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RESEARCH ARTICLE

Existence and uniqueness study for partial neutral functional fractional differential equation under Caputo derivative

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ABSTRACT

The partial neutral functional fractional differential equation described by the fractional operator is considered in the present investigation. The used fractional operator is the Caputo derivative. In the present paper, the fractional resolvent operators have been defined and used to prove the existence of the unique solution of the fractional neutral differential equations. The fixed point theorem has been used in existence investigations. For an illustration of our results in this paper, an example has been provided as well.

1. Introduction

Modeling by taking into account the memory effect is the attraction of the fractional calculus. The concept of memory is not taken by the ordinary derivative, thus modeling with the ordinary derivative gives incomplete dynamics. The works related to fractional calculus continue to impress the mathematician communities. There exist now many papers addressing the application of fractional calculus, we cite the following paper which brings information on the application of this field of mathematics to biology [1–3], engineerings [4–7], physics and applications [8–12] and fluid modeling [13–16]. Modeling with the Caputo derivative is more adequate due to the inconvenience of the Riemann-Liouville fractional derivative. It is noticed that the Riemann-Liouville derivative of the constant function does not give zero, it is a serious inconvenience in the pratic because many initial conditions are constant or null. Due to this fact, we model in this paper using the Caputo derivative. The field of fractional calculus has attracted many authors due to the diversity and existence of many fractional operators. He has the Caputo derivative [17, 18], and the Riemann-Liouville derivative version of the fractional operator also exists, see more details in the paper [19]. We have the Atangana-Baleanu derivative which has two versions, the Caputo version and the Riemann-Liouville version. The Caputo-Fabrizio derivative exists but is with the exponential kernel [20]. Note that the Antangana-Baleanu derivative has as a kernel the Mittag-Leffler function as described in the paper [21]. There exist many other derivatives as conformable derivatives, Hilfer derivatives, and others, the difference between them is not significant, just the kernel change in many of them. In this

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paper, the application of fractional calculus to neutral fractional differential equations has been considered. The problem considered in this paper is represented by the following neutral fractional differential equation

$$\begin{cases} D_t^\alpha \left(x(t) - Bx(t-h)\right) = Ax(t) + Cx(t-h) \;\; t \geq 0 \\ x_0 = \varphi \in \mathcal{C}_X. \end{cases} \tag{1}$$

The literature review concerning the fractional neutral functional differential equations as described in the equation (1) or similar to the previous equation is very large. In [22], the authors addressed the neutral differential equation using Caputo derivative, and with the utilization of the resolvent operator, the authors also used fixed point theorem to prove the existence of the solution of the considered neutral differential equation. In [23], Wen et al provide the Complete controllability of nonlinear fractional neutral functional differential equations described by the Caputo derivative. In this paper, the authors provided an interesting example of a neutral fractional differential equation to illustrate their main results. In [24], Wang, et al. presented applying an iterative technique, sufficient conditions are obtained for the existence of the solution of the nonlinear neutral fractional integrodifferential equation described by the Riemann-Liouville derivatives of different fractional orders. In this paper, the existence has been proved without the resolvent operators. In [25], using the conformable derivative Li et al. provided the existence of the unique solution of the class of the fractional Integral neutral differential equations. In [26], the author proposed the investigation in a fractional context related to the existence and uniqueness of solutions for fractional neutral Volterra-Fredholm integrodifferential equations. In [27], we can find the application of Krasnoselskii's fixed point theorem on periodicity and stability in neutral nonlinear differential equations. In integer versions many investigations have been made related to neutral differential equations in different types, the studies related to the existence, and the controllability are already made as well, see the following Ezzinbi et al papers investigations [28, 29]. In [30], Sene proposed a new fundamental result concerning the contribution of the resolvent operator for proving the existence of the unique solution of the fractional integrodifferential equation under the Caputo derivative. It is very important to model with the Caputo derivative or with integer derivative, it is also important to be sure that the investigations can be made on the considered fractional model. To make sure that the model is well defined in mathematics, it is important to prove the existence and uniqueness of the solution of the model using one known fixed point theorem. This paper's novelties can be summarized in different points. The first is to prove the existence and the uniqueness of the solution using resolvent operators. This problem is interesting because the resolvent operator in the fractional context is a new problem in the literature. The second problem is that the fractional neutral functional differential equations described by the Caputo derivative have been used. The last novel and interesting thing is that we used the fixed point theorem to prove our main results in this paper.

The present paper is organized in the following form. In Section 2, we recall the necessary tools for our investigation as the fractional operators and the fixed point theorem. In Section 3, we start with the main results concerning the existence of the solution of the fractional neutral differential equation using the resolvents operators. In Section 4, we illustrate our main results with an example to highlight our results. In Section 5 we finish with the conclusion and future direction of investigations.

2. Preliminaries

In this section we recall the preliminary definition necessary for our investigations. We begin with the fractional operator, we continue with the fractional resolvents necessary to define our solutions.

Some important results on the fixed point theorem can also be recalled because they will be used in our investigation, we mean Schauder's Fixed Point Theorem used in many papers in the literature.

Definition 1. *The Riemann-Liouville integral of order $\alpha > 0$ for a continuous function defined on $[0,1]$ is given by:*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \tag{2}$$

with $\Gamma(\alpha) := \int_0^\infty e^{-u} u^{\alpha-1} du$.

Definition 2. *If $f \in C^n([0,1], \mathbb{R})$ and $n-1 < \alpha \leq n$, then, the Caputo fractional derivative is* *given by:*

$$\begin{aligned} D^{\alpha}f(t) &= I^{n-\alpha}\frac{d^n}{dt^n}(f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}f^{(n)}(s)ds. \end{aligned} \tag{3}$$

Lemma 1. *Taking $n \in \mathbb{N}^*$ and $n-1<\alpha<n$, then the general solution of $D^{\alpha}y(t)=0$ is given by*

$$y(t)=\sum_{i=0}^{n-1}c_i t^i \tag{4}$$

such that $c_i \in \mathbb{R}, i=0,1,2,..,n-1$.

