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RESEARCH ARTICLE

New generalized integral transform via Dzherbashian–Nersesian fractional operator

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ABSTRACT

In this paper, we derive a new generalized integral transform on Dzherbashian–Nersesian fractional operator and give some special cases. We make a generalization of the application of integral transformations to different fractional operators, where several previous results can be invoked from a single relation. We also use the new results obtained to solve some fractional differential equations involving the recent revival of Dzherbashian-Nersesian fractional operators.

1. Introduction

Fractional calculus is used to investigate fractional integral operators and derivatives of any order. This is a well-known idea that has shown to be quite helpful in explaining many memory and internal process phenomena. These phenomena are extremely valuable for tackling a variety of problems related to science and technology, including physics, chemistry, biology, cybernetics, economics, electronics, and many more domains that have emerged in recent decades. We refer the readers to the following references for more details [1–7], and [8].

Despite its long history, fractional calculus still has a large number of unsolved remaining problems, from both theoretical and applied perspectives (see references [4, 9] and [10]). Over the last few years, various forms of fractional operators have been introduced in the scientific literature [11].

Obtaining generalized fractional operators is a major challenge in the present time. It was recently demonstrated that the use of a large class of fractional operators yields interesting mathematical results. However, in practice, real data is important in determining which fractional operator produces the best results for a certain kind of models.

In this context, the interesting Dzherbashian–Nersesian fractional operator was first proposed in 1968 [12], but was rarely studied. This is a generalized fractional operator for which the fractional operators; Riemann–Liouville (R–L) fractional integral, Caputo derivative and Hilfer (HF)

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are special cases, that can be obtained by a special setting of parameter values in the fractional operator of Dzherbashian - Nersesian [13, 14].

Integral transformations are mathematical operations that transform a function from one domain to another. They are tools used in various fields to simplify difficulties and discover answers. These transformations map a function from the time or space domain to the frequency or complex plane domain, making it easier to study and solve problems.

These integral transformations, on the other hand, are important for the simplification they provide, most commonly in the processing of differential equations under precise boundary conditions. A well-chosen set of integral transformations aid in the conversion of differential equation and integro-differential equation into easily solvable algebraic equations. To complete the procedure, the solution that is found is the transform of the solution of the original differential equation, which must be inverted. Many integral Laplace transformations have been introduced over the last two decades, including the Sumudu [15], Elzaki [16], Natural [17], Aboodh [18], Pourreza [19, 20], Mohand [21], Sawi [22], Shehu [23] and Kamal transformations [24].

In ref. [25], H. Jafari proposed a new integral transform in 2021. This new generalized integral transform, which will be abbreviated as NG-Transform in the remainder of this article, includes a variety of the family of Laplace transforms, and is particularly useful for solving differential equations and integro-differential equations. Recently, some authors applied this transform to some fractional operators, such as: AB fractional derivative [38], CF fractional derivative [26], R–L fractional integral and Caputo derivative of fractional order [41], Hilfer–Prabhakar fractional derivatives in [27], and Costa et al. in [28] for the k–Hilfer fractional derivative. On the other hand, others have applied this transform in several works, e.g., [29–33], and [34].

Based on the above, the motivation of this research is to investigate and demonstrate certain important properties of NG-Transform that have not before been employed as solutions to some types of Cauchy problems involving the Dzherbashian–Nersesian fractional operators.

This paper is organised into five sections: Section 2 covers important key definitions, properties, theorems and lemmas, and some useful results about the new generalised integral transform, which are used throughout the remainder of the paper. In section 3, we provide some key findings derived via the NG-Transform of the Dzherbashian–Nersesian fractional operator, and give some special cases in section 4. Various applications of the Dzherbashian–Nersesian fractional operator implementing the NG-Transform to solve some Cauchy-type problems are given in Section 5. The conclusion follows next in Section 6.

2. Preliminary

For the sake of clarity, we give in following paragraphs some fundamental definitions and concepts. There are also interesting results regarding the new transform.

Definition 1. *(see [35]). The left sided R–L fractional integral $I^{\alpha}_{0^+,t}$ of $v \in L^1([a,b])$ is given by the following formula*

$$I^{\alpha}_{0^+,t} v(t) = \int_0^t \frac{(t-\tau)^{\alpha-1} v(\tau)}{\Gamma(\alpha)} d\tau, \quad t>0, \alpha>0. \tag{1}$$

Definition 2. *(see [36]). The Caputo derivative of fractional order α for the function $v(t)$ is described as*

$$^{C}D^{\alpha}_{a^+} v(t) = \begin{cases} \int_a^t \frac{(t-\tau)^{m-\alpha-1} v^{(m)}(\tau)}{\Gamma(m-\alpha)} d\tau, \\ \text{if } 0<m-1<\alpha<m, \\ \frac{d^m}{dt^m} v(t) \ \text{ if } \alpha = m. \end{cases} \tag{2}$$

Definition 3. *(see [37]). Let $\alpha \in (0,1)$, $\sigma \in [0,1]$, $v \in L^1[a,b]$, $v * \frac{t^{(1-\sigma)(1-\alpha)}}{\Gamma((1-\sigma)(1-\alpha))} \in AC^1[a,b]$. The Hilfer fractional derivative (H–FD) is expressed as*

$$D^{\alpha,\sigma}_{a^+} v(t) = \left(I^{\sigma(1-\alpha)}_{a^+} \frac{d}{dt} \left(I^{(1-\sigma)(1-\alpha)}_{a^+} v \right) \right)(t). \tag{3}$$

Remark 1.

for $\sigma = 0$ in (3), the H–FD coincides with the fractional operator $D^{\alpha}_{0^+,t}$ (2).

