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RESEARCH ARTICLE

Controllability of nonlinear fractional integrodifferential systems involving multiple delays in control

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ABSTRACT

This work studies the existence of solutions and approximate controllability of fractional integrodifferential systems with Riemann-Liouville derivatives and with multiple delays in control. We establish suitable assumptions to prove the existence of solutions. Controllability of the system is shown by assuming a range condition on control operators and Lipschitz condition on non-linear functions. We use the concepts of strongly continuous semigroup rather than resolvent operators. Finally, an example is give to illustrate the theory.

1. Introduction

There are many problems in which the current rate of change of a function can be obtained from the past values of that function. Time delay systems are mathematical models of these types of problems. A system may have variable or constants delays eithre in control action or in the state variable or in both. Therefore it is reasonable to study the existence or controllability property of delay dynamical systems. Some of biological and physical systems having time delays are population growth, prey predator problems, mixing of liquids, equations having feedback control, etc.

In several biological, engineering and physical problems, differential systems of fractional-order are found to be suitable models. Therefore, in last twenty years, they attracted more attention from researchers. In fact, for the illustration of memory and hereditary properties, fractional derivatives provide a better instrument. For this reason, they have given a lot of applications in the areas of control theory, aerodynamics, viscoelasticity, physics, electrodynamics of complex medium, heat conduction, electricity mechanics, etc. [1–12]. For the modeling of the anomalous phenomena in the theory of complex systems as well as in nature, systems of fractional-order became more appropriate and interesting [1, 13]. Therefore, to describe diffusion in media with fractal geometry, the fractional diffusion equation was introduced in physics by substituting the first-order derivative by a fractional derivative in classical diffusion equation, which becomes appropriate for many applications.

In some areas such as dynamics of nuclear reactor and thermoelasticity, it is required to reflect the memory effect of systems in their models. In the modeling of these problems, if differential equations are utilized, which involve functions at any given space and time, the effect of previous outcomes is omitted. For this reason, to incorporate the memory effect in these differential equations,

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a term of integration is introduced, which turns to integrodifferential equation. The integrodifferential equations have given a huge applications in mechanics, viscoelastic fluid dynamics, control theory, thermoelastic contact, heat conduction, financial mathematics, industrial mathematics, biological models, aerospace systems, chemical kinetics, etc. (see [15–21]).

The existence and controllability results for different types of linear and non-linear systems are proved in many articles [14, 20–34, 36–42, 44–51, 53]. Among them, approximate controllability of fractional systems with Riemann-Liouville derivatives was proved by Liu and Li [38] assuming Lipschitz continuity. In [36], Zhu et al. analyzed the approximate controllability of fractional semilinear systems using itegral contractor. Using fractional resolvent, Ji and Yang [21] obtained the solution to fractional integrodifferential systems with Riemann-Liouville derivatives without assuming the Lipschitz condition. Ibrahim et. al. [33] analyzed approximate controllability of functional equations with Riemann-Liouville derivative by applying iterative technique. Approximate controllability for higher order fractional integrodifferential equation was discussed by Raja et al. [52]. Making use of fractional resolvent, existence and controllability of higher order Riemann-liouville fractional equations were derived in [35]. However, the controllability of fractional integrodifferential equations with multiple delays in control is still an untreated topic. Our purpose is to obtain a set of new sufficient conditions for the existence and uniqueness of solutions and approximate controllability of the following fractional integrodifferential systems:

$$\begin{cases} D_t^\kappa z(t) = Az(t) + \sum_{j=0}^{m} B_j u(t-b_j) \\ +f\left(t, z(t), \int_0^t \xi(t,s,z(s))\,ds\right), \qquad t \in (0,\hbar], \\ I_t^{1-\kappa} z(t)\big|_{t=0} = y_0 \in V,\ u(t)=0, \quad t\in[-b_m,0], \end{cases} \tag{1}$$

where $0 < \kappa \leq 1 < p\kappa$ and D_t^κ is the κ-order Riemann-Liouville derivative. The control $u \in U = L_p([0,\hbar];V')$, the state $z \in Z = L_p([0,\hbar];V)$, where V and V' are complete normed spaces. b_j $j = 0,1,2,\dots,m$, are constant delays such that $0 = b_0 < b_1 < b_2 < \cdots < b_m < \hbar$. The linear operator $A : D(A) \subseteq V \to V$ generates a C_0-semigroup $T(t)$. $B_j : U \to Z$, $j = 0,1,2,\dots,m$, are linear maps. f and ξ are V-valued non-linear functions defined on $[0,\hbar]\times V\times V$ and $\Delta\times V$, respectively; where $\Delta = \{(t_1,t_2) : 0 \leq t_2 \leq t_1 \leq \hbar\}$.

The article is structured as follows: After introduction, we have given the preliminaries in Section 2. In Section 3, the existence and uniqueness of solutions are proven. Controllability of the system is shown in Section 4. Finally, an example is given in Section 5.

2. Preliminaries

Definition 1. *The Riemann-Liouville fractional integral of order κ is given by*

$$I_t^\kappa \varphi(t) = \frac{1}{\Gamma(\kappa)}\int_0^t (t-s)^{\kappa-1}\varphi(s)\,ds, \quad \kappa > 0,$$

where Γ is the gamma function.

Definition 2. *The Riemann-Liouville fractional derivative of order κ is given by*

$$D_t^\kappa \varphi(t) = \frac{1}{\Gamma(m-\kappa)}\frac{d^m}{dt^m}\int_0^t (t-s)^{m-\kappa-1}\varphi(s)\,ds,$$

where $1 + [\kappa] = m$.

Definition 3. *The Mittag-Leffler function $E_{\kappa,\widehat{\kappa}}(\cdot)$ is given by*

$$E_{\kappa,\widehat{\kappa}}(\zeta) = \sum_{j=0}^{\infty}\frac{\zeta^j}{\Gamma(\kappa j + \widehat{\kappa})}.$$

For $\widehat{\kappa} = 1$, it is denoted by $E_\kappa(\cdot)$.
Consider the complete normed space

$$C_{1-\kappa}([0,\hbar];V) = \{\varphi : t^{1-\kappa}\varphi(t) \in C([0,\hbar];V)\}$$

with the norm

$$\|\varphi\|_{C_{1-\kappa}} = \sup_{t\in[0,\hbar]}\{t^{1-\kappa}\|\varphi(t)\|_V\},$$

where $C([0,\hbar];V)$ is the set of V-valued continuous functions defined on $[0,\hbar]$. For C_0-semigroup $T(t)$, we assume $\sup_{t\in[0,\hbar]}\|T(t)\| \leq \lambda_T < \infty$.

