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RESEARCH ARTICLE

Exploring constraint qualification-free optimality conditions for linear second-order cone programming

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ABSTRACT

Linear second-order cone programming (SOCP) deals with optimization problems characterized by a linear objective function and a feasible region defined by linear equalities and second-order cone constraints. These constraints involve the norm of a linear combination of variables, enabling the representation of a wide range of convex sets. The SOCP serves as a potent tool for addressing optimization challenges across engineering, finance, machine learning, and various other domains. In this paper, we introduce new optimality conditions tailored for SOCP problems. These conditions have the form of two optimality criteria that are obtained without the requirement of any constraint qualifications and are defined explicitly. The first criterion utilizes the concept of immobile indices of constraints. The second criterion, without relying explicitly on immobile indices, introduces a special finite vector set for assessing optimality. To demonstrate the effectiveness of these criteria, we present two illustrative examples highlighting their applicability. We compare the obtained criteria with other known optimality conditions and show the advantage of the former ones.

1. Introduction

A conic optimization problem is characterized by a constraint stipulating that the optimization variables must belong to a closed convex cone. Such problems encompass a wide spectrum of optimization problems and serve as a fundamental framework for addressing various real-world challenges. Conic problems form a broad and important class of optimization problems, since according to [1, 2], any convex optimization problem can be represented as a conic one. This universality underscores the essential significance of conic optimization in mathematical optimization theory. In recent years, conic optimization has attracted considerable attention due to its versatility and widespread applicability across diverse domains [3–5]. Among the most prominent and extensively studied subclasses of conic optimization problems are Linear Programming (LP) and convex Quadratic Programming (QP) problems. Another notable class of conic optimization problems is Semidefinite Programming (SDP), where the optimization is performed over the cone of positive semidefinite matrices. SDP has garnered significant interest owing to its utility in tackling a broad range of optimization tasks, including control theory, combinatorial optimization, and quantum information processing (see [6–8]).

Linear Second-Order Cone Programming (SOCP) deals with conic problems where the objective is to optimize a linear cost function over the intersection of an affine set and the product of the *second-order (Lorentz)* cones in a finite-dimensional vector space. The problems of LP, QP, and the quadratically constrained convex

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quadratic problems can be formulated as SOCP problems, which in turn, belong to a special class of SDP problems (see *e.g.* [9–11], and others).

The class of SOCP problems has been extensively studied in the past two decades due to its broad applicability across various fields of research, including engineering, finance, control theory, robust and combinatorial optimization. The literature dedicated to second-order problems is vast. For the applications, see, *e.g.* [10, 12, 13], and the references therein. As highlighted in [9], many of the SDP problems encountered in practical applications and considered in [7], can also be formulated as instances of SOCP problems, further emphasizing the significance and relevance of SOCP in optimization theory and practice.

Necessary and sufficient optimality conditions play an important role in optimization by providing a framework for identifying optimal solutions. By leveraging these conditions, researchers and practitioners can effectively discern the best possible outcome from the optimization process. Among the various types of optimality conditions, two prominent categories can be distinguished: the optimality conditions in *ordinary* (*punctual*) form as in, *e.g.*, [14–17] and *sequential* optimality conditions, see [18–20]. Additionally, other types of optimality conditions, such as those discussed in [21, 22], contribute to the comprehensive understanding of optimization processes and strategies.

To test ordinary optimality conditions for a primal feasible solution x^0, one has to find a finite vector y^0, which is a dual feasible solution, and check a finite number of equalities and inequalities constructed on the base of x^0 and y^0. When applying sequential optimality conditions to a feasible solution x^0, it is necessary to identify some sequences, $\{x^k\}$ and $\{y^k\}$, of vectors associated with the primal and dual variables, respectively, and check some conditions in the form of limits of functions built on the base of these sequences.

Optimality conditions are often formulated under certain additional conditions on the problem's constraints, known as *constraint qualifications* (CQ). Constraint qualifications are properties inherent in the analytical description of a feasible set ensuring that its structure around a given feasible point can be described by (first-order) approximations of the constraint functions (see *e.g.* [23]) and guarantee the Karush-Kuhn-Tucker (KKT) optimality conditions to hold at a local minimizer. The most widely used CQ for SOCP is the Slater condition (or *strict feasibility*) presupposing the existence of a feasible solution that belongs to the interior of the feasible set.