Definition 4. *(see [12]). The fractional operator Dzherbashian–Nersesian $D^{\alpha_m}_{0^+,t}$ of order m is expressed as*

$$D^{\alpha_m}_{0^+,t} v(t) = I^{1-\sigma_m}_{0^+,t} D^{\sigma_{m-1}}_{0^+,t} D^{\sigma_{m-2}}_{0^+,t} ... D^{\sigma_1}_{0^+,t} D^{\sigma_0}_{0^+,t} v(t), \quad t>0, m \in \mathbb{N}, 0<\alpha_m \le m, \tag{4}$$

where

$$\alpha_m = \sum_{r=0}^{m} \sigma_r - 1, \ \sigma_r \in (0,1]. \tag{5}$$

Special cases

(1) for $m = 1$ in (4), we have (see [12])

$$D^{\alpha_1}_{0^+,t} v(t) = I^{1-\sigma_1}_{0^+,t} D^{\sigma_0}_{0^+,t}. \tag{6}$$

(2) For $\sigma_1 = \sigma_2 = ... = \sigma_m = 1$ and $\sigma_0 = 1 + \alpha - m$, where $\sigma_0 \in (0,1)$, in (4) we have (see [14])

$$D^{\alpha_m}_{0^+,t} v(t) = \frac{d^m}{dt^m} I^{m-\alpha}_{0^+,t} v(t) = D^{\alpha}_{0^+,t} v(t). \quad (7)$$

In this case, Dzherbashian–Nersesian fractional operator reduces to the fractional operator $D^{\alpha}_{0^+,t}$ of order $m-1<\alpha\le m$ which given by the equation 2.

(3) For $\sigma_0 = \sigma_2 = ... = \sigma_{m-1} = 1$ and $\sigma_m = 1 + \alpha - m$, where $\sigma_m \in (0,1)$, in (4) we have (see [14])

$$D^{\alpha_m}_{0^+,t} v(t) = I^{m-\alpha}_{0^+,t} \frac{d^m}{dt^m} v(t) = {}^{c}D^{\alpha}_{0^+,t} v(t). \quad (8)$$

This case, Dzherbashian–Nersesian fractional operator interpolates to the CFD operator ${}^{C}D^{\alpha}_{0^+,t}$ of order α.

(4) For $\sigma_m = 1 - \sigma m - +\sigma\alpha$, $\sigma_0 = 1 + \alpha - m - \sigma_m$, where $\sigma_0, \sigma_m \in (0,1)$ and $\sigma_1 = \sigma_2 = ... = \sigma_{m-1} = 1$ in (4) we have (see [14])

$$D^{\alpha_m}_{0^+,t} v(t) = \frac{d^m}{dt^m} v(t) = D^{\alpha}_{0^+,t} v(t). \quad (9)$$

In this instance, the Dzherbashian–Nersesian fractional operator is reduced to HFD of order $m-1<\alpha<m$, and type $0<\sigma<1$.

Definition 5. *(see [25]) Let $v(t)$ be a integrable function defined for $t \ge 0$, $\zeta(p) \neq 0$ and $\psi(p)$ are positive real functions. The NG-Transform of $v(t)$ is expressed as*

$$\begin{aligned}\mathcal{T}_{\mathcal{NG}}\{v(t),p\} &= \mathfrak{F}(p)\\ &= \xi(p)\textstyle\int_0^\infty v(t)\exp(-\psi(p)t)\,dt,\end{aligned} \quad (10)$$

provided the integral exists for some $\psi(p)$.

The new transform $\mathfrak{F}(p)$ exists for all $\psi(p) > \phi$ and has numerous properties such as linear operator for example, a more detailed discussion about this can be found in [25]. Inversion formula of (10), is given by [38]

$$\begin{aligned}\mathfrak{F}^{-1}(p) &= \mathcal{T}_{\mathcal{NG}}{}^{-1}\left\{\xi(p)\textstyle\int_0^\infty v(t)\exp(-\psi(p)t)\,dt\right\}\\ &= v(t).\end{aligned} \quad (11)$$

This new integral transform can be easily implemented directly to an appropriate problem by specifically selecting $\xi(p)$ and $\psi(p)$. In the table 1, we mention some of them.

Theorem 1. *(See [25]). Let $\xi(p), \psi(p) > 0$, the NG-Transform of derivatives of $v(t)$ is defined as*

$$\begin{aligned}\mathcal{T}_{\mathcal{NG}}\left\{v^{(m)}(t),p\right\} &= (\psi(p))^m \Big[\mathfrak{F}(p)\\ &-\xi(p)\textstyle\sum_{r=0}^{m-1}(\psi(p))^{-1-r}v^{(r))}(0)\Big], \forall m \in \mathbb{N}.\end{aligned} \quad (12)$$

Theorem 2. *(See [25, 38]). Let $\mathfrak{F}_1(p) = \mathcal{T}_{\mathcal{NG}}\{v_1,p\}$ and $\mathfrak{F}_2(p) = \mathcal{T}_{\mathcal{NG}}\{v_2,p\}$, then*

$$\begin{aligned}\mathcal{T}_{\mathcal{NG}}\{v_1 \star v_2, p\} &= \textstyle\int_0^\infty v_1(t)\,v_2(t-z)\,dz\\ &= \frac{1}{\phi(s)}\mathfrak{F}_1(p).\mathfrak{F}_2(p).\end{aligned} \quad (13)$$

