\mathfrak{J}_\varepsilon) = \{0\}.$$
- (2)$\Leftrightarrow$ (3)
  i) First, let us show that: $Ker(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon) = \{0\} \Rightarrow Ker(\mathfrak{J}_\varepsilon) = \{0\}$.
  Let us consider $y \in Ker(\mathfrak{J}_\varepsilon)$, then:
$$\begin{aligned}\mathfrak{J}_\varepsilon(t)y = 0 &\Rightarrow \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon(t)y = 0\\ &\Rightarrow y \in Ker(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon)\\ &\Rightarrow y = 0.\end{aligned}$$
  Then, we get: $Ker(\mathfrak{J}_\varepsilon) = \{0\}$.
  ii) Second, let us show that: $Ker(\mathfrak{J}_\varepsilon) = \{0\} \Rightarrow Ker(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon) = \{0\}$.
  Let us consider $y \in Ker(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon)$, we have:
$$\begin{aligned}\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon(.)y, y\rangle_{\mathfrak{X}^2} &= \langle \mathfrak{J}_\varepsilon(.)y, \mathfrak{J}_\varepsilon(.)y\rangle_{L^2(]0,T];\mathfrak{O})}\\ &= \|\mathfrak{J}_\varepsilon(.)y\|^2_{L^2(]0.T];\mathfrak{O})}\cdot\end{aligned}$$
  We have $\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon(.)y = 0$, then,
  $\mathfrak{J}_\varepsilon(.)y = 0$, which implies that $y \in Ker(\mathfrak{J}_\varepsilon)$. Thus, $y = 0$.
  Consequently $Ker(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon) = \{0\}$.
- (3) $\Leftrightarrow$ (4) This is a direct consequence from the fact that:
$$\mathrm{Ker}\,(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon) = \left[\overline{Im\,(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon)}\right]^\perp$$
- (5)$\Leftrightarrow$ (2) If $\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon$ is positive definite. Then:
$$\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon u, u\rangle \geq 0, \quad \forall u \in \mathfrak{X}^2,$$
  and
$$\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon u, u\rangle = 0 \Rightarrow u = 0, \quad \forall u \in \mathfrak{X}^2\cdot$$
  We have
$$\begin{aligned}\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon u, u\rangle_{\mathfrak{X}} &= \langle \mathfrak{J}_\varepsilon{}^*u, \mathfrak{J}_\varepsilon{}^*u\rangle_{L^2(]0,T];\mathfrak{O})}\\ &= \|\mathfrak{J}_\varepsilon(.)u\|^2_{L^2(0,T;\mathfrak{O})} \geq 0, \forall u \in \mathfrak{X}^2.\end{aligned}$$
  On the other hand
$$\begin{aligned}&[\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon u, u\rangle_{\mathfrak{X}} = 0 \Longrightarrow u = 0]\\ \Leftrightarrow &\left[\|\mathfrak{J}_\varepsilon(.)u\|^2_{L^2(]0,T];\mathfrak{O})} = 0 \Longrightarrow u = 0\right]\\ \Leftrightarrow &\left[\mathfrak{J}_\varepsilon u = 0 \Longrightarrow u = 0\right]\\ \Leftrightarrow &\quad Ker(\mathfrak{J}_\varepsilon) = \{0\}\cdot\end{aligned}$$

This completes the proof. □

In this next proposition, we shed light on some characterizations regarding the exact observability of the systeme (1).

Proposition 3. *The mentioned statements are equivalent:*

(1) *The system* (1) *with* (4) *is exactly observable.*
(2) $\exists M > 0$, *such that:*
$$\|z\|_{\mathfrak{X}^2} \leq M\|\mathfrak{J}_\varepsilon(.)z\|_{L^2(]0,T],\mathfrak{O})}, \quad \forall z \in \mathfrak{X}^2.$$
(3) *The operator* $\mathfrak{J}_\varepsilon\mathfrak{J}_\varepsilon{}^*$ *is coercive .*
(4) $Im(\mathfrak{J}_\varepsilon{}^*)$ *is closed and* $Ker(\mathfrak{J}_\varepsilon) = \{0\}$.

Before proving the main proposition, we recall the following lemma:

Lemma 1. *[44] Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be three reflexive Banach spaces. Let us consider $\mathcal{N} \in \mathcal{L}(\mathcal{F};\mathcal{H})$ and $\mathcal{T} \in \mathcal{L}(\mathcal{G};\mathcal{H})$. The mentioned statements are equivalent:*

(1) $Im(\mathcal{N}) \subset Im(\mathcal{T})$,
(2) $\exists M > 0$, *such that:*
$$\|\mathcal{N}^*z^*\|_{\mathcal{F}^*} \leq M\|\mathcal{T}^*z^*\|_{\mathcal{G}^*}, \quad \forall z^* \in \mathcal{H}^*\cdot$$

Now, we give the proof of the proposition (3).

Proof. We show that $1 \Leftrightarrow 2$, $2 \Leftrightarrow 3$ and $1 \Leftrightarrow 4$.

- $1 \Leftrightarrow 2$ This is a direct application of Lemma (1) with:
  - $\mathcal{F} = \mathcal{H} = \mathfrak{X}^2$.
  - $\mathcal{G} = L^2(]0,T];\mathfrak{O})$.
  - $\mathcal{N} = Id_{\mathfrak{X}^2}$.
  - $\mathcal{T} = \mathfrak{J}_\varepsilon{}^*$.