Constraint qualifications are particularly crucial for deriving primal and primal-dual characterizations of solutions in optimization and variational problems. They are essential for studying duality relations, conducting sensitivity and stability analysis, and justifying the convergence and evaluating the convergence rate of computational methods.

Many papers are dedicated to CQ conditions for different classes of optimization problems (see [14, 15, 18, 23–26], and others). One of the main challenges in this area is that for many conic problems in general and SOCP problems in particular, the CQs needed for formulation of optimality conditions may not hold (see, for example, [9, 16, 27], and the references therein). Therefore, it is very important to search for optimality conditions that do not rely on any CQ (referred to as *CQ-free optimality conditions*). Many research is dedicated to CQ-free optimality conditions for different classes of optimization problems (see [16, 19–21, 28, 29], and others). However, to the best of the authors' knowledge, no CQ-free optimality conditions in the ordinary form specifically designed for SOCP problems have been published to date.

In this paper, new CQ-free optimality conditions in the ordinary form are derived for SOCP problems. These conditions are formulated and proven in the form of two criteria. Illustrative examples demonstrate situations where classical conditions fail to test optimality, while the optimality criteria presented in the paper allow such a test.

The paper is structured as follows. In section 2, we formulate the problem and introduce the basic notation. In section 3, we introduce the set I_0 of special constraint indices referred to as *immobile*. Here the immobility of a constraint's index means that this constraint remains active for all feasible values of the problem's variables. We utilize the set of immobile indices to prove an optimality criterion for SOCP problems. This criterion does not use any additional conditions on the feasible set of the problem under consideration, making it an CQ-free optimality criterion. However, its application may be hindered by the requirement for information about the set I_0, which may not always be available. In the subsequent section 4, we present an alternative CQ-free optimality criterion wherein the set I_0 is not explicitly utilized. At the end of the section, we provide a short discussion on two different approaches to the CQ-free optimality conditions and on the properties of the approach proposed in the paper. Illustrative examples in section 5 highlight the new optimality conditions derived in the paper particularly in scenarios where the classical KKT optimality conditions fail to suffice. In section 6, motivated by the optimality criterion obtained by Gorokhovik in [21], for a more general class of convex problems and using the *lexicographical separations approach*, we formulate the optimality criteria for SOCP. We compare this criterion with that obtained in sections 3 and 4. The paper ends with some conclusions presented in section 7.

2. Problem's statement and basic notions

Consider a linear second-order cone programming problem in the form

$$\textbf{SOCP}: \qquad \max\ b^\top x \quad \text{s.t. } A_i x + c(i) \in \mathcal{SOC}(i), i \in I,$$

where $x \in \mathbb{R}^n$ is a vector of decision variables, $b \in \mathbb{R}^n$, $c(i) \in \mathbb{R}^{m_i+1}$, $A_i \in \mathbb{R}^{(m_i+1)\times n}$, $i \in I$, are given vectors and matrices; the sets

$$\mathcal{SOC}(i) := \{z = \begin{pmatrix} z_0 \\ z_* \end{pmatrix} \in \mathbb{R}^{m_i+1}, \quad z_0 \in \mathbb{R},\ z_* \in \mathbb{R}^{m_i} : ||z_*|| \leq z_0\},\ i \in I,$$

are the second-order cones. Here $n \in \mathbb{N}$, $m_i \in \mathbb{N}$, $i \in I$; $||z_*|| = \sqrt{z_*^\top z_*}$, and the set $I \subset \mathbb{N}$ is supposed to be a finite index set.

Given $i \in I$, the second-order cone $\mathcal{SOC}(i)$ is convex, full-dimensional, *nice*, and consequently, is *facially exposed* (for definitions see *e.g.* [6]).

In what follows, for $i \in I$, we will suppose that a vector $z \in \mathcal{SOC}(i)$ has the form $z = (z_0, z_*^\top)^\top \in \mathbb{R}^{m_i+1}$, where $z_0 \in \mathbb{R}$, $z_* \in \mathbb{R}^{m_i}$.

Given $x \in \mathbb{R}^n$ and $i \in I$, denote

$$z(i, x) := A_i x + c(i).$$

For the problem (**SOCP**), the corresponding standard (Lagrangian) dual problem has the form

$$\textbf{SOCD}: \qquad \min \sum_{i \in I} c(i)^\top y(i) \quad \text{s.t.} \sum_{i \in I} A_i^\top y(i) = -b, \quad y(i) \in \mathcal{SOC}(i),\ i \in I,$$

where vectors $y(i)$, $i \in I$, are the decision variables.