Definition 6. *(See [10, 39, 40]). For any $w \in \mathbb{C}$, The usual Mittag-Leffler (ML) function is defined as*

$$E_\eta(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\eta i + 1)}, \quad w \in \mathbb{C},\ \mathrm{Re}(\eta) > 0. \quad (14)$$

where Re *denotes the real part. The 2-parameters ML function is given by [10]*

$$E_{\eta,\beta}(w) = \sum_{i=0}^{\infty} \frac{w^i}{\Gamma(\eta i + \beta)}, w \in \mathbb{C}, \quad (15)$$
$$Re(\eta) > 0, Re(\sigma) > 0,$$

such that $E_{\eta,1}(z) = E_\eta(w)$, and the 3-parameters ML function is stated as [39, 40]

$$\begin{aligned}E^{\gamma}_{\eta,\beta}(w) = \tfrac{1}{\Gamma(\gamma)}\textstyle\sum_{i=0}^{\infty}\tfrac{\Gamma(\gamma+i)}{\Gamma(\eta i+\beta)}\tfrac{w^i}{i!}, \quad w \in \mathbb{C},\\ Re(\eta) > 0, Re(\beta) > 0, Re(\gamma) > 0.\end{aligned} \quad (16)$$

Lemma 1. *(See [38]). Let $\eta \in (0,1)$ and $z \in \mathbb{R}$ such that $0 < \psi(p) < |z|^{\frac{1}{\eta}}$, then*

$$\mathcal{T}_{\mathcal{NG}}\left\{t^{\sigma-1}E^{\gamma}_{\eta,\sigma}(zt^\eta),p\right\} = \frac{\xi(p)}{\psi(p)^\sigma}\left(1 - \frac{z}{\psi(p)^\eta}\right)^{-\gamma}. \quad (17)$$

Proposition 1. *(See [38, 41]). The NG-Transform of t^α is expressed as*

$$\mathcal{T}_{\mathcal{NG}}\{t^\alpha,p\} = \Gamma(\alpha+1)\frac{\xi(p)}{\psi(p)^{\alpha+1}}, \ \alpha > 0. \quad (18)$$

Theorem 3. *(See [41]). The NG-Transform of RLF integral of v is presented as*

$$\mathcal{T}_{\mathcal{NG}}\left\{I^{\alpha}_{0^+,t}v(t),p\right\} = \frac{\mathfrak{F}(p)}{\psi(p)^\alpha}, \quad (19)$$

where $\alpha > 0$ and $\mathfrak{F}(p)$ is denoted by the NG-Transform of the function $v(t)$.

3. Main results

In the following part, we will derive the NG-Transform on the Dzherbashian–Nersesian fractional operator and discuss some particular cases. Let $v(t) \in \mathcal{A}$ with $\mathcal{T}_{\mathcal{NG}}\{v(t),p\} = \mathfrak{F}(p)$.

Lemma 2. *The NG-Transform of RLF derivative of v is given as follows*

$$\begin{aligned}\mathcal{T}_{\mathcal{NG}}\left\{D^{\alpha}_{0^+,t}v(t),p\right\} &= [\psi(p)]^\alpha\,\mathfrak{F}(p)\\ &-\xi(p)\textstyle\sum_{k=0}^{m-1}[\psi(p)]^k\left[D^{\alpha-k-1}_{0^+,t}v(t)\right]_{t=0^+},\\ &m-1<\alpha\le m.\end{aligned} \quad (20)$$

Table 1. Relation between the NG-Transform and other integral transforms.

Transform	Definition	Jafari transform
Laplace [7]	$\mathcal{L}\{v(t),p\} = \int_0^\infty v(t)e^{-pt}dt$	$\xi(p) = 1$ and $\psi(p) = p$
Sumudu [15]	$\mathcal{S}\{v(t),p\} = \frac{1}{p}\int_0^\infty v(t)e^{-pt}dt$	$\xi(p) = \psi(p) = \dfrac{1}{p}$
Elzaki [16]	$E\{v(t),p\} = p\int_0^\infty v(t)e^{\frac{t}{p}}$	$\xi(p) = 1$ and $\psi(p) = \dfrac{1}{p}$
Natural [17]	$N\{v(t),p,u\} = p\int_0^\infty v(ut)e^{-pt}dt$	$\xi(p) = u$ and $\psi(p) = \dfrac{p}{u}$
Aboodh [18]	$A\{v(t),p\} = \frac{1}{p}\int_0^\infty v(t)e^{-pt}dt$	$\xi(p) = \dfrac{1}{p}$ and $\psi(p) = 1$
Mohand [21]	$\mathcal{M}\{v(t),p\} = p^2\int_0^\infty v(t)e^{-pt}dt$	$\xi(p) = p^2$ and $\psi(p) = p$
Sawi [22]	$\mathcal{R}\{v(t),p\} = \frac{1}{p^2}\int_0^\infty v(t)e^{-\frac{t}{p}}dt$	$\psi(p) = \dfrac{1}{p}$ and $\xi(p) = \psi(p)^2$
Kamel [24]	$\mathcal{K}\{v(t),p\} = p\int_0^\infty v(t)e^{-\frac{t}{p}}dt$	$\xi(p) = 1$ and $\psi(p) = \dfrac{1}{p}$
Shehu [23]	$\mathcal{SH}\{v(t),p,u\} = \int_0^\infty v(t)e^{-\frac{pt}{u}}dt$	$\xi(p) = 1$ and $\psi(p) = \dfrac{p}{u}$