Lemma 2. *Taking $n \in \mathbb{N}^*$ and $n-1<\alpha<n$, then, we have*

$$I^{\alpha}D^{\alpha}y(t)=y(t)+\sum_{i=0}^{n-1}c_i t^i \tag{5}$$

such that $c_i \in \mathbb{R}, i=0,1,2,..,n-1$.

Definition 3. *Let X be a Banach space. Then a map $T: X \longrightarrow X$ is called a contraction mapping on X if there exists $q \in [0,1)$ such that*

$$\|T(x)-T(y)\| \le q\|x-y\|$$

for all $x, y \in X$.

Theorem 1 (Banach's fixed point theorem)**.** *[31] Let Ω be a non-empty closed subset of a Banach space X. Then, any contraction mapping T of Ω into itself has a unique fixed point.*

Theorem 2 (Schauder's fixed point theorem)**.** *[31] Let X be a Banach space, and let $N: X \longrightarrow X$ be a completely continuous operator. If the set $E=\{y \in X : y=\lambda N y \text{ for some } \lambda \in (0,1)\}$ is bounded, then N has fixed points.*

3. Main results

In this section, we give the procedure to get the analytical solution of the neutral functional differential equation including the resolvent operator. The novelty of this section will be the use of the Laplace transform to get resolvent and to get the analytical solution. We start this procedure by applying the Laplace transform we get the following form

Let's give the solution of the fractional differential equation given by Eq. (1). The form is described in the following lemma, for the simplification we consider $f(t,x_t)=Ax(t)+Cx(t-h)$.

Lemma 3. *Let the neutral fractional differential equation described by the Caputo derivative, under the initial condition described in Eq. (1), the solution are described by the following form*

$$\begin{cases} x(t)=\varphi(0)-B\varphi(-h)+Bx(t-h) \\ +\int_0^t (t-s)^{\alpha-1}\left[f(s,x_s)\right]ds \quad t\ge 0 \\ x(t)=\varphi(t) \quad\ t\in[-h,0]. \end{cases} \tag{6}$$

Proof. The procedure of the proof uses the application of the Riemann-Liouville integral to the equation Eq. (1). Using the lemma 2 in the equation 1, we get

$$x(t)-Bx(t-h)=\int_0^t (t-s)^{\alpha-1}\left[f(s,x_s)\right]ds + c_0, \tag{7}$$

where c_0 is a real constant. Using the initial condition of the equation (1), we obtain

$$\begin{cases} x(t)=\varphi(0)-B\varphi(-h)+Bx(t-h) \\ +\int_0^t (t-s)^{\alpha-1}\left[f(s,x_s)\right]ds \quad t\ge 0 \\ x(t)=\varphi(t) \quad\ t\in[-h,0]. \end{cases}$$

□

The next problem will consist to rewrite the solution described in Eq. (6) the neutral fractional differential equation in terms of the resolvent operator. We make the following lemma.

Lemma 4. *We consider that Eq. (1) are hold and then we should have the following relationship described by the following form*

$$\begin{cases} x(t)=R_{\alpha}(t)\left[\varphi(0)-B\varphi(-h)\right]+Bx(t-h) \\ +\int_0^t (t-s)^{\alpha-1}S_{\alpha}(t-s)\left[AB\right]x(s-h)ds \\ +\int_0^t (t-s)^{\alpha-1}S_{\alpha}(t-s)\left[C\right]x(s-h)ds \qquad t\ge 0 \\ x(t)=\varphi(t) \quad\ t\in[-h,0] \end{cases}$$

where the resolvent operator in our context is defined by the following expressions for simplifications.

Proof. The proof, we apply the Laplace transform to the equation represented in Eq. (6) we get the series of transformations given in the forthcoming equations. We have that

$$\begin{aligned} \bar{x} &= \frac{1}{q}\left[\varphi(0)-B\varphi(-h)\right]+B\bar{x}_h-q^{-\alpha}A\bar{x}+q^{-\alpha}C\bar{x}_h \\ &= q^{\alpha-1}\left[q^{\alpha}I+A\right]^{-1}\left[\varphi(0)-B\varphi(-h)\right] \\ &+ q^{\alpha}\left[q^{\alpha}I+A\right]^{-1}B\bar{x}_h+\left[q^{\alpha}I+A\right]^{-1}C\bar{x}_h \end{aligned} \tag{8}$$

$$x(t)=\varphi(t)\in t\in[-h,0] \tag{9}$$

For the rest of the proof, we suppose that

$$\mathcal{L}\left\{T_\alpha(t)\right\}(q) = \left[q^\alpha I + A\right]^{-1} \tag{10}$$

where the so-called in our present paper the fractional analytic semigroup $\{T_\alpha(t)\}_{t\geq 0}$, there is that there exist constant M such that $M = \sup_{t\in[0,+\infty[} |T_\alpha(t)| < \infty$ and for any $\alpha \in (0,1)$, we can found a constant C_α verifying the condition that $|A^\alpha T_\alpha(t)| \leq C_\alpha t^{-\alpha}$. Replacing Eq. (10) in Eq. (8), we get the following relationships

$$\begin{aligned}\bar{x} &= q^{\alpha-1}\int_0^\infty e^{-q^\alpha s} T_\alpha(s)\left[\varphi(0) - B\varphi(-h)\right] ds \\ &+ q^\alpha \int_0^\infty e^{-q^\alpha s} T_\alpha(s) B\bar{x}_h ds \\ &+ \int_0^\infty e^{-q^\alpha s} T_\alpha(s) C\bar{x}_h ds. \end{aligned}\tag{11}$$