A vector $x \in \mathbb{R}^n$ is a *strictly feasible solution* in the problem (**SOCP**) if $z(i, x) \in \text{int}\,\mathcal{SOC}(i)$ for all $i \in I$. A feasible solution of the problem (**SOCD**), consisting of vectors $y(i)$, $i \in I$, is called *strictly feasible* if $y(i) \in \text{int}\,\mathcal{SOC}(i)$ for all $i \in I$. Here int$S$ stands for the interior of a set S.

Lemma 1. *[Weak duality, [9]] If $\bar{x}$ is feasible in the problem* (**SOCP**) *and $(\bar{y}(i), i \in I)$ is feasible in the dual problem* (**SOCD**)*, then the value of the objective function of* (**SOCP**) *evaluated at $\bar{x}$ is less than or equal to the value of the objective function of* (**SOCD**) *evaluated at $(\bar{y}(i), i \in I)$.*

Given a primal-dual pair of optimization problems (**P**) and (**D**), let val(**P**) and val(**D**) denote the optimal values of the cost functions of these problems. The difference val(**D**) $-$ val(**P**) is called *the duality gap*.

From Lemma 1, it follows that for a pair of dual problems (**SOCP**) and (**SOCD**), the duality gap is non-negative. To guarantee that the duality gap is equal to zero, the problems should satisfy certain additional conditions.

The following theorems are proved in [9].

Theorem 1. *[Strong duality] If the second-order cone problems* (**SOCP**) *and* (**SOCD**) *have strictly feasible solutions, then they both have optimal solutions (are solvable) and val*(**SOCD**) $-$ *val*(**SOCP**) $= 0$.

Theorem 2. *[KKT optimality conditions] Suppose that* (**SOCP**) *is strictly feasible (admits a strictly feasible solution). Then a feasible solution x^0 is optimal in this problem iff there exist vectors $y^0(i)$, $i \in I$, such that*

$$\sum_{i \in I} A_i^\top y^0(i) = -b, \qquad y^0(i) \in \mathcal{SOC}(i),\ y^0(i)^\top z(i, x^0) = 0\ \forall i \in I. \tag{1}$$

Without additional conditions (CQs) on the constraints of the problem (**SOCP**), the duality gap may be positive. In this case, the KKT optimality conditions may not be met (see [9,27], and the example below).

The aim of this study is to formulate and prove for the second-order cone problem (**SOCP**) new CQ-free optimality conditions in the ordinary form.

3. An optimality criterion for the primal second-order cone problem

Denote by X the set of feasible solutions of the problem (**SOCP**):

$$X := \{x \in \mathbb{R}^n : z(i, x) \in \mathcal{SOC}(i)\ \forall i \in I\}. \tag{2}$$

Notice that the set X is convex.

Suppose that $X \neq \emptyset$ and consider a subset of the index set I:

$$I_0 := \{i \in I : \ ||z_*(i, x)|| = z_0(i, x)\ \forall x \in X\}. \tag{3}$$

This subset plays an important role in our approach. It contains the indices of constraints that can be characterized as *always active* or *immobile* in the terminology of our previous papers (see *e.g.* [16, 30], and the references therein).

The constraints of the problem (**SOCP**) are said to satisfy the *Slater condition* if the problem admits a strictly feasible solution, *i.e.* there exists a vector $\bar{x} \in \mathbb{R}^n$ such that

$$z(i,\bar{x}) \in \operatorname{int} \mathcal{SOC}(i) \ \forall i \in I. \tag{4}$$

The Slater condition is one of CQs that guarantee the existence of KKT multipliers for a given optimal solution.

It is easy to show that conditions (4) are equivalent to the inequalities

$$||z_*(i,\bar{x})|| < z_0(i,\bar{x}) \ \forall i \in I. \tag{5}$$

Therefore, in terms of (3), one can see that the constraints of the problem (**SOCP**) satisfy the Slater condition iff $I_0 = \emptyset$. Hence, the emptiness of the set I_0 can be considered as a constraint qualification.