Proof. Using the NG-Transform on both sides of (2) and subsequently the relation (12), we get

$$\begin{aligned}
&\mathcal{T_{NG}}\left\{D^{\alpha}_{0^+,t}v(t),p\right\}\\
&= \mathcal{T_{NG}}\left\{\frac{d^m}{dt^m}I^{m-\alpha}_{0^+,t}v(t),p\right\}\\
&= [\psi(p)]^m\,\mathcal{T_{NG}}\left\{I^{m-\alpha}_{0^+,t}v(t),p\right\}\\
&\quad -\xi(p)\textstyle\sum_{k=0}^{m-1}[\psi(p)]^k\\
&\quad \times\left[\left(\frac{d}{dt}\right)^{(m-1-k)}I^{m-\alpha}_{0^+,t}v(t)\right]_{t=0^+}\\
&= [\psi(p)]^{\alpha}\,\mathfrak{F}(p)\\
&\quad -\xi(p)\textstyle\sum_{k=0}^{m-1}[\psi(p)]^k\left[D^{\alpha-k-1}_{0^+,t}v(t)\right]_{t=0^+}.
\end{aligned}$$

This is the desired result (20). □

Remark 2. *Lemma (2) generalizes some results that already exist, so for example:*

(1) *The Laplace integral transform of R–L fractional operator, which is given by equation (2), is obtained when $\xi(p) = 1$ and $\psi(p) = p$ (see [35]), i.e.*

$$\mathcal{L}\left\{D^{\alpha}_{0^+,t}v(t),p\right\} = p^{\alpha}\mathcal{L}\{v(t),p\} - \textstyle\sum_{k=0}^{m-1}p^k\left[D^{\alpha-k-1}_{0^+,t}v(t)\right]_{t=0^+}.$$

(2) *If we take $\xi(p) = \psi(p) = \dfrac{1}{p}$, yields Sumudu's integral transform witch is mentioned in the work [42], i.e.,*

$$\mathcal{S}\left\{D^{\alpha}_{0^+,t}v(t),p\right\} = p^{-\alpha}\mathcal{S}\{v(t),p\} - \textstyle\sum_{k=0}^{m-1}p^{-(k+1)}\left[D^{\alpha-k-1}_{0^+,t}v(t)\right]_{t=0^+}. \tag{21}$$

The NG-Transform of the Dzherbashian–Nersesian fractional operator is obtained by the following theorem.

Theorem 4. *The NG-Transform of Dzherbashian–Nersesian fractional operator of order α_m $(0 < \alpha_m < m)$ is expressed as follows*

$$\begin{aligned}
&\mathcal{T_{NG}}\left\{D^{\alpha_m}_{0^+,t}v(t),p\right\}\\
&= [\psi(p)]^{\alpha_m}\,\mathcal{T_{NG}}\{v(t),p\}\\
&\quad -\xi(p)\sum_{r=1}^{m}[\psi(p)]^{\alpha_m-\alpha_{m-r}-1}\left[D^{\alpha_{m-r}}_{0^+,t}v(t)\right]_{t=0^+}.
\end{aligned}\tag{22}$$

Proof. Using the Dzherbashian–Nersesian fractional operator formulation, the NG-Transform of the Equation (4), and the lemma (2), we get

$$\begin{aligned}
&\mathcal{T_{NG}}\left\{D^{\alpha_m}_{0^+,t}v(t),p\right\}\\
&= \mathcal{T_{NG}}\left\{I^{1-\sigma_m}_{0^+,t}D^{1-\sigma_{m-1}}_{0^+,t}D^{1-\sigma_{m-2}}_{0^+,t}...D^{1-\sigma_1}_{0^+,t}D^{1-\sigma_0}_{0^+,t}v(t),p\right\}\\
&= [\psi(p)]^{\sigma_m-1}\,\mathcal{T_{NG}}\left\{D^{\sigma_{m-1}}_{0^+,t}D^{\sigma_{m-2}}_{0^+,t}...D^{1-\sigma_1}_{0^+,t}D^{1-\sigma_0}_{0^+,t}v(t),p\right\}\\
&= [\psi(p)]^{\sigma_m-1+\sigma_{m-1}}\,\mathcal{T_{NG}}\left\{D^{\sigma_{m-2}}_{0^+,t}...D^{\sigma_1}_{0^+,t}D^{\sigma_0}_{0^+,t}v(t),p\right\}\\
&\quad -\xi(p)[\psi(p)]^{\sigma_m-1}\left[D^{\sigma_{m-1}-1}_{0^+,t}D^{\sigma_{m-2}}_{0^+,t}...D^{\sigma_1}_{0^+,t}D^{\sigma_0}_{0^+,t}v(t)\right]_{t=0^+}.
\end{aligned}$$

Continuing in the similar manner, we have

$$\begin{aligned}
&\mathcal{T_{NG}}\left\{D^{\alpha_m}_{0^+,t}v(t),p\right\}\\
&= [\psi(p)]^{\sum_{r=0}^{m}\sigma_r-1}\,\mathcal{T_{NG}}\{v(t),p\}\\
&\quad -\xi(p)\sum_{r=0}^{m-1}[\psi(p)]^{\sigma_{m-r}}\left[D^{\sigma_{m-r}-1}_{0^+,t}v(t)\right]_{t=0^+}.
\end{aligned}$$

Using Formula $\alpha_m = \sum_{r=0}^{m}\sigma_r - 1$ and after some calculations, we finally get the desired result (22). □

4. Special cases

The following corollary presents a few important results in different cases of the Jafari integral transform of the Dzherbashian–Nersesian fractional operator.