Before beginning the simplification in the previous expression we suppose the following density of probability is well known in the literature of fractional calculus and can be found in, we have the following form

$$\begin{aligned}\varpi_\alpha(\theta) =& \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\theta^{-\alpha n-1} \\ &\times \frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha).\end{aligned}\tag{12}$$

The form of its Laplace transform can be represented by $\int_0^\infty e^{-q\theta}\varpi_\alpha(\theta)\,d\theta = e^{-q^\alpha}$, this relation will be replaced by its values in the forthcoming calculations. We now begin the simplification in Eq. (11), for the next calculations the sketch is inspired by the paper in the literature, we have to calculate the first form of Eq. (11) given in the following equation

$$\begin{aligned}&q^{\alpha-1}\int_0^\infty e^{-q^\alpha s} T_\alpha(s)\left[\varphi(0) - B\varphi(-h)\right] ds \\ &= \int_0^\infty \alpha (qt)^{\alpha-1} e^{-(qt)^\alpha} T_\alpha(t^\alpha)\left[\varphi(0) - B\varphi(-h)\right] dt \\ &= -\int_0^\infty \frac{1}{q}\frac{d}{dt}\left[e^{-(qt)^\alpha} T_\alpha(t^\alpha)\left[\varphi(0) - B\varphi(-h)\right]\right] dt\end{aligned}$$

we continue the variable change and we use the probability density described in Eq. (12), we get the following forms

$$\begin{aligned}&q^{\alpha-1}\int_0^\infty e^{-q^\alpha s} T_\alpha(s)\left[\varphi(0) - B\varphi(-h)\right] ds \\ &= -\int_0^\infty \frac{1}{q}\frac{d}{dt}\left[e^{-(qt)^\alpha} T_\alpha(t^\alpha)\left[\varphi(0) - B\varphi(-h)\right]\right] dt \\ &= \int_0^\infty\int_0^\infty \theta\varpi_\alpha(\theta)\left[e^{-qt\theta} T_\alpha(t^\alpha)\left[\varphi(0) - B\varphi(-h)\right]\right] d\theta dt \\ &= \int_0^\infty e^{-qt}\left(\int_0^\infty \varpi_\alpha(\theta)\left[T_\alpha\left(\frac{t^\alpha}{\theta^\alpha}\right)\left[\varphi(0) - B\varphi(-h)\right]\right] d\theta\right) dt.\end{aligned}$$

We take the second expression from Eq. (11) and continue the simplifications, we have the following relationships

$$\begin{aligned}&\int_0^\infty e^{-q^\alpha s} T_\alpha(s) C\bar{x}_h ds \\ &= \int_0^\infty\int_0^\infty \alpha t^{\alpha-1} e^{-(qt)^\alpha} T_\alpha(t^\alpha) C\bar{x}_h(s)\, ds dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\varpi_\alpha(\theta) e^{-(qt\theta)} T_\alpha(t^\alpha) t^{\alpha-1} C\bar{x}_h(s)\, d\theta ds dt \\ &= \int_0^\infty\int_0^\infty\int_0^\infty \alpha\varpi_\alpha(\theta) e^{-qt} T_\alpha\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha} C\bar{x}_h(s)\, d\theta ds dt \\ &= \int_0^\infty e^{-qt}\left(\alpha\int_0^\infty\int_0^\infty \varpi_\alpha(\theta) T_\alpha\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^\alpha}{\theta^\alpha} C\bar{x}_h(s)\, d\theta ds\right) dt.\end{aligned}$$

We take the last calculation for giving a more simple form of Eq. (11), the formula which we will simplify is given by the following relationships

$$\begin{aligned}&q^\alpha\int_0^\infty e^{-q^\alpha s} T_\alpha(s) B\bar{x}_h ds \\ &= \int_0^\infty\int_0^\infty \alpha q^\alpha t^{\alpha-1} e^{-(qt)^\alpha} T_\alpha(t^\alpha) B\bar{x}_h ds dt \\ &= \int_0^\infty\left[\int_0^\infty -T_\alpha(t^\alpha) B\bar{x}_h ds\right] de^{-(qt)^\alpha}\end{aligned}\tag{13}$$

Applying the integration by parts, and introducing the function described in Eq. (12), according to the calculations, we arrive at the following calculation for Eq. (13), that is

$$\begin{aligned}&q^\alpha\int_0^\infty e^{-q^\alpha s} T_\alpha(s) B\bar{x}_h ds = \int_0^\infty e^{-qt} Bx(t-h)\, dt \\ &+ \int_0^\infty e^{-qt}\left[q\int_0^t\int_0^\infty \varpi_\alpha(\theta) AT_\alpha\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right) C\frac{(t-s)^\alpha}{\theta^\alpha} d\theta ds\right] dt\end{aligned}\tag{14}$$

We now try to compute the inverse of the Laplace transform by inverting Eq. (11) by considering the simplified form described in the previous equation, we get the following form as the final expression

$$\begin{aligned}x(t) &= \int_0^\infty \phi_\alpha T_\alpha(t^\alpha\theta)\left[\varphi(0) - B\varphi(-h)\right] d\theta + Bx(t-h) \\ &+ q\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\phi_\alpha AB(s-h)\, T_\alpha\theta(t-s)^\alpha\, d\theta ds \\ &+ q\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}\phi_\alpha T_\alpha\theta(t-s)^\alpha Cx(s-h)\, d\theta ds,\end{aligned}$$

where $\phi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-\frac{1}{\alpha}}\varpi\left(\theta^{-\frac{1}{\alpha}}\right)$. With the previous representation of the solution we can now define our resolvents operators which we will consider to continue our investigation, the resolvents are represented as

$$R_\alpha(t)x = \int_0^\infty \phi_\alpha(\theta) T_\alpha(t^\alpha\theta) x d\theta, \tag{15}$$

and

$$S_\alpha(t)x = q\int_0^\infty \theta\phi_\alpha(\theta) T_\alpha(t^\alpha\theta) x d\theta. \tag{16}$$

Using Eq. (15) and Eq. (16) we get the solutions represented in the Lemma 4 when the following condition is respected $t \geq 0$. The second form of the solution is that $x(t) = \varphi(t)$, $t \in [-h, 0]$. We end the proof of our lemma. □

Lemma 5. *The resolvents operators $R_\alpha(t)$ and $S_\alpha(t)$ are strongly continuous, and furthermore verify the relationships that they are bounded operators and satisfies the conditions that*

$$\begin{cases} \|R_\alpha(t)x\| \leq M\|x\| \\ and \\ \|S_\alpha(t)x\| \leq \frac{M\alpha}{\Gamma(1+\alpha)}\|x\| \end{cases} \tag{17}$$

where M is constant.