Definition 4. *[38] A function $z \in C_{1-\kappa}([0,\hbar];V)$ is said to be a mild solution of (1) if*

$$\begin{aligned} z(t) =& t^{\kappa-1}T_\kappa(t)y_0 + \int_0^t (t-s)^{\kappa-1} \\ &\cdot T_\kappa(t-s)\Bigg(\sum_{j=0}^{m} B_j u(s-b_j) \\ &+ f\left(s, z(s), \int_0^s \xi(s,\varsigma,z(\varsigma))\,d\varsigma\right)\Bigg)ds, \end{aligned} \tag{2}$$

where

$$T_\kappa(t) = \kappa\int_0^\infty \vartheta\zeta_\kappa(\vartheta)T(t^\kappa\vartheta)d\vartheta,$$

$$\zeta_\kappa(\vartheta) = \frac{1}{\kappa}\vartheta^{-1-\frac{1}{\kappa}}\omega_\kappa\left(\vartheta^{-\frac{1}{\kappa}}\right),$$

$$\omega_\kappa(\vartheta) = \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}\Gamma(j\kappa+1)}{\vartheta^{j\kappa+1}j!}\sin(j\pi\kappa),$$
$$0<\vartheta<\infty.$$

Definition 5. *The set given by*

$$\Re_\hbar(f) = \{z_u(\hbar)\in V : u\in U\}$$

is called the reachable set of (1), where $z_u(\cdot)$ is the mild solution of (1) corresponding to u.

Definition 6. *The system (1) is said to be approximately controllable on $[0,\hbar]$ if $\overline{\Re_\hbar(f)} = V$.*

Lemma 1. *[34] For every $t\in[0,\infty)$, $T_\kappa(t)$ is continuous linear map such that*

$$\|T_\kappa(t)y\| \le \frac{\lambda_T}{\Gamma(\kappa)}\|y\| \ \forall\ y\in V.$$

Lemma 2. *[38] If the semigroup $T(t)$ generated by A is differentiable, then*

(i) $T_\kappa(t)y\in D(A)\quad \forall\ t>0$ and $y\in V$;

(ii) $T_\kappa(t_1)T_\kappa(t_2) = T_\kappa(t_2)T_\kappa(t_1)\ \forall\ t_1,t_2>0$;

(iii) $\frac{dT^2_\kappa(t)y}{dt} = 2T_\kappa(t)\frac{dT_\kappa(t)y}{dt},\ t>0,\ y\in V$;

(iv) for any $y\in D(A)$, there is a $\varphi\in Z$ such that $\int_0^\hbar(\hbar-s)^{\kappa-1}T_\kappa(\hbar-s)\varphi(s)\,ds = y$.

3. Existence and Uniqueness of Mild Solution

To derive the existence result we assume the following:

(A_1) $T(t)$ is continuous with respect to operator norm for $t>0$.

(A_2) there is a $\lambda_f>0$ satisfying
$$\|f(t,y_1,y_1^*) - f(t,y_2,y_2^*)\| \le \lambda_f\left(\|y_1-y_2\| + \|y_1^*-y_2^*\|\right)$$
for all $y_i,y_i^*\in V,\ i=1,2$,

(A_3) there is a $\wp\in L_p([0,\hbar];\mathbb{R})$, and a $\lambda'_f>0$ such that
$$\|f(t,y,y^*)\| \le \wp(t) + \lambda'_f t^{1-\kappa}\left(\|y\|+\|y^*\|\right)$$
for a.e. $t\in[0,\hbar]$ and $y,y^*\in V$,

(A_4) there is a $\lambda_\xi>0$ verifying
$$\|\xi(t,s,y_1)-\xi(t,s,y_2)\| \le \lambda_\xi\|y_1-y_2\|$$
forall $y_1,y_2\in V$;

(A_5) there is a $\Theta\in L_p([0,\hbar];\mathbb{R})$ verifying
$$\|\xi(t,s,y)\|\le\Theta(s)$$
for all $(t,s)\in\Delta$ and $y\in V$.

Theorem 1. *Suppose assumptions (A_1)-(A_5) are true. Then, for each $u\in U$, the semilinear system (1) admits exactly one mild solution in $C_{1-\kappa}([0,\hbar];V)$.*

Proof. It is enough to prove that, the function $\mathcal{E}: C_{1-\kappa}([0,\hbar];V)\to C_{1-\kappa}([0,\hbar];V)$ defined by

$$(\mathcal{E}z)(t) = t^{\kappa-1}T_\kappa(t)y_0 + \int_0^t (t-s)^{\kappa-1} \cdot T_\kappa(t-s)\Bigg(\sum_{j=0}^m B_j u(s-b_j) + f\left(s,z(s),\int_0^s \xi(s,\varsigma,z(\varsigma))\,d\varsigma\right)\Bigg)ds,$$

has exactly one fixed point in $C_{1-\kappa}([0,\hbar];V)$. Due to above assumptions, the function $\mathcal{E}$ is well defined.

Let $z,z^*\in C_{1-\kappa}([0,\hbar];V)$. Then,

$$\begin{aligned}
&t^{1-\kappa}\|(\mathcal{E}z)(t)-(\mathcal{E}z^*)(t)\|\\
&\le t^{1-\kappa}\int_0^t\Bigg\|(t-s)^{\kappa-1}T_\kappa(t-s)\cdot\Bigg(f\left(s,z(s),\int_0^s\xi(s,\varsigma,z(\varsigma))\,d\varsigma\right)\\
&\quad - f\left(s,z^*(s),\int_0^s\xi(s,\varsigma,z^*(\varsigma))\,d\varsigma\right)\Bigg)\Bigg\|ds\\
&\le\frac{\lambda_T\lambda_f}{\Gamma(\kappa)}t^{1-\kappa}\int_0^t(t-s)^{\kappa-1}\Bigg(\|z(s)-z^*(s)\| + \int_0^s\|\xi(s,\varsigma,z(\varsigma))-\xi(s,\varsigma,z^*(\varsigma))\|\,d\varsigma\Bigg)ds\\
&\le\frac{\lambda_T\lambda_f}{\Gamma(\kappa)}t^{1-\kappa}\int_0^t(t-s)^{\kappa-1}\cdot\Bigg(s^{\kappa-1}s^{1-\kappa}\|z(s)-z^*(s)\| + \lambda_\xi\int_0^s\varsigma^{\kappa-1}\varsigma^{1-\kappa}\|z(\varsigma)-z^*(\varsigma)\|\,d\varsigma\Bigg)ds\\
&\le\frac{\lambda_T\lambda_f}{\Gamma(\kappa)}t^{1-\kappa}\int_0^t(t-s)^{\kappa-1}\left(s^{\kappa-1}+\lambda_\xi\frac{s^\kappa}{\kappa}\right)ds\cdot\|z-z^*\|_{C_{1-\kappa}}\\
&=\frac{\lambda_T\lambda_f}{\Gamma(\kappa)}t^\kappa\left(\frac{(\Gamma(\kappa))^2}{\Gamma(2\kappa)}+\frac{\lambda_\xi\Gamma(\kappa)\Gamma(\kappa+1)t}{\kappa\Gamma(2\kappa+1)}\right)\cdot\|z-z^*\|_{C_{1-\kappa}}\\
&\le\frac{\Gamma(\kappa)\lambda_T\lambda_f}{\Gamma(2\kappa)}t^\kappa\left(1+\frac{\lambda_\xi\hbar}{2\kappa}\right)\|z-z^*\|_{C_{1-\kappa}}.
\end{aligned}$$