Corollary 1. *The NG-Transform of the Dzherbashian–Nersesian fractional operator of order* α_m ($0 < \alpha_m < m$) *is expressed as*

$$\begin{aligned} &\mathcal{T}_{\mathcal{NG}}\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} \\ = \; & [\psi(p)]^{\alpha_m} \Big(\mathcal{T}_{\mathcal{NG}}\{v(t), p\} \\ & - \xi(p) \sum_{r=1}^{m} [\psi(p)]^{-\alpha_{m-r}-1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}\Big). \end{aligned}$$

- *When* $\xi(p) = 1$ *and* $\psi(p) = p$, *we obtain the Laplace transform of the fractional operator of Dzherbashian–Nersesian, which is studied in [14], i.e.,*

$$\mathcal{L}\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = p^{\alpha_m} \Big(\mathcal{L}\{v(t), p\} - \sum_{r=1}^{m} p^{-\alpha_{m-r}-1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]\Big)_{t=0^+},$$

  where $\mathcal{L}\{v(t), p\}$ *denotes the Laplace transform of a function* $v(t)$.
- *The Sumudu integral transform of the Dzherbashian–Nersesian fractional operator is obtained, when* $\xi(p) = \frac{1}{p}$ *and* $\psi(p) = \frac{1}{p}$. *If we represent the Sumudu transform of a function* $v(t)$ *as* $\mathcal{S}\{v(t), p\}$, *we get*

$$\mathcal{S}\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = p^{-\alpha_m} \mathcal{S}\{v(t), p\} - \sum_{r=1}^{m} p^{\alpha_{m-r}-\alpha_m} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}. \tag{23}$$

- *When* $\xi(p) = 1$ *and* $\psi(p) = \frac{p}{u}$, *we get the formula for the Shehu transformation of the Dzherbashian–Nersesian fractional operator. If* $V(p,u)$ *denotes the Shehu transform of a function* $v(t)$, *then*

$$\mathcal{SH}\left\{D_{0^+,t}^{\alpha_m} v(t), p, u\right\} = \left(\tfrac{p}{u}\right)^{\alpha_m} \Big(V(p,u) - \sum_{r=1}^{m} \left(\tfrac{u}{p}\right)^{\alpha_{m-r}+1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}\Big). \tag{24}$$

- *The Elzaki transform of the Dzherbashian–Nersesian fractional operator is obtained, when* $\xi(p) = 1$ *and* $\psi(p) = \frac{1}{p}$, *we have*

$$E\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = p^{-\alpha_m} E\{v(t), p\} - \sum_{r=1}^{m} p^{\alpha_{m-r}-\alpha_m+1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}, \tag{25}$$

  where the Elzaki transform of $v(t)$ *is denotes by* $E\{v(t), p\}$.
- *When* $\xi(p) = \frac{1}{p^2}$ *and* $\psi(p) = \frac{1}{p}$, *the NG-Transform yields in the Sawi transform of the Dzherbashian–Nersesian fractional operator. If we designate the Sawi transform of* $v(t)$ *as* $\mathcal{R}\{v(t), p\}$, *then*

$$\mathcal{R}\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = p^{-\alpha_m} \mathcal{R}\{v(t), p\} - \sum_{r=1}^{m} p^{\alpha_{m-r}-\alpha_m-1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}. \tag{26}$$

- *When* $\xi(p) = \frac{1}{p}$ *and* $\psi(p) = 1$, *the NG-Transform produces the Aboodh integral transform of the Dzherbashian–Nersesian fractional operator. If the Aboodh transform of* $v(t)$ *is denoted by* $A\{v(t), p\}$, *we get*

$$A\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = A\{v(t), p\} - \frac{1}{p} \sum_{r=1}^{m} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}. \tag{27}$$

- *The Natural integral transform of the Dzherbashian–Nersesian fractional operator is obtained when* $\xi(p) = u$ *and* $\psi(p) = \frac{p}{u}$. *If the Natural integral transform of* $v(t)$ *is represented by* $W(p,u)$, *then*

$$N\left\{D_{0^+,t}^{\alpha_m} v(t), p, u\right\} = \left(\frac{p}{u}\right)^{\alpha_m} W(p,u) - u \sum_{r=1}^{m} \left(\frac{p}{u}\right)^{\alpha_m-\alpha_{m-r}-1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}. \tag{28}$$

- *When* $\xi(p) = p^2$ *and* $\psi(p) = p$, *the NG-Transform yields the Mohand transform to the Dzherbashian–Nersesian fractional operator. If we represent the Mohand transform of* $v(t)$ *by* $\mathcal{M}\{v(t), p\}$, *we obtain*

$$\mathcal{M}\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = p^{\alpha_m} \Big(\mathcal{M}\{v(t), p\} - \sum_{r=1}^{m} p^{1-\alpha_{m-r}} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}\Big). \tag{29}$$

- *When* $\xi(p) = 1$ *and* $\psi(p) = \frac{1}{p}$, *the NG-Transform yields the Kamal integral transform of the fractional operator of Dzherbashian–Nersesian. If we denote the Kamal integral transform of a function* $v(t)$ *by* $\mathcal{K}\{v(t), p\}$, *then*

$$\mathcal{K}\left\{D_{0^+,t}^{\alpha_m} v(t), p\right\} = p^{-\alpha_m} \mathcal{K}\{v(t), p\} - \sum_{r=1}^{m} p^{\alpha_{m-r}-\alpha_m+1} \left[D_{0^+,t}^{\alpha_{m-r}} v(t)\right]_{t=0^+}. \tag{30}$$

Proof. The preceding conclusions are clearly demonstrated by the Equation (22) and the table 1.