Now, we are ready to prove the existence of the mild solution of the neutral fractional differential equation defined in Eq. (1). We make a certain number of assumptions necessary in our investigations.

(A1) The resolvent operators $R_\alpha(t)$ and $S_\alpha(t)$ are compact operators for every $t \geq 0$.

(A2) The function $Cx(t-h)$ is messurable, continuous and satifies the condition that there exists $q \in (0,1)$ and $m \in L^{1/q}([0,T], \mathbb{R}^+)$, we have that $|Cx(t-h)| \leq m(t)\rho(\|x_t\|)$ for all $x \in C$ and furtermore almost all $t \in [0,T]$.

(A3) Let for the function $Bx(t-h)$ and we have existence of a constant $\beta \in (0,1)$ and two constant k and k_1 satisfying the condition that $Bx(t-h) \in D\left(A^\beta\right)$ and for $x, y \in C$ and $t \in [0,a]$ we have $\left\|A^\beta Bx - A^\beta By\right\| \leq k\|x-y\|$ and $\left\|A^\beta Bx\right\| \leq k_1(\|x_t\|+1)$.

For the main results of our present paper, we make the following theorem. This theorem proves the existence of the mild solution. In our investigation, we use Schauder Fixed Point Theorem, which is more appropriate for this study.

Theorem 3. *Under the hypotheses (A1), (A2) and (A3) the problem (1) has at least one mild solution.*

Proof. We begin by proving the boundedness of some mathematical expressions. Let the function that

$$\begin{aligned} &\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) ABx(s-h)ds\right\| \\ &\leq \int_0^t \left|(t-s)^{\alpha-1} A^{1-\beta} S_\alpha(t-s) A^\beta Bx(s-h)\right| ds \\ &\leq \int_0^t (t-s)^{\alpha-1} \left|A^{1-\beta} S_\alpha(t-s) A^\beta Bx(s-h)\right| ds \end{aligned}$$

We use the assumption described by (A3) and the statement posed in lemma 5, the next established results are well known in fundamental mathematics as the Lebesgue integrability of the function into the integration, we get the following relationships

$$\begin{aligned} &\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) ABx(s-h)ds\right\| \\ &\leq \int_0^t (t-s)^{\alpha-1} \left|A^{1-\beta} S_\alpha(t-s) A^\beta Bx(s-h)\right| ds \\ &= \int_0^t (t-s)^{\alpha-1} \frac{\alpha\Gamma(1+\beta) C_{1-\beta}}{\Gamma(1+\alpha\beta)(t-s)^{\alpha(1-\beta)}} k_1(\|x_t\|+1) ds \\ &= \frac{\alpha\Gamma(1+\beta) C_{1-\beta}}{\Gamma(1+\alpha\beta)} k_1(\|x_t\|+1) \int_0^t (t-s)^{\alpha-1} ds \\ &= \frac{\alpha\Gamma(1+\beta) C_{1-\beta}}{\Gamma(1+\alpha\beta)} k_1(\|x_t\|+1) T^{\alpha\beta} \end{aligned} \tag{18}$$

As in the previous bound we also continue the simplification by trying to find a bound for the next integration, we have the following relationship

$$\begin{aligned} &\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s) Cx(s-h)ds\right\| \\ &\leq \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^t \left|(t-s)^{\alpha-1} Cx(s-h)\right| ds. \end{aligned}$$

Applying Holder inequality and using the assumption described in (A2), we get the following relationships

$$\left\|\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)\, Cx(s-h)ds\right\|$$
$$\leq \frac{M\alpha}{\Gamma(1+\alpha)}\left[\int_0^\infty (t-s)^{\frac{\alpha-1}{1-\beta}}ds\right]^{1-\beta}\|m\|_{L^{1/\beta}[0,t]}\,\rho(\|x\|)$$
$$\leq \frac{M\alpha}{\Gamma(1+\alpha)}\frac{\alpha MNa^{\left(1+\frac{\alpha-1}{1-\beta}\right)(1-\beta)}\rho(\|x\|)}{\Gamma(1+\alpha)\left(1+\frac{\alpha-1}{1-\beta}\right)^{1-\beta}}.$$

These two previous relationships will help us in the application of the fixed point theorem which we want to illustrate. The application of the fixed point need to defined an operator as the following form $\Phi : B_r \longrightarrow C([-h,a], X)$ such that

$$\Phi x(t) = R_\alpha(t)[\varphi(0) - B\varphi(-h)] + Bx(t-h)$$
$$+\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[AB]\,x(s-h)ds$$
$$+\int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[C]\,x(s-h)ds$$
$$respect\quad to\quad t\geq 0 \tag{19}$$
$$\Phi x(t) = \varphi(t) \in t\in[-h,0] \tag{20}$$

Note that the set B_r is defined as all $x_t \in B_r$ satisfy the condition that $\|x_t\| \leq r$. The proof should de be divided into three parts, In the first part the operator Φ maps to itself. The first step is denoted by step1. We have the following.