  Where $Id_{\mathfrak{X}^2}$ is the identity function of $\mathfrak{X}^2$.
- $3 \Leftrightarrow 2$ If $\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon$ is coercive, then there exists $M > 0$ such that $\forall (z_0, z_1) \in \mathfrak{X}^2$
$$\left\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\rangle_{\mathfrak{X}^2} \geq M \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|^2_{\mathfrak{X}^2}$$

$$\Leftrightarrow \left\| \mathfrak{J}_\varepsilon(.) \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|^2_{L^2(]0,T];\mathfrak{O})} \geq M \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|^2_{\mathfrak{X}^2}$$

$$\Leftrightarrow \left\| \mathfrak{J}_\varepsilon(.) \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|_{L^2(]0,T];\mathfrak{O})} \geq \sqrt{M} \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|_{\mathfrak{X}^2}.$$

- $1 \Leftrightarrow 4$
  (1) $1 \Rightarrow 4$
  Since system (1) with (4) is exactly observable it also approximately observable . Then,
  $$\operatorname{Im}(\mathfrak{J}_\varepsilon{}^*) = \mathfrak{X}^2 \textit{ and } \overline{\operatorname{Im}(\mathfrak{J}_\varepsilon{}^*)} = \mathfrak{X}^2.$$
  Then $Im(\mathfrak{J}_\varepsilon{}^*)$ is closed. By using the proposition (2) we get $\operatorname{Ker}(\mathfrak{J}_\varepsilon) = \{0\}$.
  Thus, $Im(\mathfrak{J}_\varepsilon{}^*)$ is closed and $\operatorname{Ker}(\mathfrak{J}_\varepsilon) = \{0\}$.
  (2) $4 \Rightarrow 1$
  We have $\operatorname{Ker}(\mathfrak{J}_\varepsilon) = \{0\}$, Hence, by using the proposition (2), we get $\overline{\operatorname{Im}(\mathfrak{J}_\varepsilon{}^*)} = \mathfrak{X}^2$, this, together with the fact that $Im(\mathfrak{J}_\varepsilon{}^*)$ is closed, we obtain $\operatorname{Im}(\mathfrak{J}_\varepsilon{}^*) = \mathfrak{X}^2$.

This completes the proof. □

We give the following proposition.

Proposition 4. *If the system* (1) *with* (4) *exactly observable then the application* $(\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon)^{-1}$ *is continuous .*

Proof. We know that a linear operator $\mathfrak{B}$ is invertible and of continuous inverse if and only if:

$$\exists m > 0, \textit{ such that } m\|z\| \leq \|\mathfrak{B}z\|, \tag{8}$$

see [44].

If the system (1) with (4) exactly observable, then:

$\exists m > 0,$ *such that*, $\forall (z_0, z_1) \in \mathfrak{X}^2$,

$$\left\langle \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}, \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\rangle_{\mathfrak{X}^2} \geq m \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|^2_{\mathfrak{X}^2}$$

$\Longrightarrow \exists m > 0,$ *such that*, $\forall (z_0, z_1) \in \mathfrak{X}^2$,

$$m \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|^2_{\mathfrak{X}^2} \leq \left\| \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon(.) \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|_{L^2(0,T;\mathfrak{O})} \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|_{\mathfrak{X}^2}$$

$\Longrightarrow \exists m > 0,$ *such that*, $\forall (z_0, z_1) \in \mathfrak{X}^2$,

$$m \left\| \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|_{\mathfrak{X}^2} \leq \left\| \mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon(.) \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} \right\|_{L^2(0,T;\mathfrak{O})}$$

$\Longrightarrow (\mathfrak{J}_\varepsilon{}^*\mathfrak{J}_\varepsilon)^{-1}$ *is continus.*

This completes the proof. □

By the conditions 3, it is well known that A is symmetric and $-A$ is uniformly elliptic. In this case, it is well known that $-A$ has a set of eigenvalues $(\lambda_j)_{j\geq 1}$, such that:

$0 < \lambda_1 < \lambda_2 < \lambda_3 < \lambda_4 < \cdots \lambda_{j+1} < \cdots \to +\infty$. Each eigenvalue λ_j corresponds with r_j eigenfunctions $\{\varphi_{jk}\}_{1\leq k\leq r_j}$, where $r_j \in \mathbb{N}^*$ is the multiplicity of λ_j, such that $A\varphi_{jk} = \lambda_j\varphi_{jk}$ and $\varphi_{jk} \in \mathcal{D}(A), \forall j \in \mathbb{N}^*$ and $1 \leq k \leq r_j$. In addition, the set $\{\varphi_{jk}\}_{\substack{j\geq 1 \\ 1\leq r_j\leq j}}$ form an orthogonal basis of $\mathfrak{X}$, [45].

We give the following proposition.

Proposition 5. *Let us consider* $\Theta_0, \Theta_1 \in \mathfrak{X}$ *and* $t \in]0, T]$, *we have:*

1. $$R_q(t)\Theta_1(x) = \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j} t^{2q-1}E_{2q,2q}\left(\lambda_j t^{2q}\right) \times \langle \Theta_1, \varphi_{jk}\rangle_{\mathfrak{X}}\, \varphi_{jk}(x).$$
2. $$M_q(t)\Theta_0(x) = \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j} t^{2q-2}E_{2q,2q-1}(\lambda_j t^{2q}) \times \langle \Theta_0, \varphi_{jk}\rangle_{\mathfrak{X}}\, \varphi_{jk}(x).$$

Proof. Let us consider $\Theta_0, \Theta_1 \in \mathfrak{X}$ and $t \in]0, T]$. From (1), we can write the cosine family $(\mathfrak{W}(t))_{t\geq 0}$ in the following form:

$$\mathfrak{W}(t)\Theta_0(x) = \sum_{n=0}^{+\infty}\sum_{j=1}^{+\infty}\sum_{k=1}^{r_j} \frac{t^{2n}\lambda_j^n}{2n!} \langle \Theta_0, \varphi_{jk}\rangle_{\mathfrak{X}}\, \varphi_{jk}(x).$$