□

Remark 3. *If $\sigma_1 = \sigma_2 = ... = \sigma_m = 1$ and $\sigma_0 = 1 + \alpha - m$, where $\sigma_0 \in (0,1)$, then, the NG-Transform of RLF derivative of order $\alpha \in (m-1,m)$ is obtained , i.e.,*

$$\mathcal{T}_{\mathcal{NG}}\left\{D^{\alpha}_{0^+,t}v(t),p\right\} = [\psi(p)]^{\alpha}\,\mathcal{T}_{\mathcal{NG}}\{v(t),p\} \\ -\xi(p)\sum_{r=0}^{m-1}[\psi(p)]^{m-r-1}\left[D^{r}_{0^+,t}I^{m-\alpha}_{0^+,t}v(t)\right]_{t=0^+}. \tag{31}$$

Remark 4. *When $\sigma_0 = \sigma_2 = ... = \sigma_{m-1} = 1$ and $\sigma_m = 1+\alpha-m$, where $\sigma_m \in (0,1)$, Equation (22) interpolates The NG-Transform of CFD operator of order $\alpha \in (m-1,m)$, i.e.,*

$$\mathcal{T}_{\mathcal{NG}}\left\{{}^{c}D^{\alpha}_{0^+,t}v(t),p\right\} = [\psi(p)]^{\alpha}\,\mathcal{T}_{\mathcal{NG}}\{v(t),p\} \\ -\xi(p)\sum_{r=0}^{m-1}[\psi(p)]^{\alpha-r-1}\left[D^{r}_{0^+,t}v(t)\right]_{t=0^+}. \tag{32}$$

In this particular case, it was studied in [41] .

Remark 5. *For $\sigma_m = 1+\sigma(\alpha-m)$, $\sigma_0 = 1+(\alpha-m)(1-\sigma)$, where $\sigma_0,\sigma_m \in (0,1)$ and $\sigma_1 = \sigma_2 = ... = \sigma_{m-1} = 1$ in (4, we gain The NG-Transform of Hilfer–fractional derivative of order $\alpha \in (m-1,m)$ and $0<\sigma<1$, i.e.,*

$$\mathcal{T}_{\mathcal{NG}}\left\{D^{\alpha}_{0^+,t}v(t),p\right\} = [\psi(p)]^{\alpha}\,\mathcal{T}_{\mathcal{NG}}\{v(t),p\} \\ -\xi(p)\sum_{r=0}^{m-1}[\psi(p)]^{r-\sigma(m-\alpha)} \\ \times\left[D^{m-r-1}_{0^+,t}I^{(m-\alpha)(1-\sigma)}_{0^+,t}v(t)\right]_{t=0^+}. \tag{33}$$

5. Applications

In this part, we will illustrate how to resolve certain Cauchy-type problems employing the Dzherbashian–Nersesian fractional operator and the NG-Transform.

Example 1. *Consider the following problem (See [12]):*

$$\begin{cases} D^{\alpha_m}v(t) = u(t), & t\in(0,b),\quad b>0, \\ D^{\alpha_r}v(t)|_{t=0} = v_r^0, & r = 0,1,...,m-1. \end{cases} \tag{34}$$

$u(t)$ denotes an arbitrary function such that $I^{\alpha_m}_{0^+,t}u(t)$ exists and $\{v_r^0\}_{r=0}^{m-1}$ is a specified set of real numbers. In the work [12], the existence of the solution and its uniqueness are proved, and they found the solution explicitly in a direct way. Here we will try to find the solution by employing The NG-Transform of the fractional operator of Dzherbashian–Nersesian.

To start, we apply the NG-Transform to both sides of (34) *to get*

$$\mathcal{T}_{\mathcal{NG}}\{D^{\alpha_m}v(t),p\} = \mathcal{T}_{\mathcal{NG}}\{u(t),p\},$$

which yields

$$[\psi(p)]^{\alpha_m}\,\mathcal{T}_{\mathcal{NG}}\{v(t),p\} \\ -\xi(p)\sum_{r=1}^{m}[\psi(p)]^{\alpha_m-\alpha_{m-r}-1}\left[D^{\alpha_m}v(t)\right]_{t=0} \\ = \mathcal{T}_{\mathcal{NG}}\{u(t),p\}$$

or

$$\mathcal{T}_{\mathcal{NG}}\{v(t),p\} = \xi(p)\sum_{r=1}^{m}[\psi(p)]^{-\alpha_{m-r}-1}v_m^0 \\ +[\psi(p)]^{-\alpha_m}\,\mathcal{T}_{\mathcal{NG}}\{u(t),p\}. \tag{35}$$

When implementing the general inverse transformation of (35) *with* (18) *and* (19)*, we obtain*

$$v(t) = \mathcal{T}_{\mathcal{NG}}{}^{-1}\left(\sum_{r=1}^{m}\frac{\xi(p)}{[\psi(s)]^{\alpha_{m-r}+1}}v_m^0\right) \\ +\mathcal{T}_{\mathcal{NG}}{}^{-1}\left(\frac{\mathcal{T}_{\mathcal{NG}}\{u(t),p\}}{[\psi(s)]^{\alpha_m}}\right) \\ = \sum_{r=1}^{m}v_m^0\frac{t^{\alpha_{m-r}}}{\Gamma(\alpha_{m-r}+1)} + I^{\alpha_m}_{0^+,t}u(t).$$

which is the exact solution of (34)*.*

Example 2. *Consider the following problem (See [12]):*

$$D^{\alpha_m}v(t) = \lambda v(t), \tag{36}$$

subject to,

$$D^{\alpha_k}v(t)|_{t=0} = \begin{cases} 1, & k=j, \\ 0, & k=0,1,...,j-1,j+1 \end{cases}, \tag{37}$$

where λ is an arbitrary parameter.