Step 1: Let us prove that Φ maps to itself. We suppose that the following relationships will play important role in the proof-by-contradiction process, we have that

$$\lim_{r\to\infty}\left(\frac{\theta(r)}{r}+\frac{K}{r}\int_0^\infty \rho(s)\,ds\right)<1. \tag{21}$$

We begin the proof by contradiction by applying the norm used in our paper to the function defined in Eq. (19), we also assume that all the assumptions have been verified as well, we have the following form

$$\|\Phi x(t)\| \leq M\|\varphi\| + M\left|A^{-\beta}\right|k_1(\|\varphi\|+1)$$
$$+\left|A^{-\beta}\right|k_1(\|x_t\|+1)$$
$$+\frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}k_1(\|x_t\|+1)T^{\alpha\beta}$$
$$+\frac{M\alpha}{\Gamma(1+\alpha)}\frac{\alpha MNa^{\left(1+\frac{\alpha-1}{1-\beta}\right)(1-\beta)}}{\Gamma(1+\alpha)\left(1+\frac{\alpha-1}{1-\beta}\right)^{1-\beta}}\int_0^t \rho(s)\,ds$$

Let's consider that the function and the constant that

$$\theta(\|x_t\|) = \left|A^{-\beta}\right|k_1\|x_t\| + \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}k_1\|x_t\|T^{\alpha\beta} \tag{22}$$

and

$$K = \frac{M\alpha}{\Gamma(1+\alpha)}\frac{\alpha MNa^{\left(1+\frac{\alpha-1}{1-\beta}\right)(1-\beta)}}{\Gamma(1+\alpha)\left(1+\frac{\alpha-1}{1-\beta}\right)^{1-\beta}} \tag{23}$$

And then the previous equation can be written in the form that

$$\|\Phi x(t)\| \leq M\|\varphi\| + M\left|A^{-\beta}\right|k_1(\|\varphi\|+1)$$
$$+\left|A^{-\beta}\right|k_1 + \frac{\alpha\Gamma(1+\beta)C_{1-\beta}}{\Gamma(1+\alpha\beta)}k_1T^{\alpha\beta}$$
$$+\theta(\|x_t\|) + K\int_0^t \rho(s)\,ds$$

Let that for each strictly positive constant r, there exist exist $x \in B_r$, such that $\phi x \notin B_r$. For simplification in the calculations, we add a further constant notation that is

$$M_1 = M\|\varphi\| + M\left|A^{-\beta}\right|k_1(\|\varphi\|+1) + \left|A^{-\beta}\right|k_1 \tag{24}$$

The previous assumption can be written mathematically by the condition described in the following form

$$r < \|\phi x(t)\| \leq M_1 + \theta(r) + K\int_0^t \rho(s)\,ds \tag{25}$$

The next step consists to divide the previous Eq. (25) by our constant r, we get the following relationships

$$1 < \|\phi x(t)\| \leq \frac{M_1}{r} + \frac{\theta(r)}{r} + \frac{K}{r}\int_0^t \rho(s)\,ds \tag{26}$$

Applying the limit respect to r at infinity, we get the following relationship which will contradict our preliminary assumptions, we have that

$$1 < \|\phi x(t)\| \leq \liminf_{r\to\infty}\left[\frac{\theta(r)}{r} + \frac{K}{r}\int_0^t \rho(s)\,ds\right] \tag{27}$$

We notice that Eq. (27) is in contradiction with the assumption reported in Eq. (21). This means that $\|\Phi x(t)\| \leq r$, we conclude that Φ maps to itself.

Step 2: In the second step, we will prove that the operator $\Phi : B_r \longrightarrow B_r$ is continuously using the classical method for proving the continuity. We set that $\{x^n\} \subseteq B_r$ respect to the property that $x^n \to x$ on the set B_r. In our present context using the assumption that (A) and the fact that $x_t^n \to x_t$, we have in particular the following intermediary condition that $Cx^n(t-h) \to Cx(t-h)$ a:e $t \in [0,T]$ when $n \to \infty$. Furthermore with the assumption (A), we have in particular that $\|Cx^n(s-h) - Cx(s-h)\| \leq m(s)\|\rho(x^n(s)) - \rho(x(s))\|$. We notice with the previous condition that, the fact that the function ρ is Lipschitz continuous implies the convergence to zero of the previous relationship. In addition, using the classical dominated convergence theorem, we get the following transformation and convergence, which are

$$\begin{aligned}
&\|\Phi x^n(t) - \Phi x(t)\| \leq \|Bx^n(t-h) - Bx(t-h)\|_X \\
&+ \int_0^t (t-s)^{\alpha-1} |S_\alpha(t-s)[ABx^n(s-h) - ABx(s-h)]| \, ds \\
&+ \int_0^t (t-s)^{\alpha-1} |S_\alpha(t-s)[Cx^n(s-h) - Cx(s-h)]| \, ds \\
&\leq (k+1)) A^{-\beta} |B| \|x^n(t) - x(t)\|_X \\
&+ \int_0^t (t-s)^{\alpha-1} |S_\alpha(t-s)[ABx^n(s-h) - ABx(s-h)]| \, ds \\
&+ \int_0^t (t-s)^{\alpha-1} |S_\alpha(t-s)[Cx^n(s-h) - Cx(s-h)]| \, ds
\end{aligned}$$

Using the previously established results and the Lipchitz property in the assumption (A), and adding the condition in Lemma 5, we have the following form

$$\begin{aligned}
&\|\Phi x^n(t) - \Phi x(t)\| \leq (k+1)) A^{-\beta} |B| \|x^n(t) - x(t)\|_X \\
&+ \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} |[ABx^n(s-h) - ABx(s-h)]| \, ds \\
&+ \frac{M\alpha}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} |[Cx^n(s-h) - Cx(s-h)]| \, ds
\end{aligned} \tag{28}$$

We observe by using Eq. (28) that $\|\Phi x^n(t) - \Phi x(t)\| \to 0$ as $n \to \infty$ where it follows that the continuity of the operator Φ. The next section will be consecrated to prove that the set $\{\phi x : x \in B_r\}$ is relatively compact.