Thus,

1. $R_q(t)\Theta_1(x)$
$$\begin{aligned}
&= t^{q-1}\int_0^\infty q\theta S_q(\theta)\mathfrak{U}(t^q\theta)\Theta_1(x)d\theta \\
&= t^{q-1}\int_0^\infty q\theta S_q(\theta)\int_0^{t^q\theta}\mathfrak{W}(s)\Theta_1(x)dsd\theta \\
&= t^{q-1}\int_0^\infty \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{+\infty}\frac{(t^q\theta)^{2n+1}\lambda_j^n}{2n!(2n+1)} \\
&\quad \times q\theta S_q(\theta)\langle \Theta_1, \varphi_{jk}\rangle_{\mathfrak{X}}\, \varphi_{jk}(x)d\theta \\
&= t^{q-1}\sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{+\infty}\langle \Theta_1, \varphi_{jk}\rangle_{\mathfrak{X}}\, \varphi_{jk}(x) \\
&\quad \times \int_0^\infty q\theta S_q(\theta)\frac{\lambda_j^n(t^q\theta)^{2n+1}}{2n!(2n+1)}d\theta \\
&= t^{q-1}\sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{+\infty}\langle \Theta_1, \varphi_{jk}\rangle_{\mathfrak{X}}\, \varphi_{jk}(x) \\
&\quad \times \frac{q\lambda_j^n(t^q)^{2n+1}}{(2n+1)!}\int_0^\infty S_q(\theta)\theta^{2n+2}d\theta
\end{aligned}$$

$$= t^{q-1}\sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\sum_{n=0}^{+\infty}\langle\Theta_1,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x)$$
$$\times\frac{q\lambda_j^n(t^q)^{2n+1}}{(2n+1)!}\times\frac{\Gamma(1+2n+2)}{\Gamma(1+q(2n+2))}$$
$$= t^{2q-1}\sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\langle\Theta_1,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x)$$
$$\times\sum_{n=0}^{+\infty}\frac{(\lambda_j t^{2q})^n}{\Gamma(2q+2qn)}$$
$$= \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}t^{2q-1}E_{2q,2q}\left(\lambda_j t^{2q}\right)$$
$$\times\langle\Theta_1,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x).$$

And,

2. $$M_q(t)\Theta_0(x) = \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\langle\Theta_0,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x)$$
$$\times\frac{d}{dt}t^{2q-1}E_{2q,2q}\left(\lambda_j t^{2q}\right)$$
$$= \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\langle\Theta_0,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x)$$
$$\times\sum_{n=0}^{+\infty}\frac{(2qn+2q-1)\lambda_j^n t^{2qn+2q-2}}{\Gamma(2q+2qn)}$$
$$= \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}\langle\Theta_0,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x)$$
$$\times\sum_{n=0}^{+\infty}\frac{\lambda_j^n t^{2qn+2q-2}}{\Gamma(2qn+2q-1)}$$
$$= \sum_{j=1}^{+\infty}\sum_{k=1}^{r_j}t^{2q-2}E_{2q,2q-1}(\lambda_j t^{2q})$$
$$\times\langle\Theta_0,\varphi_{jk}\rangle_{\mathfrak{X}}\varphi_{jk}(x).$$

This completes the proof. □

3. The steps of HUM approach

In this section, we give an approach that allows the reconstruction of the initial state. First, we will give a new version of Green's fractional formula, which is of major importance in the field of control theory [29].

Proposition 6. *(Fractional Green's formula of order $1<\alpha<2$.)*

For any $\psi\in C^\infty\overline{(\Omega\times[0,T])}$, we have:

$$\int_0^T\int_\Omega\left({}^{RL}D_{0^+}^\alpha\Theta(x,t)+A\Theta(x,t)\right)\psi(x,t)dxdt$$
$$=\int_0^T\int_\Omega({}^{C}D_{T^-}^\alpha\psi(x,t)+A^*\psi(x,t))\Theta(x,t)dxdt$$
$$-\int_\Omega I_{0^+}^{2-\alpha}\Theta(x,T)\frac{\partial}{\partial t}\psi(x,T)dx$$
$$+\int_\Omega\frac{\partial}{\partial t}I_{0^+}^{2-\alpha}\Theta(x,T)\psi(x,T)dx$$
$$+\int_\Omega\lim_{t\to0^+}I_{0^+}^{2-\alpha}\Theta(x,t)\frac{\partial}{\partial t}\psi(x,0)dx$$
$$-\int_\Omega\lim_{t\to0^+}\frac{\partial}{\partial t}I_{0^+}^{2-\alpha}\Theta(x,t)\psi(x,0)dx$$
$$+\int_0^T\int_{\partial\Omega}\left(\Theta(\xi,t)\frac{\partial\psi(\xi,t)}{\partial\nu_{A^*}}-\psi(\xi,t)\frac{\partial\Theta(\xi,t)}{\partial\nu_A}\right)d\xi dt.$$

Proof. For any $\psi\in C^\infty\overline{(\Omega\times[0,T])}$, we have:

$$\int_0^T\int_\Omega\left({}^{RL}\mathcal{D}_{0^+}^\alpha\Theta(x,t)+A\Theta(x,t)\right)\psi(x,t)dxdt$$
$$=\int_0^T\int_\Omega\left({}^{RL}\mathcal{D}_{0^+}^\alpha\Theta(x,t)\right)\psi(x,t)dxdt$$
$$+\int_0^T\int_\Omega\left(A\Theta(x,t)\right)\psi(x,t)dxdt, \tag{9}$$

Now, we recall the two useful properties. The first one is the fractional integration by parts formula, see [46],

$$\int_0^T{}^{RL}\mathcal{D}_{0^+}^\alpha\Theta(t)\psi(t)dt=\int_0^T{}^{C}\mathcal{D}_{T^-}^\alpha\psi(t)\Theta(t)dt$$
$$-\left[\mathcal{I}_{0^+}^{2-\alpha}\Theta(t)\frac{d}{dt}\psi(t)-\frac{d}{dt}\mathcal{I}_{0^+}^{2-\alpha}\Theta(t)\psi(t)\right]_0^T$$
$$=\int_0^T{}^{C}\mathcal{D}_{T^-}^\alpha\psi(t))\Theta(t)dt-\mathcal{I}_{0^+}^{2-\alpha}\Theta(T)\frac{d}{dt}\psi(T)$$
$$+\frac{d}{dt}\mathcal{I}_{0^+}^{2-\alpha}\Theta(T)\psi(T)+\lim_{t\to0}\mathcal{I}_{0^+}^{2-\alpha}\Theta(t)\frac{d}{dt}\psi(0)$$
$$-\lim_{t\to0}\frac{d}{dt}\mathcal{I}_{0^+}^{2-\alpha}\Theta(t)\psi(0). \tag{10}$$

The second property is,

$$\int_\Omega A\Theta(x,t)\psi(x,t)dx=\int_\Omega\Theta(x,t)A^*\psi(x,t)dx$$
$$+\int_{\partial\Omega}\Theta(\xi,t)\frac{\partial\psi(\xi,t)}{\partial\nu_{A^*}}d\xi-\int_{\partial\Omega}\psi(\xi,t)\frac{\partial\Theta(\xi,t)}{\partial\nu_A}d\xi, \tag{11}$$

and it can be found in [47].