Repeating the above process, we can get

$$\begin{aligned}
&t^{1-\kappa}\|(\mathcal{E}^n z)(t)-(\mathcal{E}^n z^*)(t)\|\\
&\le\frac{\Gamma(\kappa)(\lambda_T\lambda_f)^n}{\Gamma((n+1)\kappa)}t^{n\kappa}\left(\prod_{i=1}^n\left(1+\frac{\lambda_\xi\hbar}{(i+1)\kappa}\right)\right)\cdot\|z-z^*\|_{C_{1-\kappa}}
\end{aligned}$$

$$\leq \frac{\Gamma(\kappa)\left(\lambda_T\lambda_f\hbar^\kappa\left(1+\frac{\lambda_\xi\hbar}{2\kappa}\right)\right)^n}{\Gamma((n+1)\kappa)}\|z-z^*\|_{C_{1-\kappa}}.$$

Therefore,

$$\|\mathcal{E}^n z-\mathcal{E}^n z^*\|_{C_{1-\kappa}} \leq \frac{\Gamma(\kappa)\left(\lambda_T\lambda_f\hbar^\kappa\left(1+\frac{\lambda_\xi\hbar}{2\kappa}\right)\right)^n}{\Gamma((n+1)\kappa)}\|z-z^*\|_{C_{1-\kappa}}.$$

We know that the Mittag-Leffler series

$$E_{\kappa,\kappa}\left(\lambda_T\lambda_f\hbar^\kappa\left(1+\frac{\lambda_\xi\hbar}{2\kappa}\right)\right) = \sum_{i=0}^{\infty}\frac{\left(\lambda_T\lambda_f\hbar^\kappa\left(1+\frac{\lambda_\xi\hbar}{2\kappa}\right)\right)^i}{\Gamma((i+1)\kappa)}$$

is convergent. Therefore, for sufficiently large value of n,

$$\frac{\left(\lambda_T\lambda_f\hbar^\kappa\left(1+\frac{\lambda_\xi\hbar}{2\kappa}\right)\right)^n}{\Gamma((n+1)\kappa)}<\frac{1}{\Gamma(\kappa)}.$$

Thus, from Banach contraction principle $\mathcal{E}$ has exactly one fixed point in $C_{1-\kappa}([0,\hbar];V)$. ☐

4. Controllability analysis

Define the operator $\Psi_f : C_{1-\kappa}([0,\hbar];V) \to Z$ given by

$$(\Psi_f(\omega))(t) = f\left(t,\omega(t),\int_0^t \xi(t,s,\omega(s))\,ds\right), \quad \omega\in C_{1-\kappa}([0,\hbar];V)$$

and the bounded linear operator $\Phi : Z\to V$ given by

$$\Phi(\omega)=\int_0^\hbar(\hbar-s)^{\kappa-1}T_\kappa(\hbar-s)\omega(s)\,ds,\ \omega\in Z.$$

Remark 1. *From Definition 6, the system (1) is approximately controllable if and only if for each $\varepsilon>0$ and a $\widehat{y}\in V$, there exists a control $u_\varepsilon\in U$ such that the mild solution z_ε corresponding to u_ε satisfies*

$$\left\|\widetilde{y}-\Phi(\Psi_f(z_\varepsilon))-\Phi\left(\sum_{j=0}^m B_j u_\varepsilon(\cdot-b_j)\right)\right\|\leq\varepsilon,$$

where $\widetilde{y}=\widehat{y}-\hbar^{\kappa-1}T_\kappa(\hbar)y_0$.

To prove the controllability of original system, we assume the following:

(A_6) there is a $\widehat{\lambda}_f>0$ verifying
$$\|f(t,y_1,y_1^*)-f(t,y_2,y_2^*)\| \leq \widehat{\lambda}_f t^{1-\kappa}\left(\|y_1-y_2\|+\|y_1^*-y_2^*\|\right)$$
forall $y_i,y_i^*\in V,\ i=1,2$;

(A_7) there is a $\widehat{\lambda}_\xi>0$ verifying
$$\|\xi(t,s,y_1)-\xi(t,s,y_2)\|\leq\widehat{\lambda}_\xi s^{1-\kappa}\|y_1-y_2\|$$
forall $y_i\in V,\ i=1,2$;

(A_8) for given $\varepsilon>0$ and a $z\in Z$, we can get a $u\in U$ such that
$$\|\Phi(z)-\Phi(B_0u)\|_V\leq\varepsilon$$
and
$$\|B_0u\|_Z\leq\lambda_0\|z\|_Z,$$
where λ_0 is constant and it does not dependent on z;

(A_9) $0<\frac{\lambda_T\widehat{\lambda}_f\lambda_0\lambda_p\hbar\left(1+\widehat{\lambda}_\xi\hbar^{2-\kappa}\right)E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)}{\Gamma(\kappa)-\lambda_T\widehat{\lambda}_f\widehat{\lambda}_\xi\hbar^{3-\kappa}\kappa^{-1}E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)}<1$, where $\lambda_p=\left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}}$;

(A_{10}) $R(B_0)\supseteq R(B_1)\supseteq\cdots\supseteq R(B_m)$, where R stands for the range of operators.

Remark 2. *Note that (A_2) and (A_4) are weaker assumptions than (A_6) and (A_7), respectively. Thus, by Theorem 1, the semilinear system (1) admits a unique solution in $C_{1-\kappa}([0,\hbar];V)$ for fixed $u\in U$ if assumptions (A_1), (A_3) and (A_5)-(A_7) are true.*

We derive the following lemma:

Lemma 3. *Under assumptions (A_1), (A_3), (A_5)-(A_7) and (A_9) any mild solutions of (1) satisfy the following*

(i)
$$\|z\|_{C_{1-\kappa}}\leq k_1E_\kappa\left(\lambda_T\lambda_f'\hbar\right),\ u\in U,$$

(ii)
$$\|z_1-z_2\|_{C_{1-\kappa}}\leq k_2E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)\left\|\sum_{j=0}^m B_ju_1(\cdot-b_j)-\sum_{j=0}^m B_ju_2(\cdot-b_j)\right\|_Z,\quad u_1,u_2\in U,$$

where

$$k_1=\frac{\lambda_T}{\Gamma(\kappa)}\left[\|y_0\|+\lambda_p\left(\left\|\sum_{j=0}^m B_ju(\cdot-b_j)\right\|_Z+\|\wp\|_{L_p}\right)\hbar^{1-\frac{1}{p}}+\lambda_f'\hbar^{3-\kappa-\frac{1}{p}}\kappa^{-1}\|\Theta\|_{L_p}\right]$$

and

$$k_2=\frac{\lambda_T\lambda_p\hbar^{1-\frac{1}{p}}}{\Gamma(\kappa)-\lambda_T\widehat{\lambda}_f\widehat{\lambda}_\xi\hbar^{3-\kappa}\kappa^{-1}E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)}.$$

Proof. Let $z \in C_{1-\kappa}([0,\hbar];V)$ be a mild solution of (1) for $u \in U$, then

$$z(t) = t^{\kappa-1}T_\kappa(t)y_0 + \int_0^t (t-s)^{\kappa-1} \cdot T_\kappa(t-s)\left(\sum_{j=0}^m B_j u(s-b_j) + f\left(s, z(s), \int_0^s \xi(s,\varsigma,z(\varsigma))\,d\varsigma\right)\right) ds.$$