Step 3: As recalled at the end of the previous step in this part we try to prove that the set described by $\{\phi x : x \in B_r\}$ is relatively compact. Let $x \in B_r$ and $t_1 \leq t_2 \leq T$. We use two sub-operators, we have the following form

$$\phi_a x = R_\alpha(t)[\varphi(0) - B\varphi(-h)] + Bx(t-h) \tag{29}$$

We have the following expressions by applying the norm used in our space

$$\begin{aligned}
&\|\Phi x(t_2) - \Phi x(t_1)\| \\
&\leq \|(R_\alpha(t_2) - R_\alpha(t_1))[\varphi(0) - B\varphi(-h)]\|_X \\
&+ \|Bx(t_2-h) - Bx(t_1-h)\|_X \\
&\leq \|(R_\alpha(t_2) - R_\alpha(t_1))\|_X [\varphi(0) - B\varphi(-h)] \\
&+ \|Bx(t_2-h) - Bx(t_1-h)\|_X
\end{aligned}$$

We first use that the resolvent operator R_α is strongly continuous, and then we get that when $t_2 \to t_1$ thus $\|\Phi x(t_2) - \Phi x(t_1)\| \to 0$. We set the second operator as the following form

$$\phi_b x = \int_0^t (t-s)^{\alpha-1} S_\alpha(t-s)[AB]x(s-h)ds \tag{30}$$

Let $x \in B_r$ and $t_1 \leq t_2 \leq T$, to evaluate the convergence as in the previous section, we have the following relationships

$$\begin{aligned}
&\|\Phi_b x(t_2) - \Phi_b x(t_1)\| \\
&= \| \int_0^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]x(s-h)ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_1-s)[AB]x(s-h)ds\| \\
&\leq \| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]x(s-h)ds\| \\
&+ \| \int_0^{t_1} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]x(s-h)ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_2-s)[AB]x(s-h)ds\| \\
&+ \| \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_2-s)[AB]x(s-h)ds \\
&- \int_0^{t_1} (t_1-s)^{\alpha-1} S_\alpha(t_1-s)[AB]x(s-h)ds\| \\
&\leq \| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} S_\alpha(t_2-s)[AB]x(s-h)ds\| \\
&+ \| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] S_\alpha(t_2-s)[AB]x(s-h)ds\| \\
&+ \| \int_0^{t_1} (t_1-s)^{\alpha-1} [S_\alpha(t_2-s) - S_\alpha(t_1-s)][AB]x(s-h)ds\|
\end{aligned} \tag{31}$$

The previous relation in Eq. (31) can be rewritten in terms of three integral denotes here by the following form

$$\|\Phi_b x(t_2) - \Phi_b x(t_1)\| \leq I_1 + I_2 + I_3, \tag{32}$$

we have the following relationships for the simplification of our expressions, we have that

$$I_1 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \tag{33}$$

$$\begin{aligned} I_2 = \Bigg\| \int_0^{t_1} & \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \\ & \times S_\alpha (t_2 - s) [AB] x(s-h) ds \Bigg\| \end{aligned} \tag{34}$$

$$\begin{aligned} I_3 = \Bigg\| \int_0^{t_1} & (t_1 - s)^{\alpha-1} \left[S_\alpha (t_2 - s) - S_\alpha (t_1 - s) \right] \\ & \times [AB] x(s-h) ds \Bigg\| \end{aligned} \tag{35}$$

We now proceed to the calculations of the expressions represented in Eq. (33), Eq. (34) and Eq. (35), we have the following calculations

$$\begin{aligned} I_1 &= \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \\ &\leq \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \left| A^{1-\beta} S_\alpha (t_2 - s) A^\beta B x(s-h) \right| ds \\ &= \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} \frac{\alpha \Gamma (1+\beta) C_{1-\beta}}{\Gamma (1+\alpha\beta) (t-s)^{\alpha(1-\beta)}} k_1(\|x_t\| + 1) ds \\ &= \frac{\alpha \Gamma (1+\beta) C_{1-\beta}}{\Gamma (1+\alpha\beta)} k_1(\|x_t\| + 1) \int_{t_1}^{t_2} (t_2 - s)^{\alpha\beta - 1} ds \\ &= \frac{\Gamma (1+\beta) C_{1-\beta}}{\Gamma (1+\alpha\beta)} k_1(\|x_t\| + 1) (t_2 - t_1)^{\alpha\beta} \end{aligned} \tag{36}$$

We continue with the second expression represented by the variable I_2 in Eq. (34), we have the following calculations

$$\begin{aligned} I_2 &= \left\| \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] S_\alpha (t_2 - s) [AB] x(s-h) ds \right\| \\ &\leq \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] \left| A^{1-\beta} S_\alpha (t_2 - s) A^\beta B x(s-h) \right| ds \\ &\leq \frac{\alpha M C_{1-\beta}}{\Gamma (1+\alpha)} \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] k_1(\|x_t\| + 1) ds \\ &\leq \frac{\alpha M C_{1-\beta}}{\Gamma (1+\alpha)} k_1(\|x_t\| + 1) \int_0^{t_1} \left[(t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1} \right] ds \\ &\leq \frac{\alpha M C_{1-\beta}}{\Gamma (1+\alpha)} k_1(\|x_t\| + 1) \left[|(t_2 - t_1)^\alpha| \right] \end{aligned}$$

where we are assumed that $\left| A^{1-\beta} \right| \leq C_{1-\beta}$. We continue with the third integral, we have the following bound

$$\begin{aligned} I_3 &= \left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[S_\alpha (t_2 - s) - S_\alpha (t_1 - s) \right] [AB] x(s-h) ds \right\| \\ &\leq \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\left| A^{1-\beta} S_\alpha (t_2 - s) - A^{1-\beta} S_\alpha (t_1 - s) \right| \right] \left| \left[A^\beta B \right] x(s-h) \right| ds \\ &\leq \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[\left| A^{1-\beta} S_\alpha (t_2 - s) - A^{1-\beta} S_\alpha (t_1 - s) \right| \right] k_1(\|x_t\| + 1) ds \\ &\leq \frac{t_1^\alpha k_1(\|x_t\| + 1)}{\alpha} \sup_{s \in [0, t_1]} \left[\left| A^{1-\beta} S_\alpha (t_2 - s) - A^{1-\beta} S_\alpha (t_1 - s) \right| \right] \end{aligned} \tag{37}$$