Using (10) and (11), we obtain:

$$\begin{aligned}
&\int_0^T\int_\Omega \left({}^{RL}\mathcal{D}^\alpha_{0^+}\Theta(x,t)+A\Theta(x,t)\right)\psi(x,t)dxdt\\
&=\int_0^T\int_\Omega ({}^{C}\mathcal{D}^\alpha_{T^-}\psi(x,t)+A^*\psi(x,t))\Theta(x,t)dxdt\\
&+\int_0^T\int_{\partial\Omega}\left(\Theta(\xi,t)\frac{\partial\psi(\xi,t)}{\partial\nu_{A^*}}-\psi(\xi,t)\frac{\partial\Theta(\xi,t)}{\partial\nu_A}\right)d\xi dt\\
&\quad-\int_\Omega \lim_{t\to T}\mathcal{I}^{2-\alpha}_{0^+}\Theta(x,t)\frac{\partial}{\partial t}\psi(x,t)dx\\
&\quad+\int_\Omega \lim_{t\to T}\frac{\partial}{\partial t}\mathcal{I}^{2-\alpha}_{0^+}\Theta(x,t)\psi(x,t)dx\\
&\quad+\int_\Omega \lim_{t\to 0^+}\mathcal{I}^{2-\alpha}_{0^+}\Theta(x,t)\frac{\partial}{\partial t}\psi(x,t)dx\\
&\quad-\int_\Omega \lim_{t\to 0^+}\frac{\partial}{\partial t}\mathcal{I}^{2-\alpha}_{0^+}\Theta(x,t)\psi(x,t)dx.
\end{aligned}$$

Thus, we finally get that:

$$\begin{aligned}
&\int_0^T\int_\Omega \left({}^{RL}D^\alpha_{0^+}\Theta(x,t)+A\Theta(x,t)\right)\psi(x,t)dxdt\\
&=\int_0^T\int_\Omega ({}^{C}D^\alpha_{T^-}\psi(x,t)+A^*\psi(x,t))\Theta(x,t)dxdt\\
&-\int_\Omega I^{2-\alpha}_{0^+}\Theta(x,T)\frac{\partial}{\partial t}\psi(x,T)dx\\
&+\int_\Omega \frac{\partial}{\partial t}I^{2-\alpha}_{0^+}\Theta(x,T)\psi(x,T)dx\\
&+\int_\Omega \lim_{t\to 0^+}I^{2-\alpha}_{0^+}\Theta(x,t)\frac{\partial}{\partial t}\psi(x,0)dx\\
&-\int_\Omega \lim_{t\to 0^+}\frac{\partial}{\partial t}I^{2-\alpha}_{0^+}\Theta(x,t)\psi(x,0)dx\\
&+\int_0^T\int_{\partial\Omega}\left(\Theta(\xi,t)\frac{\partial\psi(\xi,t)}{\partial\nu_{A^*}}-\psi(\xi,t)\frac{\partial\Theta(\xi,t)}{\partial\nu_A}\right)d\xi dt.
\end{aligned}$$

This completes the proof. □

Let us now present the steps of the method that we use in order to reconstruct the initial state. This method is an extension of the Hilbert uniqueness method (HUM) presented by Lions in [29].

For any $(\beta_0,\beta_1)\in\mathfrak{X}\times\mathfrak{X}$. Consider the system:

$$\begin{cases}{}^{RL}\mathcal{D}^\varepsilon_{0^+}\beta(x,t)=A\beta(x,t) & in\ Q,\\ \lim\limits_{t\to0^+}\mathcal{I}^{2-\varepsilon}_{0^+}\beta(x,t)=\beta_0(x) & in\ \Omega,\\ \lim\limits_{t\to0^+}\dfrac{d}{dt}\mathcal{I}^{2-\varepsilon}_{0^+}\beta(x,t)=\beta_1(x) & in\ \Omega,\\ \beta(\xi,t)=0 & on\ \Sigma,\end{cases}\tag{12}$$

which has a unique mild solution:

$$\varphi(x,t)=M_q(t)\beta_0(x)+R_q(t)\beta_1(x).\tag{13}$$

We introduce the semi-norm on $\mathfrak{X}^2$:

$$\begin{aligned}\|\cdot\|_{\mathcal{S}_{\rm II}}:\quad &\mathfrak{X}^2\longrightarrow\mathbb{R}^+\\ (\beta_0,\beta_1)\quad&\longmapsto\|(\beta_0,\beta_1)\|_{\mathcal{S}_{\rm II}}=\sqrt{\int_0^T\|\mathfrak{C}\beta(x,t)\|^2_{\mathbb{O}}dt}.\end{aligned}\tag{14}$$

We then have the following results:

Lemma 2. *If the system* (12) *together with the output equation* (4) *is approximately observable then the semi-norm* $\|\cdot\|_{\mathcal{S}_{\rm II}}$ *becomes a norm .*

Proof. We need to show that: $\forall\,(\beta_0,\beta_1)\in\mathfrak{X}^2$, we have:

$$\|\,(\beta_0,\beta_1)\,\|_{\mathcal{S}_{\rm II}}=0\ \Rightarrow\ \beta_0=\beta_1=0.$$