Therefore

$$\begin{aligned}
t^{1-\kappa}\|z(t)\|_V &\le \|T_\kappa(t)y_0\| + t^{1-\kappa}\int_0^t \left\|(t-s)^{\kappa-1} \cdot T_\kappa(t-s)\left(\sum_{j=0}^m B_j u(s-b_j) + f\left(s, z(s), \int_0^s \xi(s,\varsigma,z(\varsigma))\,d\varsigma\right)\right)\right\| ds \\
&\le \frac{\lambda_T}{\Gamma(\kappa)}\Bigg[\|y_0\| + t^{1-\kappa}\int_0^t (t-s)^{\kappa-1} \cdot \left\|\sum_{j=0}^m B_j u(s-b_j)\right\| ds + t^{1-\kappa}\int_0^t (t-s)^{\kappa-1} \\
&\quad \left(\wp(s) + \lambda'_f s^{1-\kappa}\|z(s)\|_V + \lambda'_f s^{1-\kappa}\int_0^s \Theta(\varsigma)\,d\varsigma\right) ds\Bigg] \\
&\le \frac{\lambda_T}{\Gamma(\kappa)}\Bigg[\|y_0\| + \left(\frac{p-1}{p\kappa-1}\right)^{1-\frac{1}{p}} \cdot \left(\left\|\sum_{j=0}^m B_j u(\cdot-b_j)\right\|_Z + \|\wp\|_{L_p}\right)\hbar^{1-\frac{1}{p}} \\
&\quad + \lambda'_f \hbar^{3-2\kappa-\frac{1}{p}}\int_0^t (t-s)^{\kappa-1}\,ds\|\Theta\|_{L_p} + \lambda'_f\hbar^{1-\kappa}\int_0^t (t-s)^{\kappa-1}s^{1-\kappa}\|z(s)\|_V\,ds\Bigg] \\
&\le k_1 + \frac{\lambda_T\lambda'_f\hbar^{1-\kappa}}{\Gamma(\kappa)}\int_0^t (t-s)^{\kappa-1}s^{1-\kappa}\|z(s)\|_V\,ds.
\end{aligned}$$

From Corollary 2 of [43], we obtain

$$t^{1-\kappa}\|z(t)\|_V \le k_1 E_\kappa\big(\lambda_T\lambda'_f\hbar\big).$$

Therefore,

$$\|z\|_{C_{1-\kappa}} \le k_1 E_\kappa\big(\lambda_T\lambda'_f\hbar\big).$$

Next, let $z_i \in C_{1-\kappa}([0,\hbar];V)$ be the mild solution of (1) for $u_i \in U, \quad i = 1,2$. Then

$$z_i(t) = t^{\kappa-1}T_\kappa(t)y_0 + \int_0^t (t-s)^{\kappa-1} \cdot T_\kappa(t-s)\left(\sum_{j=0}^m B_j u_i(s-b_j) + f\left(s, z_i(s), \int_0^s \xi(s,\varsigma,z_i(\varsigma))\,d\varsigma\right)\right) ds.$$

We have

$$\begin{aligned}
t^{1-\kappa}\|z_1(t)-z_2(t)\|_V &\le \frac{\lambda_T}{\Gamma(\kappa)}t^{1-\kappa}\Bigg[\int_0^t (t-s)^{\kappa-1}\left\|\sum_{j=0}^m B_j u_1(s-b_j) - \sum_{j=0}^m B_j u_2(s-b_j)\right\| ds + \int_0^t (t-s)^{\kappa-1} \\
&\quad \cdot \left\|f\left(s, z_1(s), \int_0^s \xi(s,\varsigma,z_1(\varsigma))\,d\varsigma\right) - f\left(s, z_2(s), \int_0^s \xi(s,\varsigma,z_2(\varsigma))\,d\varsigma\right)\right\| ds\Bigg] \\
&\le \frac{\lambda_T\lambda_p}{\Gamma(\kappa)}\hbar^{1-\frac{1}{p}}\left\|\sum_{j=0}^m B_j u_1(\cdot-b_j) - \sum_{j=0}^m B_j u_2(\cdot-b_j)\right\|_Z + \frac{\lambda_T\widehat{\lambda}_f}{\Gamma(\kappa)}\hbar^{1-\kappa} \\
&\quad \cdot \int_0^t (t-s)^{\kappa-1}s^{1-\kappa}\left(\|z_1(s)-z_2(s)\| + \widehat{\lambda}_\xi\int_0^s \varsigma^{1-\kappa}\|z_1(\varsigma)-z_2(\varsigma)\|\,d\varsigma\right) ds \\
&\le \frac{\lambda_T\lambda_p}{\Gamma(\kappa)}\hbar^{1-\frac{1}{p}}\left\|\sum_{j=0}^m B_j u_1(\cdot-b_j) - \sum_{j=0}^m B_j u_2(\cdot-b_j)\right\|_Z + \frac{\lambda_T\widehat{\lambda}_f}{\Gamma(\kappa)}\hbar^{1-\kappa} \\
&\quad \cdot \left(\int_0^t (t-s)^{\kappa-1}s^{1-\kappa}\|z_1(s)-z_2(s)\|\,ds + \widehat{\lambda}_\xi\int_0^t (t-s)^{\kappa-1}\hbar^{2-\kappa}\,ds\|z_1-z_2\|_{C_{1-\kappa}}\right) \\
&\le \frac{\lambda_T\lambda_p}{\Gamma(\kappa)}\hbar^{1-\frac{1}{p}}\left\|\sum_{j=0}^m B_j u_1(\cdot-b_j) - \sum_{j=0}^m B_j u_2(\cdot-b_j)\right\|_Z + \frac{\lambda_T\widehat{\lambda}_f\widehat{\lambda}_\xi}{\Gamma(\kappa)}\hbar^{3-\kappa}\kappa^{-1} \\
&\quad \cdot \|z_1-z_2\|_{C_{1-\kappa}} + \frac{\lambda_T\widehat{\lambda}_f}{\Gamma(\kappa)}\hbar^{1-\kappa}\int_0^t (t-s)^{\kappa-1} \cdot s^{1-\kappa}\|z_1(s)-z_2(s)\|\,ds.
\end{aligned}$$

From Corollary 2 of [43], we obtain

$$\begin{aligned} t^{1-\kappa}\|z_1(t)-z_2(t)\|_V &\le \frac{\lambda_T}{\Gamma(\kappa)}\Bigg[\lambda_p \hbar^{1-\frac{1}{p}}\Bigg\|\sum_{j=0}^{m} B_j u_1(\cdot-b_j) \\ &\quad -\sum_{j=0}^{m} B_j u_2(\cdot-b_j)\Bigg\|_Z + \widehat{\lambda}_f\widehat{\lambda}_\xi \hbar^{3-\kappa}\kappa^{-1} \\ &\quad \cdot \|z_1-z_2\|_{C_{1-\kappa}}\Bigg] E_\kappa\big(\lambda_T\widehat{\lambda}_f\hbar\big). \end{aligned}$$