In what follows, we will use the following notation for $i \in I$:

$$\mathcal{R}_i := \begin{pmatrix} 1 & 0 & ... & 0 \\ 0 & -1 & .... & 0 \\ ... & ... & ... & ... \\ 0 & 0 & ... & -1 \end{pmatrix} \in \mathbb{R}^{(m_i+1)\times(m_i+1)};$$

$$\operatorname{int} \mathcal{SOC}(i) := \{z = \begin{pmatrix} z_0 \\ z_* \end{pmatrix} \in \mathbb{R}^{m_i+1} : \ ||z_*|| < z_0\},$$

$$\operatorname{bd}^+ \mathcal{SOC}(i) := \{z = \begin{pmatrix} z_0 \\ z_* \end{pmatrix} \in \mathbb{R}^{m_i+1} : \ ||z_*|| = z_0, \ z_0 > 0\}.$$

Then, for any $i \in I$, it holds

$$\mathcal{SOC}(i) = \operatorname{int} \mathcal{SOC}(i) \cup \operatorname{bd}^+ \mathcal{SOC}(i) \cup \{\mathbf{0}\}, \tag{6}$$

where $\mathbf{0}$ is the null vector in the corresponding real space $\mathbb{R}^{m_i+1}$.

Since $X \neq \emptyset$, then it is easy to show that there exists a vector $\tilde{x} \in \mathbb{R}^n$ such that

$$||z_*(i,\tilde{x})|| < z_0(i,\tilde{x}) \ \forall i \in I \setminus I_0, \quad ||z_*(i,\tilde{x})|| = z_0(i,\tilde{x}) \ \forall i \in I_0. \tag{7}$$

A vector $\tilde{x}$ satisfying (7), is called a *minimally active feasible solution* of the problem (**SOCP**).

For $i \in I$, let $z \in \mathbb{R}^{m_i+1}$ and $y \in \mathbb{R}^{m_i+1}$ be complementary, *i.e.* satisfy the following *complementarity conditions*:

$$z^\top y = 0, \ z \in \mathcal{SOC}(i), \ y \in \mathcal{SOC}(i). \tag{8}$$

Then (see [9]) one of the next conditions takes a place:

a⁰) $z \in \operatorname{int} \mathcal{SOC}(i) \implies y = \mathbf{0}$;
b⁰) $z \in \operatorname{bd}^+ \mathcal{SOC}(i) \implies y = \alpha \mathcal{R}_i z, \ \alpha \geq 0$;
c⁰) $z = \mathbf{0} \implies \forall y \in \mathcal{SOC}(i)$.

Proposition 1. *Let $\tilde{x}$ be a minimally active feasible solution of the problem* (**SOCP**). *Then for $i \in I_0$ and $x \in X$, there exists a corresponding number $\alpha_i = \alpha_i(x)$, such that*

$$z(i,x) = \alpha_i z(i,\tilde{x}), \ \alpha_i \geq 0. \tag{9}$$

Proof. Let $i \in I_0$ and $x \in X$. It follows from the convexity of the set X that $0.5(\tilde{x}+x) \in X$. From this inclusion and the definition of the index set I_0, one can conclude:

$$||z_*(i,0.5(\tilde{x}+x))|| = z_0(i,0.5(\tilde{x}+x)), \quad ||z_*(i,\tilde{x})|| = z_0(i,\tilde{x}), \ ||z_*(i,x)|| = z_0(i,x). \tag{10}$$

Consequently,

$$0.5||z_*(i,\tilde{x}) + z_*(i,x)|| = 0.5z_0(i,\tilde{x}) + 0.5z_0(i,x) = 0.5||z_*(i,\tilde{x})|| + 0.5||z_*(i,x)||.$$

The equality $||z_*(i,\tilde{x}) + z_*(i,x)|| = ||z_*(i,\tilde{x})|| + ||z_*(i,x)||$ obtained above can be rewritten as follows: $(z_*(i,\tilde{x}) + z_*(i,x))^\top(z_*(i,\tilde{x}) + z_*(i,x)) = ||z_*(i,\tilde{x})||^2 + 2||z_*(i,\tilde{x})||\cdot||z_*(i,x)|| + ||z_*(i,x)||^2$, wherefrom we obtain $z_*(i,\tilde{x})^\top z_*(i,x) = ||z_*(i,\tilde{x})|| \cdot ||z_*(i,x)||$.