Applying the NG-Transform to both sides of Eq.(36), yields

$$[\psi(p)]^{\alpha_m}\,\mathcal{T}_{\mathcal{NG}}\{v(t),p\} \\ -\xi(p)\sum_{r=1}^{m}[\psi(p)]^{\alpha_m-\alpha_{m-r}-1}\left[D^{\alpha_m}v(t)\right]_{t=0} \\ = \lambda\mathcal{T}_{\mathcal{NG}}\{v(t),p\}.$$

Therefore

$$\mathcal{T}_{\mathcal{NG}}\{v(t),p\} \\ = \frac{\xi(p)}{[\psi(p)]^{\alpha_m}-\lambda}\sum_{r=1}^{m}[\psi(p)]^{\alpha_m-\alpha_{m-r}-1}\left[D^{\alpha_m}v(t)\right]_{t=0}, \\ = \sum_{r=1}^{m}\frac{\xi(p)}{[\psi(p)]^{\alpha_{m-r}+1}}\frac{1}{1-\frac{\lambda}{[\psi(p)]^{\alpha_m}}}\left[D^{\alpha_m}v(t)\right]_{t=0}, \\ = \sum_{k=0}^{m-1}\frac{\xi(p)}{[\psi(p)]^{\alpha_k+1}}\frac{1}{1-\frac{\lambda}{[\psi(p)]^{\alpha_m}}}\left[D^{\alpha_k}v(t)\right]_{t=0}.$$

Using the initial conditions (37), *we get*

$$\mathcal{T}_{\mathcal{NG}}\left\{v\left(t\right),p\right\}=\frac{\xi\left(p\right)}{\left[\psi\left(p\right)\right]^{\alpha_j+1}}\left(1-\frac{\lambda}{\left[\psi\left(p\right)\right]^{\alpha_m}}\right)^{-1}. \tag{38}$$

By taking the inverse transform $\mathfrak{F}^{-1}$ *of both sides of the eq.*(38) *and using relation* (17) *we get*

$$v\left(t\right)=t^{\alpha_{m-r}}E_{\alpha_m,\alpha_{m-r}+1}\left(\lambda t^{\alpha_m}\right).$$

6. Conclusion

New results of the NG-Transform on the Dzherbashian–Nersesian fractional operator are presented in this paper. First, the expression for the NG transform of fractional Dzherbashia–Nersesian operator is constructed. Then the expression for Laplace transform of fractional Dzherbashian-Nersesian operator of [14] is shown to be a special case of this new results. It has also been shown that the Riemann-Liouville, Caputo, and Hilfer derivatives are special cases of the fractional Dzherbashian–Nersesian operator [13] .

It is possible to deduce many expressions which relate integral transforms to the various operators from the relation that give the NG-transform of the Dzherbashian-Nersesian fractional operator.

The initial-boundary value problems for a fourth-order differential equation within the powerful fractional Dzherbashian–Nersesian operator (FDNO) are investigated in the research [13] . The current study, illustrates how it can be used to solve some Cauchy-type fractional differential equations with the Dzherbashian-Nersesian fractional operator.

In the near future, we will try to use this operator to model phenomena related to the real world, especially modeling diseases and social pests.

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References

[1] Agarwal, P., Baleanu, D., Chen, Y., Momani, S., & Machado, J.A.T. (1998). Fractional Calculus, *ICFDA 2018, Springer Proceedings in Mathematics and Statistics*, 303.

[2] Baleanu, D. & Agarwal, P. (2014). A composition formula of the Ppathway integral transform operator, *Note di Matematica*, 34(2), 145-155.

[3] Baleanu, D. & Agarwal, P. (2014). On generalized fractional integral operators and the generalized gauss hypergeometric functions, *Abstract and Applied Analysis*, 2014.

[4] Baleanu, D., Diethelm, K., Scalas, E., & Trujillo, JJ. (2017). *Fractional Calculus: Models and Numerical Method.* 2nd ed. World Scientific, New York.

[5] Mainard, F. (2010). *Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models.* World Scientific.

[6] Miller, KS., & Ross, B. (1993). *An Introduction to the Fractional Integrals and Derivatives Theory and Applications.* John Willey and Sons, New York.

[7] Oldham, K., & Spanier, J. (1974). *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*, Elsevier.

[8] Aychluh, M., Purohit, S. D., Agrawal, P., & Suthar, D. L. (2022). Atangana–Baleanu derivative-based fractional model of COVID-19 dynamics in Ethiopia. *Applied Mathematics in Science and Engineering* , 30(1), 635-660.

[9] Magin, RL. (2006). *Fractional Calculus in Bioengineering.* Begell House Publishers.

[10] Podlubny, I. (1998). *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications,* Academic Press, San Diego.

[11] Baleanu, D., & Fernandez, A. (2019). On fractional operators and their classifications. *Mathematics,* 7(9), 830.

[12] Dzhrbashyan, M.M., & Nersesyan, A.B. (2020). Fractional derivatives and the cauchy problem for fractional differential equations. *Fractional Calculus and Applied Analysis*, 23(6), 1810-1836.

[13] Ahmad, A., & Baleanu, D. (2023). On two backward problems with Dzherbashian-Nersesian operator. *AIMS Mathematics,* 8(1), 887-904.

[14] Ahmad, A., Ali, M., & Malik, SA. (2021). Inverse problems for diffusion equation with fractional Dzherbashian–Nersesian operator. *Fractional Calculus and Applied Analysis,* 24, 1899-1918.