Therefore,

$$\begin{aligned} \|z_1-z_2\|_{C_{1-\kappa}} &\le \frac{\lambda_T}{\Gamma(\kappa)}\Bigg[\lambda_p \hbar^{1-\frac{1}{p}}\Bigg\|\sum_{j=0}^{m} B_j u_1(\cdot-b_j) \\ &\quad -\sum_{j=0}^{m} B_j u_2(\cdot-b_j)\Bigg\|_Z + \widehat{\lambda}_f\widehat{\lambda}_\xi \hbar^{3-\kappa}\kappa^{-1} \\ &\quad \cdot \|z_1-z_2\|_{C_{1-\kappa}}\Bigg] E_\kappa\big(\lambda_T\widehat{\lambda}_f\hbar\big). \end{aligned}$$

This gives

$$\begin{aligned} \|z_1-z_2\|_{C_{1-\kappa}} &\le \frac{\lambda_T\lambda_p\hbar^{1-\frac{1}{p}}E_\kappa\big(\lambda_T\widehat{\lambda}_f\hbar\big)}{\Gamma(\kappa)-\lambda_T\widehat{\lambda}_f\widehat{\lambda}_\xi\hbar^{3-\kappa}\kappa^{-1}E_\kappa\big(\lambda_T\widehat{\lambda}_f\hbar\big)} \\ &\quad \cdot \Bigg\|\sum_{j=0}^{m} B_j u_1(\cdot-b_j)-\sum_{j=0}^{m} B_j u_2(\cdot-b_j)\Bigg\|_Z \\ &= k_2 E_\kappa\big(\lambda_T\widehat{\lambda}_f\hbar\big) \\ &\quad \cdot \Bigg\|\sum_{j=0}^{m} B_j u_1(\cdot-b_j)-\sum_{j=0}^{m} B_j u_2(\cdot-b_j)\Bigg\|_Z. \end{aligned}$$

□

Theorem 2. *Under assumptions (A_1), (A_3) and (A_5)-(A_{10}), the semilinear system (1) is approximately controllable if the semigroup $T(t)$ is differentiable.*

Proof. First we prove that for each $u^* \in U$, there is a $u \in U$ such that

$$\begin{aligned} B_0u^*(\cdot) &= B_0u(\cdot)+B_1u(\cdot-b_1) \\ &\quad +\cdots+B_mu(\cdot-b_m). \qquad (3) \end{aligned}$$

For this, set $\hbar = b_{m+1}$ and $r=\min\{b_j-b_{j-1} : j = 1,2,\dots,m+1\}$. Since $0=b_0<b_1<b_2<\cdots<b_m<b_{m+1}$ therefore for each b_{j+1} there exist a positive integer n_j and a constant $\vartheta_j \in [0,r)$ such that $b_{j+1}=b_j+n_jr+\vartheta_j$, $j=1,2,\dots,m$. For $t\in[0,b_1]$, we have

$$\begin{aligned} &B_0u^*(\cdot)-B_1u(\cdot-b_1)-\cdots-B_mu(\cdot-b_m) \\ &= B_0u^*(\cdot). \end{aligned}$$

Take $u(t)=u^*(t)$ for $t\in[0,b_1]$. For $t\in(b_1,b_1+r]$, we have $(t-b_1)\in(0,r]\subset(0,b_1]$ and

$$\begin{aligned} &B_0u^*(\cdot)-B_1u(\cdot-b_1)-\cdots-B_mu(\cdot-b_m) \\ &= B_0u^*(\cdot)-B_1u^*(\cdot-b_1)=B_0u_{11}(\cdot)(say), \end{aligned}$$

where $u_{11}(\cdot)$ is known. Take $u(t)=u_{11}(t)$ for $t\in(b_1,b_1+r]$.
Now, if $t\in(b_1+r,b_1+2r]$, then $(t-b_1)\in(r,2r]\subset(0,b_1+r]$ and $u(\cdot-b_1)$ is known. Therefore, in this case

$$\begin{aligned} &B_0u^*(\cdot)-B_1u(\cdot-b_1)-\cdots-B_mu(\cdot-b_m) \\ &= B_0u^*(\cdot)-B_1u(\cdot-b_1)=B_0u_{12}(\cdot)(say), \end{aligned}$$

where $u_{12}(\cdot)$ is known. Take $u(t)=u_{12}(t)$ for $t\in(b_1+r,b_1+2r]$. Similarly, we can find $u_{13}(\cdot),u_{14}(\cdot),\dots,u_{1n_1}(\cdot)$ for the intervals $(b_1+2r,b_1+3r],(b_1+3r,b_1+4r],\dots,(b_1+(n_1-1)r,b_1+n_1r]$; respectively. If $\vartheta_1>0$, then we can also find $u_{1\overline{n_1+1}}(\cdot)$ for the next interval $(b_1+n_1r,b_1+n_1r+\vartheta_1]$. Thus $u(\cdot)$ is completely known for $t\in(b_1,b_1+n_1r+\vartheta_1]=(b_1,b_2]$. Denote $u(\cdot)$ by $u_1(\cdot)$ for $t\in(b_1,b_2]$.
Repeating the above process, one can obtain $u_2(\cdot),u_3(\cdot),\dots,u_m(\cdot)$ for the intervals $(b_2,b_3],(b_3,b_4],\dots,(b_m,b_{m+1}]$; respectively. Hence the control function $u(\cdot)\in U$, given by

$$u(t)=\begin{cases} u^*(t), & t\in[0,b_1]; \\ u_j(t), & t\in(b_j,b_{j+1}], \quad j=1,2,\dots,m \end{cases}$$

is completely known and it satisfies

$$\begin{aligned} &B_0u^*(\cdot)-B_1u(\cdot-b_1)-\cdots-B_mu(\cdot-b_m) \\ &= B_0u(\cdot). \end{aligned}$$

Next, we prove that $D(A)\subseteq\overline{\Re_\hbar(f)}$, that is, for any $\varepsilon>0$ and $\widehat{y}\in D(A)$, we are able to find a control $u_\varepsilon\in U$ satisfying

$$\left\|\widetilde{y}-\Phi(\Psi_f(z_\varepsilon))-\Phi\left(\sum_{j=0}^{m}B_ju_\varepsilon(\cdot-b_j)\right)\right\|_V \le \varepsilon,$$

where $\widetilde{y}=\widehat{y}-\hbar^{\kappa-1}T_\kappa(\hbar)y_0$ and $z_\varepsilon(t)=z_{u_\varepsilon}(t)$. By Lemma 2, there is a $\wp\in Z$ such that $\Phi(\wp)=\widetilde{y}$.
Let $\varepsilon>0$ be given and $v_1\in U$. Then by assumption (A_8) and (3), there is a control $v_2\in U$ such that

$$\left\|\widetilde{y}-\Phi(\Psi_f(z_1))-\Phi\left(\sum_{j=0}^{m}B_jv_2(\cdot-b_j)\right)\right\|_V \le \frac{\varepsilon}{3^2},$$

where $z_1(t)=z_{v_1}(t)$. Denote $z_2(t)=z_{v_2}(t)$, in view of (A_8) and (3), there is a control $\omega_2\in U$