Taking into account the latter equality and the well-known relation $a^\top b = \cos(\varphi)||a||\cdot||b||$, where φ is the angle between the vectors a and b, we obtain that the cosine of the angle between vectors $z_*(i,x)$ and $z_*(i,\tilde{x})$ is equal to 1, and, hence, these vectors are collinear. This implies that

$$z_*(i,x) = \alpha_i z_*(i,\tilde{x}) \ \text{ with some } \alpha_i \geq 0. \tag{11}$$

It follows from (10) and (11) that

$$z_0(i,x) = ||z_*(i,x)|| = \alpha_i||z_*(i,\tilde{x})|| = \alpha_i z_0(i,\tilde{x}).$$

The equality obtained, $z_0(i,x) = \alpha_i z_0(i,\tilde{x})$, together with (11) imply that relations (9) hold true for $i \in I_0$ and $x \in X$. □

Let us fix a minimally active feasible solution $\tilde{x}$ of the problem (**SOCP**) and denote

$$\gamma(i) := z(i,\tilde{x}) \ \forall i \in I_0.$$

Then it follows from Proposition 1 that for an immobile index $i \in I_0$ and for a feasible solution $x \in X$, the non-linear condition

$$z(i,x) \in \mathcal{SOC}(i) \iff ||z_*(i,x)|| \leq z_0(i,x)$$

can be replaced by (m_i+1) linear equalities $z(i,x) = \alpha_i z(i,\tilde{x})$ with one additional variable $\alpha_i \geq 0$. Based on this, it is easy to show that $X = \bar{X}$, where

$$\bar{X} := \{x \in \mathbb{R}^n : \ z(i,x) \in \mathcal{SOC}(i) \ \forall i \in I \setminus I_0, \quad z(i,x) = \alpha_i \gamma(i) \text{ with some } \alpha_i \geq 0 \ \forall i \in I_0\}.$$

It follows from the considerations above that the problem (**SOCP**) is equivalent to the following one:

$$\mathbf{P}_*: \quad \max\ b^\top x$$
$$\text{s.t. } A_i x + c(i) = z(i),\ z(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_0;$$
$$A_i x + c(i) = \alpha_i \gamma(i),\ \alpha_i \geq 0\ \forall i \in I_0,$$

where the decision variables are vector $x \in \mathbb{R}^n$ and numbers α_i, $i \in I_0$.

Notice that in the problem ($\mathbf{P}_*$), there is a finite number of equality and inequality constraints

$$A_i x + c(i) = \alpha_i \gamma(i),\ \alpha_i \geq 0\ \forall i \in I_0,$$

that are linear w.r.t. $x \in \mathbb{R}^n$ and $\alpha_i \in \mathbb{R}$, $i \in I_0$. Moreover, there exists a feasible solution $\tilde{x}$ of the problem (**SOCP**) such that the feasible solution

$$\tilde{x},\ \tilde{\alpha}_i = 1,\ i \in I_0,\ \tilde{z}(i) = z(i, \tilde{x}),\ i \in I \setminus I_0,$$

of the problem ($\mathbf{P}_*$) satisfies the following strict inequalities:

$$\tilde{\alpha}_i > 0,\ i \in I_0,\ ||\tilde{z}_*(i)|| < \tilde{z}_0(i),\ i \in I \setminus I_0.$$

Hence, the constraints of this problem satisfy the generalized Slater condition (see [31]), and one can use the classical KKT optimality conditions for testing optimality of its feasible solution (x^0, α_i^0, $i \in I_0$).

Taking into account the equivalence of the problems (**SOCP**) and ($\mathbf{P}_*$), we obtain the following result.

Theorem 3. *[Optimality criterion 1] A feasible solution $x^0 \in X$ of the problem* (**SOCP**) *is optimal in this problem iff there exist vectors $y(i) \in \mathbb{R}^{m_i+1}$, $i \in I$, such that the following relations hold true:*

$$\sum_{i \in I} A_i^\top y(i) = -b,\ z(i, x^0)^\top y(i) = 0\ \forall i \in I; \quad (12)$$

$$y(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_0; \qquad y(i)^\top \gamma(i) \geq 0 \quad \forall i \in I_0. \quad (13)$$

Conditions (12), (13) are similar to the KKT conditions (1) but simpler than them. The difference is as follows: the conic conditions $y^0(i) \in \mathcal{SOC}(i)$, $i \in I_0$, in (1) are replaced by the linear ones $y(i)^\top \gamma(i) \geq 0$, $i \in I_0$, in (12), (13). Therefore, finding a solution to system (12), (13) is no more difficult than finding a solution to the KKT system (1).