Let us consider $(\beta_0,\beta_1)\in\mathfrak{X}^2$, then:

$$\begin{aligned}\|(\beta_0,\beta_1)\|^2_{\mathcal{S}_{\rm II}}=0\Leftrightarrow&\int_0^T\|\mathfrak{C}\beta(x,t)\|^2_{\mathfrak{O}}dt=0\\ \Leftrightarrow&\quad\|\mathfrak{C}\beta(x,t)\|_{\mathfrak{O}}=0,\quad\forall t\in]0,T]\\ \Leftrightarrow&\ \mathfrak{C}\beta(x,t)=0,\quad\forall t\in]0,T]\\ \Leftrightarrow&\ \mathfrak{I}_\varepsilon(t)\begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}=0,\quad\forall t\in]0,T]\\ \Leftrightarrow&\ \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}\in Ker(\mathfrak{I}_\varepsilon),\end{aligned}$$

using the fact that (1) with (4) is approximately observable, we get that:

$$\beta_0=\beta_1=0.$$

This completes the proof. □

We define the retrograded system of (12) as follows:

$$\begin{cases}{}^{C}\mathcal{D}^\varepsilon_{T^-}\psi(x,t)=A^*\psi(x,t)-\mathfrak{C}^*\mathfrak{C}\beta(x,t) & inQ,\\ \psi(x,T)=0,\qquad \dfrac{\partial}{\partial t}\psi(x,T)=0 & in\ \Omega,\\ \psi(\xi,t)=0 & on\ \Sigma,\end{cases}\tag{15}$$

which has a unique mild solution in $C\left(]0,T];\mathfrak{X}\right)$, see: [48].

$$\psi(x,t)=-\int_t^T(\tau-t)^{q-1}P^*_q(\tau-t)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau,\tag{16}$$

where

$$P^*_q(t)=\int_0^\infty qyS_q(y)\mathfrak{U}^*(t^qy)dy.$$

We now present a new alternative result that plays an important role in solving the reconstruction problem.

Proposition 7. *The mild solution ψ of system* (15) *satisfies:*

$$\begin{aligned} &1. \quad \psi(x,0) = -\int_0^T R_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau. \\ &2. \quad \frac{\partial}{\partial t}\psi(x,0) = \int_0^T M_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau. \end{aligned} \tag{17}$$

Proof. 1) Let ψ a mild solution of system (15), then:

$$\psi(x,t) = -\int_t^T (\tau-t)^{q-1}P_q^*(\tau-t)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau.$$

Thus, if $t=0$, we obtain:

$$\begin{aligned} \psi(x,0) &= -\int_0^T (\tau)^{q-1}P_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau, \\ &= -\int_0^T R_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau. \end{aligned}$$

2) Using equation (16), we obtain:

$$\begin{aligned} &\frac{\partial}{\partial t}\psi(x,t) \\ &= \frac{\partial}{\partial t}\left[-\int_t^T (\tau-t)^{q-1}P_q^*(\tau-t)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau\right] \\ &= \left[-\int_t^T \frac{\partial}{\partial t}((\tau-t)^{q-1}P_q^*(\tau-t))\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau\right] \\ &\quad - \lim_{\tau\longrightarrow T} R_q(\tau-t)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)\frac{\partial}{\partial t}(T) \\ &\quad + \lim_{\tau\longrightarrow t} R_q(\tau-t)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)\frac{\partial}{\partial t}(t) \\ &= \int_t^T M_q^*(\tau-t))\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau, \end{aligned}$$

thus, if $t=0$, we obtain:

$$\frac{\partial}{\partial t}\psi(x,0) = \int_0^T M_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(x,\tau)d\tau.$$

This completes the proof. □

If (β_0,β_1) is chosen such that $C\beta(.,t) = \varpi(t)$ in Ω, then, by using the fractional Green's formula, the following system can be seen as the adjoint system of(1) with (4).

$$\begin{cases} {}^{C}\mathcal{D}^{\varepsilon}_{T^-}\Upsilon(x,t) = A^*\Upsilon(x,t) - \mathfrak{C}^*\varpi(.,t) & \text{in } Q, \\ \Upsilon(x,T) = 0, \quad \dfrac{\partial}{\partial t}\Upsilon(x,T) = 0 & \text{in } \Omega, \\ \Upsilon(\xi,t) = 0 & \text{on } \Sigma. \end{cases} \tag{18}$$

We define the mapping:

$$\begin{aligned} \Lambda: \quad & \mathfrak{X}^2 \longrightarrow \mathfrak{X}^2 \\ & (\beta_0,\beta_1) \longrightarrow \Lambda(\beta_0,\beta_1) = \left(\frac{\partial}{\partial t}\psi(x,0), -\psi(x,0)\right). \end{aligned} \tag{19}$$

Then, the problem of reconstruction is reduced to solve the following equation:

$$\Lambda(\beta_0,\beta_1) = \left(\frac{\partial}{\partial t}\Upsilon(x,0), -\Upsilon(x,0)\right). \tag{20}$$

We then have the following theorem.

Theorem 1. *If the system* (12) *is approximately observable, then the equation* (20) *has a unique solution which corresponds with the initial state in* Ω.

Proof. We need to prove that Λ is coercive: (i.e) $\exists\, C>0$, such that

$$\langle \Lambda(\beta_0,\beta_1), (\beta_0,\beta_1)\rangle_{\mathfrak{X}^2} \geq C \ \ \|(\beta_0,\beta_1)\|^2_{\mathfrak{X}^2}. \tag{21}$$

Then:

$$\begin{aligned} &\langle \Lambda(\beta_0,\beta_1), (\beta_0,\beta_1)\rangle_{\mathfrak{X}^2} \\ &= \left\langle \left(\frac{\partial}{\partial t}\Upsilon(x,0), -\Upsilon(x,0)\right), (\beta_0,\beta_1)\right\rangle_{\mathfrak{X}^2} \\ &= \left\langle \int_0^T M_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(\tau)d\tau, \beta_0\right\rangle_{\mathfrak{X}} \\ &\quad + \left\langle \int_0^T R_q^*(\tau)\mathfrak{C}^*\mathfrak{C}\beta(\tau)d\tau, \beta_1\right\rangle_{\mathfrak{X}} \\ &= \int_0^T \langle \mathfrak{C}\beta(\tau), \mathfrak{C}\beta(\tau)\rangle_{\mathfrak{D}}\, d\tau \\ &= \|(\beta_0,\beta_1)\|^2_{\mathfrak{X}^2}. \end{aligned}$$