such that

$$\left\|\Phi(\Psi_f(z_2)-\Psi_f(z_1)) - \Phi\left(\sum_{j=0}^{m} B_j\omega_2(\cdot-b_j)\right)\right\|_V \le \frac{\varepsilon}{3^3}$$

and

$$\begin{aligned}
&\left\|\sum_{j=0}^{m} B_j\omega_2(\cdot-b_j)\right\|_Z \\
&\le \lambda_0 \left\|\Psi_f(z_2)-\Psi_f(z_1)\right\|_Z \\
&= \lambda_0\left[\int_0^{\hbar}\left\|f\left(t, z_2(t), \int_0^t \xi(t,\varsigma,z_2(\varsigma))\,d\varsigma\right) - f\left(t, z_1(t), \int_0^t \xi(t,\varsigma,z_1(\varsigma))\,d\varsigma\right)\right\|_V^p dt\right]^{\frac{1}{p}} \\
&\le \lambda_0\widehat{\lambda}_f\left[\int_0^{\hbar}\left(t^{1-\kappa}\|z_2(t)-z_1(t)\| + \widehat{\lambda}_\xi t^{1-\kappa}\int_0^t \varsigma^{1-\kappa}\|z_2(\varsigma)-z_1(\varsigma)\|\,d\varsigma\right)^p dt\right]^{\frac{1}{p}} \\
&\le \lambda_0\widehat{\lambda}_f\left(\int_0^{\hbar}\left(1+\widehat{\lambda}_\xi\hbar^{2-\kappa}\right)^p dt\right)^{\frac{1}{p}}\|z_2-z_1\|_{C_{1-\kappa}} \\
&= \lambda_0\widehat{\lambda}_f\hbar^{\frac{1}{p}}\left(1+\widehat{\lambda}_\xi\hbar^{2-\kappa}\right)\|z_2-z_1\|_{C_{1-\kappa}} \\
&\le \lambda_0\widehat{\lambda}_f\hbar^{\frac{1}{p}}\left(1+\widehat{\lambda}_\xi\hbar^{2-\kappa}\right)k_2E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)\cdot\left\|\sum_{j=0}^{m} B_ju_1(\cdot-b_j)-\sum_{j=0}^{m} B_ju_2(\cdot-b_j)\right\|_Z \\
&= \frac{\lambda_T\widehat{\lambda}_f\lambda_0\lambda_p\hbar\left(1+\widehat{\lambda}_\xi\hbar^{2-\kappa}\right)E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)}{\Gamma(\kappa)-\lambda_T\widehat{\lambda}_f\widehat{\lambda}_\xi\hbar^{3-\kappa}\kappa^{-1}E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)}\cdot\left\|\sum_{j=0}^{m} B_jv_1(\cdot-b_j)-\sum_{j=0}^{m} B_jv_2(\cdot-b_j)\right\|_Z.
\end{aligned}$$

Now, if we define

$$v_3(t)=v_2(t)-\omega_2(t), \quad v_3\in U,$$

then

$$\begin{aligned}
&\left\|\widetilde{y}-\Phi(\Psi_f(z_2))-\Phi\left(\sum_{j=0}^{m} B_jv_3(\cdot-b_j)\right)\right\|_V \\
&\le \left\|\widetilde{y}-\Phi(\Psi_f(z_1))-\Phi\left(\sum_{j=0}^{m} B_jv_2(\cdot-b_j)\right)\right\|_V \\
&\quad + \left\|\Phi(\Psi_f(z_2)-\Psi_f(z_1)) - \Phi\left(\sum_{j=0}^{m} B_j\omega_2(\cdot-b_j)\right)\right\|_V \\
&\le \left(\frac{1}{3^2}+\frac{1}{3^3}\right)\varepsilon.
\end{aligned}$$

By inductions, we get a sequence $\{v_n\}$ in U satisfying

$$\begin{aligned}
&\left\|\widetilde{y}-\Phi(\Psi_f(z_n))-\Phi\left(\sum_{j=0}^{m} B_jv_{n+1}(\cdot-b_j)\right)\right\|_V \\
&\le \left(\frac{1}{3^2}+\frac{1}{3^3}+\cdots+\frac{1}{3^{n+1}}\right)\varepsilon,
\end{aligned}$$

where $z_n(t)=z_{v_n}(t)$, and

$$\begin{aligned}
&\left\|\sum_{j=0}^{m} B_jv_{n+1}(\cdot-b_j)-\sum_{j=0}^{m} B_jv_n(\cdot-b_j)\right\|_Z \\
&\le \frac{\lambda_T\widehat{\lambda}_f\lambda_0\lambda_p\hbar\left(1+\widehat{\lambda}_\xi\hbar^{2-\kappa}\right)E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)}{\Gamma(\kappa)-\lambda_T\widehat{\lambda}_f\widehat{\lambda}_\xi\hbar^{3-\kappa}\kappa^{-1}E_\kappa\left(\lambda_T\widehat{\lambda}_f\hbar\right)} \cdot\left\|\sum_{j=0}^{m} B_jv_n(\cdot-b_j)-\sum_{j=0}^{m} B_jv_{n-1}(\cdot-b_j)\right\|_Z,
\end{aligned}$$

which shows that the sequence

$$\left\{\sum_{j=0}^{m} B_jv_n(\cdot-b_j) : n=1,2,\dots\right\}$$

is Cauchy in Z. Since Z

is complete and Φ is bounded therefore, the sequence

$$\left\{\Phi\left(\sum_{j=0}^{m} B_jv_n(\cdot-b_j)\right) : n=1,2,\dots\right\}$$

is Cauchy in V. Thus, we can get a $n_0\in\mathbb{N}$ such that

$$\left\|\Phi\left(\sum_{j=0}^{m} B_jv_{n_0+1}(\cdot-b_j)\right) - \Phi\left(\sum_{j=0}^{m} B_jv_{n_0}(\cdot-b_j)\right)\right\|_V \le \frac{\varepsilon}{3}.$$

Now,

$$
\begin{aligned}
&\left\| \widetilde{y} - \Phi(\Psi_f(z_{n_0})) - \Phi\left(\sum_{j=0}^{m} B_j v_{n_0}(\cdot - b_j) \right) \right\|_V \\
&\leq \left\| \widetilde{y} - \Phi(\Psi_f(z_{n_0})) \right. \\
&\qquad \left. - \Phi\left(\sum_{j=0}^{m} B_j v_{n_0+1}(\cdot - b_j) \right) \right\|_V \\
&\qquad + \left\| \Phi\left(\sum_{j=0}^{m} B_j v_{n_0+1}(\cdot - b_j) \right) \right. \\
&\qquad\qquad \left. - \Phi\left(\sum_{j=0}^{m} B_j v_{n_0}(\cdot - b_j) \right) \right\|_V \\
&\qquad \leq \left(\frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n_0+1}} \right) \varepsilon + \frac{\varepsilon}{3} \\
&\qquad < \varepsilon.
\end{aligned}
$$