[15] Watugala, G.K. (1993). Sumudu transform: a new integral Transform to solve differential equations and control engineering problems. *International Journal of Mathematical Education in Science and Technology*, 24, 35-43.

[16] Elzaki, T.M. (2011). The new integral transform Elzaki transform. *Global Journal of Pure and Applied Mathematics*, 7, 57-64.

[17] Khan, M., Salahuddin, T., Malik, M.Y., Alqarni, M.S., & Alkahtani, A.M. (2020). Numerical mobeling end analysis of bioconvection on MHD flow due to an upper paraboloid surface of revolution. *Physica A: Statistical Mechanics and its Applications* 553, Article 124231.

[18] Aboodh, K.S. (2023). The new integral transform. *Global Journal of Pure and Applied Mathematics,* 9(1), 35-43.

[19] Ahmadi, S.A.P., Hosseinzadeh, H., & Cherat, A.Y. (2019). A new integral transform for solving higher order linear ordinary differential equations. *Nonlinear Dynamics and Systems Theory*, 19(2), 243-252.

[20] Ahmadi, S.A.P., Hosseinzadeh, H., & Cherat A.Y. (2019). A new integral transform for solving higher order linear ordinary Laguerre and Hermite differential equations. *International Journal of Applied and Computational Mathematics.* 5, 142.

[21] Mohand, M., & Mahgoub, A. (2017). The new integral transform Mohand transform. *Advances in Theoretical and Applied Mathematics,* 12(2), 113-20.

[22] Mahgoub, MA., & Mohand, M. (2019). The new integral transform " Sawi Transform ". *Advances in Theoretical and Applied Mathematics*, 14(1), 81-87.

[23] Maitama, S., & Zhao, W. (2019). New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations. *International Journal of Analysis and Applications*, 17(2), 167-190.

[24] Kamal, H., & Sedeeg, A. (2016). The new integral transform Kamal transform. *Advances in Theoretical and Applied Mathematics*, 11(4), 451-8.

[25] Jafari, H. (2021). A new general integral transform for solving integral equations. *Journal of Advanced Research*, 32, 133-138.

[26] Hussein, M.A., & Hussein A. (2022). A review on integral transforms of the fractional derivatives of Caputo Fabrizio and Atangana-Baleanu. *Eurasian Journal of Media and Communications,* 7,17-23.

[27] Khalid, M., & Alha, S. (2023). New generalized integral transform on Hilfer-Prabhakar fractional derivatives and its applications. *Authorea.*

[28] Costa, F.S., Soares, J.C.A., Jarosz, S., & Sousa, J. (2022). Integral transforms of the hilfer-type fractional derivatives. *Asian Research Journal of Mathematics,* 18 (12), 57-74.

[29] El-Mesady, A.I., Hamed, Y.S., & Alsharif, A.M. (2021). Jafari transformation for solving a system of ordinary differential equations with medical application. *Fractal and Fractional*, 5(3), 130.

[30] Meddahi, M., Jafari, H., & Yang X.J. (2021). Towards new general double integral transform and its applications to differential equations. *Mathematical Methods in the Applied Sciences*, 1-18.

[31] Mansour, E.A., & Meftin, N.K. (2021). Mathematical modeling for cryptography using Jafari transformation method. *Periodicals of Engineering and Natural Sciences*, 9(4), 892-897.

[32] Rashid, S., Sultana, S., Ashraf, R., & Kaabar, M.K.A. (2021). On comparative analysis for the black-scholes model in the generalized fractional derivatives sense via Jafari transform. *Journal of Function Spaces,* 2021, 1-22.

[33] Rashid, S., Ashraf, R., & Jarad, F. (2022). Strong interaction of Jafari decomposition method with nonlinear fractional-order partial differential equations arising in plasma via the singular and nonsingular kernels. *AIMS Mathematics,* 7(5), 7936-7963.

[34] Rashid, S., Ashraf, R., & Bonyah, E. (2022). On analytical solution of time-fractional biological population model by means of generalized integral transform with their uniqueness and convergence analysis. *Journal of Function Spaces*, Article ID 7021288, 29 pages.

[35] Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (2006). *Theory and Applications of Fractional Differential Equations.* vol.204, North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam.

[36] Caputo, M., & Fabrizio, M. (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation and Applications,* 73,1-13.

[37] Hilfer, R. (2008). *Threefold Introduction to Fractional Derivative, Anomalous Transport:Foundations and Application.* Wiley-VCH, Weinheim, Germany.

[38] Meddahi, M., Jafari, H., & Ncube, M.M. (2021). New general integral transform via Atangana-Baleanu derivatives. *Advances in Continuous and Discrete Models,* 385, 2021.

[39] Prabhakar, T.R. (1971). A singular integral equation with a generalized Mittag-Leffler

function in the kernel. *Yokohama Mathematical Journal*, 19 , 7-15.

[40] Garra, R., & Garrappa, R. (2018). The Prabhakar or three parameter Mittag–Leffler function: Theory and application. *Communications in Nonlinear Science and Numerical Simulation*, 56, 314-329.

[41] Khalouta, A. (2023). A new general integral transform for solving Caputo fractional-order differential equations. *International Journal of Nonlinear Analysis and Applications*, 14(1), 67-78.

[42] Bodkhe, D.S., & Panchal, S.K. (2016). On sumudu transform of fractional derivatives and its applications to fractional differential equations. *Asian Journal of Mathematics and Computer Research,* 11(1), 69-77.

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