This completes the proof. □

4. Numerical approach

In this section, we present an approach that gives the initial state in cases where pointwise and zonal sensors are used. Let Ω be an open bounded subset on $\mathbb{R}^n$, we consider the abstract time-fractional system:

$$\begin{cases} {}^{RL}\mathcal{D}^{\varepsilon}_{0^+}\beta(x,t) = A\beta(x,t) & \text{in } Q, \\ \lim\limits_{t\to 0^+} \mathcal{I}^{2-\varepsilon}_{0^+}\beta(x,t) = \beta_0(x) & \text{in } \Omega, \\ \lim\limits_{t\to 0^+} \dfrac{\partial}{\partial t}\mathcal{I}^{2-\varepsilon}_{0^+}\beta(x,t) = \beta_1(x) & \text{in } \Omega, \\ \beta(\xi,t) = 0 & \text{on } \Sigma, \end{cases} \tag{22}$$

for simplicity, we can safely assume that the eigenvalues of $-A$ are of multiplicity equal to 1, eventhough this is not always the case. The reason behind this consideration is to work with a single iterator in the index of the eigenfunctions in order to simplify the mathematical expressions, which is always possible since we can always find a possible rearrangement of the eigenvalues and eigenfunctions which makes this possible. Note that in the new arrangement, many eigenvalues will have the same value. We denote the eigenvalues of $-A$ by λ_j, for every $j \in \mathbb{N}^*$, and the corresponding normalized eigenfunctions by $\varphi_j(x)$, for every $j \in \mathbb{N}^*$.

Hence, the system (22) has a unique mild solution given by the following formula:

$$\beta(x,t) = \sum_{j=1}^{+\infty} t^{2q-2} \langle \beta_0(.), \varphi_j \rangle_{\mathfrak{X}} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) + t^{2q-1} \langle \beta_1(.), \varphi_j \rangle_{\mathfrak{X}} E_{2q,2q}\left(\lambda_j t^{2q}\right) \varphi_j(x).$$

The adjoint system of (22) is written as:

$$\begin{cases} {}^{C}\mathcal{D}^{\varepsilon}_{T^-} \Upsilon(x,t) = A\Upsilon(x,t) - \mathfrak{C}^*\varpi(t) & in\ Q, \\ \Upsilon(x,T) = 0, \qquad \dfrac{\partial}{\partial t}\Upsilon(x,T) = 0 & in\ \Omega, \\ \Upsilon(\xi,t) = 0 & on\ \Sigma. \end{cases} \tag{23}$$

Hence, by using (17) and proposition (5) , we get:

1. $$-\Upsilon(x,0) = \int_0^T R_q(t)\mathfrak{C}^*\varpi(t)dt = \int_0^T \sum_{j=1}^{+\infty} t^{2q-1} E_{2q,2q}\left(\lambda_j t^{2q}\right) \times \langle \mathfrak{C}^*\varpi(t), \varphi_j \rangle_{\mathfrak{X}} \varphi_j(x)dt,$$
2. $$\frac{\partial}{\partial t}\Upsilon(x,0) = \int_0^T M_q(t)\mathfrak{C}^*\varpi(t)dt = \int_0^T \sum_{j=1}^{+\infty} t^{2q-2} E_{2q,2q-1}(\lambda_j t^{2q}) \times \langle \mathfrak{C}^*\varpi(t), \varphi_j \rangle_{\mathfrak{X}} \varphi_j(x)dt.$$

For any $(\beta_0, \beta_1) \in \mathfrak{X}^2$, we also have:

$$\langle \Lambda(\beta_0,\beta_1), (\beta_0,\beta_1) \rangle_{\mathfrak{X}^2} = \int_0^T \|\mathfrak{C}\beta(x,t)\|^2_{\mathfrak{O}} dt, \tag{24}$$

by using the expression of $\beta(.,.)$, we obtain:

$$\begin{aligned} \int_0^T \|\mathfrak{C}\beta(x,t)\|^2_{\mathfrak{O}} dt = & \\ \sum_{j,i=1}^{+\infty} \int_0^T & t^{4q-4} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \\ & \times \mathfrak{C}\varphi_j(x)\mathfrak{C}\varphi_i(x) \langle \beta_0, \varphi_j \rangle_{\mathfrak{X}} \langle \beta_0, \varphi_i \rangle_{\mathfrak{X}} \\ + \int_0^T & t^{4q-3} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \mathfrak{C}\varphi_j(x) \\ & \times \mathfrak{C}\varphi_i(x) \langle \beta_1, \varphi_j \rangle_{\mathfrak{X}} \langle \beta_0, \varphi_i \rangle_{\mathfrak{X}} \\ + \int_0^T & t^{4q-3} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \mathfrak{C}\varphi_j(x) \\ & \times \mathfrak{C}\varphi_i(x) \langle \beta_0, \varphi_j \rangle_{\mathfrak{X}} \langle \beta_1, \varphi_i \rangle_{\mathfrak{X}} \\ + \int_0^T & t^{4q-2} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \mathfrak{C}\varphi_j(x) \\ & \times \mathfrak{C}\varphi_i(x) \langle \beta_1, \varphi_j \rangle_{\mathfrak{X}} \langle \beta_1, \varphi_i \rangle_{\mathfrak{X}}. \end{aligned} \tag{25}$$

Then, the problem (24) can be approached by the linear systems:

$$\sum_{j=1}^{N} \begin{pmatrix} A_{i,j} & B_{i,j} \\ C_{i,j} & D_{i,j} \end{pmatrix} \begin{pmatrix} \beta_0^j \\ \beta_1^j \end{pmatrix} = \begin{pmatrix} \Upsilon_1^i(x,0) \\ \Upsilon_0^i(x,0) \end{pmatrix}, \tag{26}$$

for i=1,2,3,. . .,N . Where:

- $A_{i,j} = \displaystyle\int_0^T t^{4q-4} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \times \mathfrak{C}\varphi_j(x)\mathfrak{C}\varphi_i(x).$
- $B_{i,j} = \displaystyle\int_0^T t^{4q-3} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \times \mathfrak{C}\varphi_j(x)\mathfrak{C}\varphi_i(x).$
- $C_{i,j} = \displaystyle\int_0^T t^{4q-3} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \times \mathfrak{C}\varphi_j(x)\mathfrak{C}\varphi_i(x).$
- $D_{i,j} = \displaystyle\int_0^T t^{4q-2} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \times \mathfrak{C}\varphi_j(x)\mathfrak{C}\varphi_i(x).$
- $\beta_0^j = \langle \beta_0, \varphi_j \rangle_{\mathfrak{X}}$.
- $\beta_1^j = \langle \beta_1, \varphi_j \rangle_{\mathfrak{X}}$.
- $\Upsilon_0^i(x,0) = \langle -\Upsilon(.,0), \varphi_i \rangle_{\mathfrak{X}}$.
- $\Upsilon_1^i(x,0) = \left\langle \dfrac{\partial}{\partial t}\Upsilon(.,0), \varphi_i \right\rangle_{\mathfrak{X}}$.

In the case where the output function is given by a pointwise sensor $\varpi(t) = \beta(b,t)$, where $b \in \Omega$. Then, we get:

- $\Upsilon(x,0) = -\displaystyle\sum_{j=1}^{+\infty} \int_0^T t^{2q-1} E_{2q,2q}\left(\lambda_j t^{2q}\right) \beta(b,t)dt \times \varphi_j(b)\varphi_j(x).$
- $\dfrac{\partial}{\partial t}\Upsilon(x,0) = \displaystyle\sum_{j=1}^{+\infty} \int_0^T t^{2q-2} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) \beta(b,t)dt \times \varphi_j(b)\varphi_j(x).$

And

- $A_{i,j} = \displaystyle\int_0^T t^{4q-4} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \times \varphi_j(b)\varphi_i(b).$
- $B_{i,j} = \displaystyle\int_0^T t^{4q-3} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \times \varphi_j(b)\varphi_i(b).$
- $C_{i,j} = \displaystyle\int_0^T t^{4q-3} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \times \varphi_j(b)\varphi_i(b).$
- $D_{i,j} = \displaystyle\int_0^T t^{4q-2} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \times \varphi_j(b)\varphi_i(b).$

In the case where the output function is given by a zonal sensor (g, D), where $g(x) = \chi_D(x)$ is the spatial distribution of the sensor and D is the spatial domain of g. Then, we get:

- $\Upsilon(x,0) = -\sum_{j=1}^{+\infty} \int_0^T t^{2q-1} E_{2q,2q}\left(\lambda_j t^{2q}\right) \varpi(t) dt \times \langle g(.), \varphi_j(.)\rangle_{L^2(D)} \varphi_j(x).$
- $\frac{\partial}{\partial t}\Upsilon(x,0) = \sum_{j=1}^{+\infty} \int_0^T t^{2q-2} E_{2q,2q-1}(\lambda_j t^{2q}) \varpi(t) dt \times \langle g(.), \varphi_j\rangle_{L^2(D)} \varphi_j(x).$

Where:

- $A_{i,j} = \int_0^T t^{4q-4} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \times \langle g, \varphi_j(.)\rangle_{L^2(D)} \langle g, \varphi_i(.)\rangle_{L^2(D)}.$
- $B_{i,j} = \int_0^T t^{4q-3} E_{2q,2q-1}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \times \langle g, \varphi_j(.)\rangle_{L^2(D)} \langle g, \varphi_i(.)\rangle_{L^2(D)}.$
- $C_{i,j} = \int_0^T t^{4q-3} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q-1}\left(\lambda_i t^{2q}\right) dt \times \langle g, \varphi_j(.)\rangle_{L^2(D)} \langle g, \varphi_i(.)\rangle_{L^2(D)}.$
- $D_{i,j} = \int_0^T t^{4q-2} E_{2q,2q}\left(\lambda_j t^{2q}\right) E_{2q,2q}\left(\lambda_i t^{2q}\right) dt \, \langle g, \varphi_j(.)\rangle_{L^2(D)} \langle g, \varphi_i(.)\rangle_{L^2(D)}.$

5. Simulations

In this section, we present two numerical simulations illustrating the previous results, to make it easier to understand, let's focus on the one-dimensional case. Throughout this section, we work with we consider $A = \Delta = \frac{\partial^2}{\partial x^2}$ to be the Laplace operator with a domain:

$$\begin{aligned} \mathcal{D}(\Delta) &= \{\sigma \in H_1^0(\Omega)\ ,\ \Delta\sigma \in L^2(\Omega)\} \\ &= H^2(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

The operator Δ in this case generates a uniformly bounded strongly continuous cosine family $\{\mathfrak{W}(t)\}_{t\geq 0}$. It is well known that $-\Delta$ has a discrete spectrum and its eigenvalues are λ_j for every $j \in \mathbb{N}^*$, with the corresponding normalized eigenvectors $\varphi_j(x)$ for every $j \in \mathbb{N}^*$.

Example 1. *Consider the abstract time-fractional system:*

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\frac{3}{2}}\Theta(x,t) = \Delta\Theta(x,t) & in\ Q, \\ \lim\limits_{t\to 0^+} \mathcal{D}_{0^+}^{\frac{1}{2}}\Theta(x,t) = \Theta_0(x) = \frac{1}{2} \times x(1-x) & in\ \Omega, \\ \lim\limits_{t\to 0^+} \frac{\partial}{\partial t}\mathcal{D}_{0^+}^{\frac{1}{2}}\Theta(x,t) = \Theta_1(x) & in\ \Omega, \\ \Theta(\xi,t) = 0 & on\ \Sigma. \end{cases} \tag{27}$$

where $Q =]0,1[\times]0,3], \Sigma = \{0,1\} \times]0,3]$.
The output function is given by a pointwise sensor $\varpi(t) = \Theta(0.09,t)$, $\forall t \in]0,3]$. According to the numerical algorithm, we obtain the figure (1).