Note that from the continuity of the resolvent operator S_α, follows also the continuity of the operator $A^{1-\beta} S_\alpha$. Then from Eq.(36) to Eq.(38), we observe that $t_2 \to t_1$ thus $\|\Phi_b x(t_2) - \Phi_b x(t_1)\|_X \to 0$. We finish this sub-section with the term represented by

$$\phi_c x = \int_0^t (t - s)^{\alpha-1} S_\alpha (t - s) [C] x(s-h) ds \tag{38}$$

Let $x \in B_r$ and $t_1 \leq t_2 \leq T$, to evaluate the convergence as in the previous section, we have the following relationships and referring to the previous section we have the following relationships

$$\|\Phi_c x(t_2) - \Phi_c x(t_1)\|_X \leq I_1 + I_2 + I_3 \tag{39}$$

where

$$I_1 = \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) C x(s-h) ds \right\| \tag{40}$$

$$\begin{aligned} I_2 = \Bigg\| \int_0^{t_1} & \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \\ & \times S_\alpha (t_2 - s) C x(s-h) ds \Bigg\| \end{aligned} \tag{41}$$

$$\begin{aligned} I_3 = \Bigg\| \int_0^{t_1} & (t_1 - s)^{\alpha-1} \left[S_\alpha (t_2 - s) - S_\alpha (t_1 - s) \right] \\ & \times C x(s-h) ds \Bigg\| \end{aligned} \tag{42}$$

We do the same as the previous sub-section but here the Holder inequality is used many times, we begin with the expression represented by

$$\begin{aligned} I_1 &= \left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} S_\alpha (t_2 - s) C x(s-h) ds \right\| \\ &\leq \frac{\alpha M \rho (\|x_t\|)}{\Gamma (1+\alpha)} \left[\int_{t_1}^{t_2} (t_2 - s)^{\frac{1-\alpha}{1-\eta}} ds \right]^{1-\eta} \|m\|_{L^{1/\eta}[t_1, t_2]} . \end{aligned} \tag{43}$$

For simplification in the rest of the calculutaion we take that $L_1 = \|m\|_{L^{1/\eta}[t_1,t_2]}$ and $\kappa = \frac{1-\alpha}{1-\eta}$, and then we get the following relationship

$$I_1 \leq \frac{\alpha M \rho(\|x_t\|) L_1 (t_2 - t_1)^{(1+\kappa)(1-\eta)}}{\Gamma(1+\alpha)(1+\kappa)^{1-\eta}}.$$

We now continue with the expression represented by the I_2 in Eq. (34), here also the Holder inequality is used for the simplification of the upbound, we have the following relationships

$$\begin{aligned} I_2 &= \Big\| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \\ &\times S_\alpha (t_2 - s) Cx(s-h) ds \Big\| \\ &\leq \tfrac{\alpha M \rho(\|x_t\|)}{\Gamma(1+\alpha)} \left[\int_0^{t_1} (t_1 - s)^{\frac{1-\alpha}{1-\eta}} - (t_2 - s)^{\frac{1-\alpha}{1-\eta}} ds \right]^{1-\eta} \\ &\times \|m\|_{L^{1/\eta}[t_1,t_2]} \\ &= \tfrac{\alpha M L_1 \rho(\|x_t\|)}{\Gamma(1+\alpha)(1+\kappa)^{1-\eta}} \left[t_1^{1+\kappa} - t_2^{1+\kappa} + (t_2 - t_1)^{1+\kappa} \right]^{1-\eta} \\ &\leq \tfrac{\alpha M L_1 \rho(\|x_t\|)}{\Gamma(1+\alpha)(1+\kappa)^{1-\eta}} (t_2 - t_1)^{(1+\kappa)(1-\eta)} \end{aligned} \tag{44}$$

We finish by repeating the same calculations with the expression described in I_3 at Eq. (35). We have to do the following results after the application of the Holder,

$$\begin{aligned} I_1 &= \Big\| \int_0^{t_1} (t_1 - s)^{\alpha-1} \left[S_\alpha (t_2 - s) - S_\alpha (t_1 - s) \right] \\ &\times Cx(s-h) ds \Big\| \\ &\leq \tfrac{L_1 \rho(\|x_t\|) t_1^{(1+\kappa)(1-\eta)}}{(1+\kappa)^{1-\eta}} \\ &\times \sup_{s\in[0,t_1]} \left[S_\alpha (t_2 - s) - S_\alpha (t_1 - s) \right]. \end{aligned} \tag{45}$$

The first remark is that the resolvent operator S_α, follows also the continuity of the operator $A^{1-\beta} S_\alpha$. Then from Eq.(43) to Eq.(45), we observe that $t_2 \to t_1$ thus $\|\Phi_c x(t_2) - \Phi_c x(t_1)\|_X \to 0$. That ends the proof of the third step by concluding that the $\{\phi x : x \in B_r\}$ is relatively compact. □

4. Illustrative example

In this section we add an illustrative example to illustrate the findings of our paper, we take the partial neutral functional fractional differential equation under Caputo derivative described by the form that

$$D^\alpha \left[x(t,z) - \int_0^\pi g(z,y) x_t(\theta, y) dy \right] = \frac{\partial^2 x(t,z)}{\partial z^2} + f(t, x_t) \tag{46}$$

$$x(t,0) = x(t,\pi) = 0, \quad 0 < t \leq 1, \tag{47}$$

$$x(\theta, z) = \phi(\theta, z), \quad -r \leq \theta \leq 0, \tag{48}$$

where the function g is an continuous function and measurable, $x_t(\theta,z) = x(t+\theta, z)$, $\phi(\theta,z)$ is also assumed to be continuous and the function f is specified later in the example. The next section will be to write the previous equation in terms of Eq. (1) representing our mean result. The second step will be to verify all the assumptions considered in this paper.