As $D(A)$ is dense in V therefore, $\overline{\mathfrak{R}_{\hbar}(f)} = V$. □

5. Example

For $x \in [0, \pi]$, consider the system with the given boundary condition

$$
\begin{cases}
D_t^{\frac{2}{3}} z(t,x) = \frac{\partial^2}{\partial x^2} z(t,x) + \sum_{j=0}^{m} u(t - b_j) \\
+ f\left(t, z(t,x), \int_0^t \xi(t,s,z(s,x))\, ds\right), \quad 0 < t \leq 1, \\
I_t^{\frac{1}{3}} z(t,x)\big|_{t=0} = y_0(x), u(t) = 0, \quad -b_m \leq t \leq 0, \\
z(t,0) = z(t,\pi) = 0, \qquad 0 < t \leq 1,
\end{cases} \tag{4}
$$

where $0 = b_0 < b_1 < \cdots < b_m < 1$.
Take $V = V' = L_2[0,\pi]$ and $A : D(A) \subset V \to V$ given by

$$Ay = y_{xx}$$

with the domain

$$
\begin{aligned}
D(A) = \{ y \in V : y,\ y_x \quad &\text{are absolutely continuous} \\
&\text{and } y_x \in V,\ y(0) = 0 = y(\pi) \}.
\end{aligned}
$$

Then, A has the spectral representation

$$Ay = \sum_{n=1}^{\infty} (-n^2) \langle y, q_n \rangle q_n, \quad y \in D(A),$$

which generates a semigroup $T(t)$ given by

$$T(t)y = \sum_{n=1}^{\infty} \exp(-n^2 t) \langle y, q_n \rangle q_n, \quad y \in V$$

with

$$\|T(t)\| \leq \exp(-1) < 1;$$

where $q_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ are eigen functions of A associated with the eigenvalues $\lambda_n = -n^2$, $n \in \mathbb{N}$ and the set $\{q_n : n \in \mathbb{N}\}$ form an orthonormal basis for V.
If we take

$$z^*(t,x) = \int_0^t \xi(t,s,z(s,x))\, ds$$

and

$$
\begin{aligned}
&f\left(t, z(t,x), z^*(t,x)\right) \\
&= f\left(t, z(t,x), \int_0^t \xi(t,s,z(s,x))\, ds\right) \\
&= (1+t^2) + k_0 t^{a_0} \Big(z(t,x) + \int_0^t k_1 \left(t^2 + s^2\right) s^{a_1} \\
&\quad \cdot \cos(ts) \cos(1 + z(s,x))\, ds \Big),
\end{aligned}
$$

where

$$\xi(t,s,z(s,x)) = k_1 \left(t^2 + s^2\right) s^{a_1} \cos(ts) \cos(1+z(s,x))$$

and $a_i \geq 1 - \kappa$, $i = 0, 1$. Then (A_2), (A_3) and (A_6) are satisfied with $\lambda_f = \lambda_f' = \widehat{\lambda}_f = |k_0|$. Also, the conditions (A_4) and (A_7) are satisfied with $\lambda_\xi = \widehat{\lambda}_\xi = 4|k_1|$.
Now,

$$
\begin{aligned}
\|\xi(t,s,z(s,\cdot))\| &\leq |k_1| \left(1 + s^2\right) s^{a_1} \\
&= \Theta(s) \in L_p([0,1]; \mathbb{R}^+).
\end{aligned}
$$

Hence (A_5) is satisfied. If we choose the constants k_0 and k_1 small enough so that (A_9) is satisfied, then from Theorem 2, the approximate controllability of (4) follows.

References

[1] Kilbas, A.A., Srivastava, H.M. & Trujillo J.J. (2006). *Theory and Applications of Fractional Differential Equations.* North-Holland Math. Stud., 204, Elsevier Science, Amsterdam.

[2] Heymans, N. & Podlubny, I. (2006). Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheologica Acta,* 45, 765-771.

[3] Galucio, A.C., Deu, J.F. & Ohayon, R. (2005). A fractional derivative viscoelastic model for hybrid active-passive damping treatments in time domain-application to sandwich beams. *Journal of Intelligent Material Systems and Structures,* 16(1), 33-45.

[4] Baleanu, D. & Golmankhaneh, A.K. (2010). On electromagnetic field in fractional space. *Nonlinear Analysis: Real World Applications,* 11(1), 288-292.

[5] Hilfer, R. (2000). *Applications of Fractional Calculus in Physics.* Singapore: World Scientfic Publ Co.

[6] Jia, J.H., Shen, X.Y. & Hua, H.X. (2007). Viscoelastic behavior analysis and application of the fractional derivative Maxwell model. *Journal of Vibration and Control,* 13(4), 385-401.

[7] Koeller, R.C. (1984). Applications of fractional calculus to the theory of viscoelasticity. *Journal of Applied Mechanics,* 51(2), 299-307.

[8] Li, J., Liu, F., Feng, L. & Turner I. (2017). A novel finite volume method for the Riesz space distributed-order diffusion equation. *Computers & Mathematics with Applications,* 74, 772-783.

[9] Liu, X.Y., Liu, Z.H. & Fu, X. (2014). Relaxation in nonconvex optimal control problems described by fractional differential equations. *Journal of Mathematical Analysis and Applications,* 409(1), 446-458.

[10] Liu, Z.H., Zeng, S.D. & Bai, Y.R. (2016). Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications.*Fractional Calculus and Applied Analysis,* 19(1), 188-211.

[11] Liu, Z.H. & Zeng, S.D. (2017). Differential variational inequalities in infinite Banach spaces. *Acta Mathematica Scientia,* 37B(1), 26-32.

[12] Samko, S.G., Kilbas, A.A. & Marichev, O.I. (1993). *Fractional Integral and Derivatives: Theory and Applications*, Gordon and Breach, New York.

[13] Podlubny, I. (1999). *Fractional Differential Equations,* Academic Press, San Diego, CA.

[14] Balachandran, K., Govindaraj, V. , Rivero, M. & Trujillo J.J. (2015). Controllability of fractional damped dynamical systems. *Applied Mathematics and Computation,* 257, 66-73.

[15] Liu, Z.H., Sun, J.H. & Szanto, I. (2013). Monotone iterative technique for Riemann-Liouville fractional integrodifferential equations with advanced arguments. *Results in Mathematics,* 63, 1277–1287.

[16] Hosseini, S.M. & Shahmorad, S. (2003). Numerical solution of a class of integrodifferential equations by the tau method with an error estimation.*Applied Mathematics and Computation,* 136(2-3), 559-570.

[17] Shakeri, F. & Dehghan, M. (2013). A high order finite volume element method for solving elliptic partial integrodifferential equations. *Applied Numerical Mathematics,* 65, 105-118.

[18] Dehghan, M. & Salehi, R. (2012). The numerical solution of the non-linear integrodifferential equations based on the meshless method. *Journal of Computational and Applied Mathematics,* 236(9), 2367-2377.

[19] Dehghan, M. (2006). Solution of a partial integrodifferential equation arising from viscoelasticity. *International Journal of Computer Mathematics,* 83(1), 123-129.

[20] Wang, L. (2009). Approximate controllability of integrodifferential equations with multiple delays. *Journal of Optimization Theory and Applications,* 143, 185-206.

[21] Ji, S. & Yang, D. (2019). Solution to Riemann-Liouville fractional integrodifferential equations via fractional resolvents. *Advances in Difference Equations,* 524, 1-17.