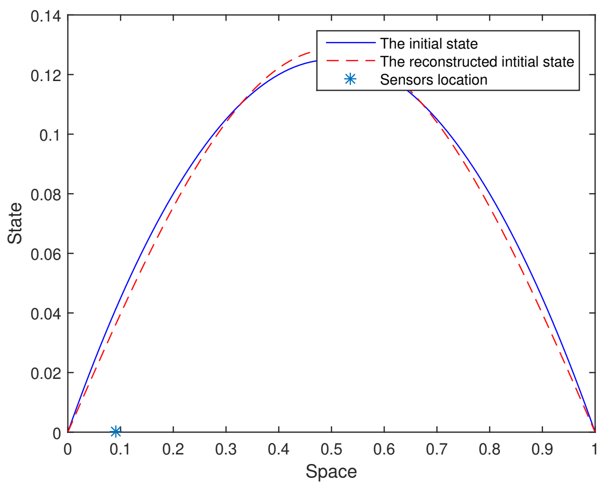

Figure 1. The initial state is represented by a continuous line, while the reconstructed initial state is shown as a dashed line within the interval $]0,1[$.

In figure (1), the starting state of the system is established with a reconstruction error of $\|\Theta_0 - \beta_0\|_{L^2(]0,1[)} = 3.5 \times 10^{-3}$, and it can be observed that the starting state is close to the estimated starting state in the interval $]0.1[$.

Example 2. *Consider the abstract time-fractional system:*

$$\begin{cases} {}^{RL}\mathcal{D}_{0^+}^{\frac{3}{2}}\Theta(x,t) = \Delta\Theta(x,t) & in\ Q, \\ \lim\limits_{t\to 0^+} \mathcal{I}_{0^+}^{\frac{1}{2}}\Theta(x,t) = \Theta_0(x) = \eta \times x(1-x) & in\ \Omega, \\ \lim_{t\to 0^+} \frac{\partial}{\partial t}\mathcal{I}_{0^+}^{\frac{1}{2}}\Theta(x,t) = \Theta_1(x) & in\ \Omega, \\ \Theta(\xi,t) = 0 & on\ \Sigma. \end{cases} \tag{28}$$

where $Q =]0,1[\times]0,6], \Sigma = \{0,1\} \times]0,3]$. In this example, we utilize a zonal sensor (g,D) where $D = \left]\frac{1}{4}, \frac{1}{2}\right[$ is the spatial domain and $g(x) = \chi_D(x)$ is the spatial distribution of the sensor. The performance of the sensor is measured by the function $\varpi(t) = \langle g, \beta\rangle_{L^2(D)}$. The numerical *process requires the selection of η to ensure that $\Theta_0(x)$ has a suitable magnitude. As a result, we can see the outcome displayed in figure* (2).

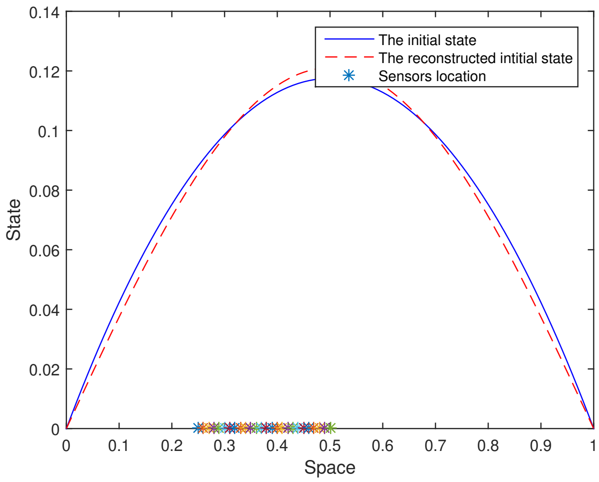

Figure 2. The starting state (continous line) and the reconstructed starting state (dashed line) in]0,1[.

As depicted in Figure (2), the actual initial state is nearly equivalent to the estimated initial state in]0.1[. The error in the reconstruction is $\|\Theta_0 - \beta_0\|_{L^2(]0,1[)} = 3.2 \times 10^{-3}$.

The figures (1) and (2) clearly demonstrate the effectiveness of the approach being considered (i.e. the HUM method and numerical algorithm are suited to solving reconstruction problems).

6. Conclusion

In this article, we have discussed some characterizations concerning the exact and approximate observability of the time-fractional system under consideration. We focused on the steps of the Hilbert uniqueness method to globally reconstruct the initial state for a specific class of linear time-fractional systems, where the Riemann-Liouville derivative has an order of ε in the interval $]1, 2[$. The difficulty of this process lies in transforming the reconstruction problem into a solvability issue of equation (20), which requires precision in finding its solution. Numerical methods must therefore be employed. To demonstrate the effectiveness of this method, two successful numerical examples were presented at the end of the study.

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Hamza Ben Brahim *is a researcher at TSI Team, Faculty of Sciences, Moulay Ismail University, Meknes, Morocco. His fields of research are: regional observability, existence of solutions for fractional systems, fractional calculus.*

https://orcid.org/0009-0000-0602-6242

Fatima-Zahrae El Alaoui *obtained her Ph.D. degree in Analysis and Control of Systems, in 2011, from the Faculty of Sciences, University Moulay Ismail, Meknes, Morocco. She is currently working as a professor at the Department of Mathematics, Faculty of Sciences, University Moulay Ismail, Meknes, Morocco. She is an active member of TSI team. Her fields of research are optimal control, regional observability, regional controllability, fractional calculus, and applied mathematics. She has published many papers in these areas.*

https://orcid.org/0000-0001-8912-4031

Khalid Zguaid *obtained his Ph.D. degree in Analysis and Control of Systems, in 2022, from the Faculty of Sciences, University Moulay Ismail, Meknes, Morocco. He is currently a professor at the Higher School of Education and Training of Agadir (ESEFA), Ibn Zohr University, Agadir, Morocco. He is also an active member of TSI team (Systems Theory & Informatics). His fields of research are optimal control, regional observability, fractional calculus, and applied mathematics, and he has published many papers in these areas.*

https://orcid.org/0000-0003-3027-8049