For the rest we suppose that $X = L^2([0,\pi])$. We define an operator $A : D(A) \subset X \to X$ such that $Av = v''$ where the considered domain is defined by the set

$$\begin{aligned} D(A) = \{ v \in X : v; v' \text{are absolutely continuous;} \\ v'' \in X : v(0) = v(\pi) = 0 \}. \end{aligned} \tag{49}$$

Thus and the operator defined by A generates a compact semigroup $T(t)$ in X and it is having some properties summarized as the following properties. We have that $T(t)v = \sum_{n=1}^\infty e^{n^2 t}(v, e_n) e_n$ where $v \in X$. The second properties is that for each $v \in X$, we have that $A^{-1/2} v = \sum_{n=1}^\infty \frac{1}{n}(v, e_n) e_n$. The third properties is that the operator $A^{1/2}$ can be obtained by the form that $A^{1/2} v = \sum_{n=1}^\infty n(v, e_n) e_n$ where where the set $D(A^{1/2}) = \{ v \in X : \sum_{n=1}^\infty n(v, e_n) e_n \in X \}$. Note that in the previous part we works with $e_n(z) = \sqrt{\frac{2}{\pi}} \sin(nz)$ where $0 \leq z \leq \pi$. It is not hard to see that the family $\{e_n\}$ with $n = 1, 2, 3, \ldots$ represent an orthonormal base for our set X. Let consider that $Bx(t-h)(z) = \int_0^\pi g(z,y) x_t(\theta, y) dy$. We assume that g is continuously differentiable and satisfies the condition that $b(t,.,0) = b(t,.,\pi) = 0$. Let the function f is Lipschitz continuous according to the following properties that $\|f(t,\xi_1) - f(t,\xi_2)\| \leq a \|\xi_1 - \xi_1\|$ where $\xi_1, \xi_2 \in R$. Finally, the fractional differential equation represented by Caputo derivative of order $\alpha = 0.5$ can be presented as the form

$$D_t^{\alpha}\left[x\left(t\right)-Bx\left(t-h\right)\right]$$
$$=Ax\left(t\right)+f\left(t,x_t\right)\quad t\geq 0 \tag{50}$$
$$x_0=\varphi\in\mathcal{C}_X \tag{51}$$

where $f\left(t,x_t\right) = \frac{1}{t^{1/3}}\sin x_t$. we can see that the function f is Lipchitz continuous, and it is trivial to see the assumption (A2) is satisfied. The verification of the last assumption assumption (A3), the sketch of the proof can be found in, we proceed as the following, let that $\int_0^{\pi}\int_0^{\pi} g^2(z,y)dydz < \infty$, furthermore we consider that $\frac{\partial}{\partial z}g(z,y)$ is measurable too and satisfying the conditions that $g(0,y)=g(\pi,y)=0$ and $\delta = \left(\int_0^{\pi}\int_0^{\pi}\left[\frac{\partial}{\partial z}g(z,y)\right]^2 dydz\right)^{\frac{1}{2}} < \infty$. For simplification in our calculations we let that $\int_0^{\pi} g(z,y)v(z)dy = U_h v\left(z\right)$. Using the fact that $\int_0^{\pi}\int_0^{\pi} g^2(z,y)dydz < \infty$ generate that U_h is bounded into the set X and we have that $U_h v\in D\left(A^{1/2}\right)$ and $\left\|A^{1/2}U_h\right\|<\infty$. The second assumption $\delta = \left(\int_0^{\pi}\int_0^{\pi}\left[\frac{\partial}{\partial z}g(z,y)\right]^2 dydz\right)^{\frac{1}{2}} < \infty$ and using the definition of e_n, we get the following relationship

$$\begin{aligned}(U_h(v),e_n) &= \int_0^{\pi} e_n\left(\int_0^{\pi} g(z,y)x_t\left(\theta,y\right)dy\right)dz\\ &= \frac{1}{n}\sqrt{\frac{2}{\pi}}\left(\int_0^{\pi}\frac{\partial}{\partial z}g(z,y)x_t dy,\cos nz\right)\end{aligned} \tag{52}$$

Thus $\left\|\int_0^{\pi}\frac{\partial}{\partial z}g(z,y)v(z)dy\right\| \leq \delta$ and then we get $\left\|A^{1/2}U_h\left(v\right)\right\| \leq \delta$, wchich generate the satisfaction of the assumption $(A3)$. The present results can be compared with the results in the same direction in [25, 32, 33]. The difference is the used resolvent operators. The nature of the resolvent depends on the used problem and the used fractional operators. It is important to see that the resolvent operators defined in this paper for existence depend on the order of the fractional operator. But in general, the works are in good agreement.

5. Conclusion

In this paper, we have focussed on the existence of the unique solution of the partial neutral functional fractional differential equation described by the Caputo derivative. The novelties of this work were the use of the fractional resolvents operators to arrive to prove existence via fixed point theorem. The present investigation can be made with the other operators by in our idea it should depend on new fractional resolvent operators. Their definitions should differ according to the fractional operators. This idea can be an open problem for future investigations.

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