[22] Sheng, J. & Jiang, W. (2017). Existence and uniqueness of the solution of fractional damped dynamical systems. *Advances in Continuous and Discrete Models,* 16, 1-14.

[23] Davies, I. & Jackreece, P. (2005). Controllability and null controllability of linear systems. *Journal of Applied Sciences and Environmental Management,* 9, 31-36.

[24] Haq, A. & Sukavanam, N. (2020). Controllability of second-order nonlocal retarded semilinear systems with delay in control. *Applicable Analysis,* 99(16), 2741-2754.

[25] Klamka, J. (2009). Constrained controllability of semilinear systems with delays. *Nonlinear Dynamics,* 56, 169-177.

[26] Liu, S., Debbouche, A. & Wang, J. (2018). ILC method for solving approximate controllability of fractional differential equations with noninstantaneous impulses. *Journal of Computational and Applied Mathematics,* 339, 343-355.

[27] Kumar, S. & Sukavanam, N. (2012). Approximate controllability of fractional order semilinear systems with bounded delay. *Journal of Differential Equations,* 252, 6163-6174.

[28] Rykaczewski, K. (2012). Approximate controllability of differential inclutions in Hilbert spaces. *Nonlinear Analysis,* 75, 2701-2702.

[29] Wang, J.R. & Zhou, Y. (2011). A class of fractional evolution equations and optimal controls. *Nonlinear Analysis: Real World Application,* 12, 262-272.

[30] Yang, M. & Wang, Q. (2016). Approximate controllability of Riemann-Liouville fractional differential inclusions. *Applied Mathematics and Computation,* 274, 267-281.

[31] Mahmudov, N.I. & McKibben, M.A. (2015). On the Approximate controllability of fractional evolution equations with generalized Riemann-Liouville fractional derivative. *Jouranl of Function Spaces,* 2015, 1-9.

[32] Li, K., Peng, J. & Jia, J. (2012). Cauchy problems for fractional differential equations with Riemann-Liouville fractional derivatives. *Journal of Functional Analysis,* 263, 476-510.

[33] Ibrahim, BHE., Fan, Z. & Li, G. (2017). Approximate controllability for functional equations with Riemann-Liouville derivative by iterative and approximate method. *Journal of Function Spaces,* 2017, 1-7.

[34] Haq, A. (2022). Partial-approximate controllability of semi-linear systems involving two Riemann-Liouville fractional derivatives. *Chaos, Solitons & Fractals,* 157, 111923. https://doi.org/10.1016/j.chaos.2022.111923

[35] Haq, A. & Sukavanam, N. (2022). Existence and controllability of higher-order nonlinear fractional integrodifferential systems via fractional resolvent, *Mathematical Methods in the Applied Sciences,* 45(16), 9034-9048.

[36] Zhu, S., Fan, Z. & Li, G. (2018). Approximate controllability of Riemann-Liouville fractional evolution equations with integral contractor assumption. *Journal of Applied Analysis & Computation,* 8, 532-548.

[37] Chang, Y.K., Pereira, A. & Ponce, R. (2017). Approximate controllability for fractional differential equations of sobolev type via properties on resolvent operators. *Fractional Calculus and Applied Analysis,* 20(4), 963-987.

[38] Liu, Z. & Li, X. (2015). Approximate controllability of fractional evolution systems with Riemann–Liouville fractional derivatives. *SIAM Journal on Control Optimization,* 53(1), 1920-1933.

[39] He, B., Zhou, H. & Kou, C. (2016). The controllability of fractional damped dynamical systems with control delay. *Communications in Nonlinear Science and Numerical Simulation,* 32, 190-198.

[40] Debbouche, A. & Antonov, V. (2017). Approximate controllability of semilinear Hilfer fractional differential inclusions with impulsive control inclusion conditions in Banach spaces. *Chaos, Solitons & Fractals,* 102, 140-148.

[41] Li, X., Liu, Z., Li, J. & Tisdell, C. (2019). Existence and controllability for non-linear fractional control systems with damping in Hilbert spaces. *Acta Matematica Scientia,* 39B(1), 229-242.

[42] Aimene, D., Baleanu, D. & Seba, D. (2019). Controllability of semilinear impulsive Atangana-Baleanu fractional differential equations with delay.*Chaos, Solitons & Fractals,* 128, 51-57.

[43] Ye, H.P., Gao, J.M. & Ding, Y.S. (2007). A generalized Gronwall inequality and its application to a fractional differential equation. *Journal of Mathematical Analysis and Applications,* 328, 1075-1081.

[44] Haq, A. & Sukavanam, N. (2022). Mild solution and approximate controllability of second-order retarded systems with control delays and nonlocal conditions. *Bulletin of the Iranian Mathematical Society,* 48(2), 447-464.

[45] Haq, A. & Sukavanam, N. (2021). Mild solution and approximate controllability of retarded semilinear systems with control delays and nonlocal conditions. *Numerical Functional Analysis and Optimization,* 42(6), 721-737.

[46] Sharma, M. (2021). Solvability and optimal control of nonautonomous fractional dynamical systems of neutral-type with nonlocal conditions. *Iranian Journal of Science and Technology, Transaction A: Science,* 45, 2121-2133. https://doi.org/10.1007/s40995-021-01215-z

[47] Patel, R., Shukla, A. & Jadon, S.S. (2020). Existence and optimal control problem for semilinear fractional order $(1, 2]$ control system. *Mathematical Methods in the Applied Sciences,* https://doi.org/10.1002/mma.6662.

[48] Shukla, A., Vijayakumar, V. & Nisar, K.S. (2021). A new exploration on the existence and approximate controllability for fractional semilinear impulsive control systems of order $r \in (1, 2)$. *Chaos, Solitons & Fractals,* 1-20.

[49] Dineshkumar, C., Udhayakumar, R., Vijayakumar, V., Shukla, A. & Nisar, K.S. (2021). A note on approximate controllability for nonlocal fractional evolution stochastic integrodifferential inclusions of order $r \in (1, 2)$ with delay. *Chaos, Solitons & Fractals,* 153, 111565.

[50] Shukla, A., Sukavanam, N. & Pandey, D.N. (2015). Complete controllability of semi-linear stochastic system with delay. *Rendiconti del Circolo Matematico di Palermo,* 64, 209-220.

[51] Sahijwani, L. & Sukavanam, N. (2023). Approximate controllability for Riemann-Liouville fractional differential equations. *International Journal of Optimization & Control: Theories & Applications,* 13, 59-67.

[52] Raja, M.M., Vijayakumar, V., Shukla, A., Nisar, K.S. & Baskonus, H.M. (2022). On the approximate controllability results for fractional integrodifferential systems of order $1 < r < 2$ with sectorial operators. *Journal of Computational and Applied Mathematics,* 415, 114492. https://doi.org/10.1016/j.cam.2022.114492

[53] Shukla, A., Sukavanam, N. & Pandey, D.N. (2015). Approximate controllability of semilinear fractional control systems of order $\alpha \in (1, 2]$. *SIAM Proceedings of the Conference on Control and its Applications.* https://doi.org/10.1137/1.9781611974072.2

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