It is evident that if $I_0 = \emptyset$, then conditions (12), (13) coincide with (1).

It should be noted here that the optimality criterion in the form of Theorem 3 does not use any additional conditions on the feasible set of the problem (**SOCP**) and is therefore an CQ-free optimality criterion. The only possible difficulty in its application is the need to know the set of immobile indices I_0.

In the next section, we will demonstrate an alternative CQ-free optimality criterion that does not explicitly rely on any knowledge of I_0.

4. An alternative CQ-free optimality criterion for the second-order cone programming

The optimality criterion presented in this section is based on the following idea used in literature for convex optimization problems (see, for example [22]).

For a given convex problem, at the first step, one attempts to obtain an *exact extended dual* problem (EEDP) explicitly formulated in terms of the data of the original primal problem (see [32–35]). The exact (strong) duality property entails that when the primal problem and its corresponding dual are consistent, their optimal values are equal, and the dual problem attains its optimal value.

The dual problem (EEDP) has an extended set of dual decision variables compared to the Lagrangian dual. Note that some regularization procedure is necessary to justify the exactness of this dual problem.

At the second step, taking into account the exactness of the extended dual problem (EEDP), attempts are made to formulate CQ-free optimality conditions for a feasible solution to the original primal problem using an optimal solution to this dual problem.

Below, we utilize this idea to derive an CQ-free optimality criterion for the problem (**SOCP**). Taking into account the specific nature of the problem under consideration, we are able to formulate the optimality conditions without an explicit representation of the corresponding exact extended dual problem. The regularization procedure associated with this formulation is implicitly embedded within the proof of the criterion.

It is worth noting that the KKT optimality conditions (see Theorem 2) are also based on a similar idea: these conditions are formulated using the set of vectors $y^0(i)$, $i \in I$, (the KKT multipliers for a given optimal solution) which, in fact, represents an optimal solution of the Lagrangian dual problem (**SOCD**). However, in the formulation of these conditions, this fact is not explicitly mentioned.

We commence by formally introducing a set of vectors that, in essence, constitutes a feasible solution of the exact extended dual problem.

Having fixed $i \in I$ and $k_0 \in \mathbb{N}$, $0 \le k_0 \le |I|$, consider the following set of vectors:

$$\{\pi(k,i) \in \mathbb{R}^{m_i+1},\ k = 0,1,\dots,k_0\}. \tag{14}$$

If $\pi(k,i) \not\equiv \mathbf{0}$ for all $k = 0,1,\dots,k_0$, denote

$$q_i = \min\{k : 0 \le k \le k_0,\ \pi(k,i) \ne \mathbf{0}\}.$$

We say that for a given $i \in I$, the set of vectors (14) satisfies Condition (A) if one of the following conditions is true:

A1) $\pi(k,i) \equiv \mathbf{0}$ for all $k = 0,1,\dots,k_0$;
A2) $\pi(q_i,i) \in \mathcal{SOC}(i)$, $\pi(k,i)^\top \mathcal{R}_i \pi(q_i,i) \ge 0$ for all $k = q_i+1,\dots,k_0$.

Here and in what follows, the set of indices $\{k = q, q+1,\dots,s\}$ is assumed to be empty if $s < q$.

Let us prove a technical proposition.

Proposition 2. *Suppose that $i \in I$ and that the set of vectors (14) satisfies Condition (A). Then for any $z \in \mathcal{SOC}(i)$, there exists $\bar\theta = \bar\theta(z) > 0$ such that*

$$\sum_{k=0}^{k_0} \theta^{k_0-k} z^\top \pi(k,i) \ge 0 \quad \forall \theta \ge \bar\theta. \tag{15}$$

Proof. If $\pi(k,i) \equiv \mathbf{0}$ for all $k = 0,1,\dots,k_0$, then inequalities (15) are trivially satisfied with any $\bar\theta > 0$.

Suppose that $\pi(k,i) \not\equiv \mathbf{0}$ for $k = 0,1,\dots,k_0$. In this case, we have

$$\sum_{k=0}^{k_0} \theta^{k_0-k} z^\top \pi(k,i) = \sum_{k=q_i}^{k_0} \theta^{k_0-k} z^\top \pi(k,i), \tag{16}$$

where $z^\top \pi(q_i,i) \ge 0$ since $z \in \mathcal{SOC}(i)$ and $\pi(q_i,i) \in \mathcal{SOC}(i)$.

If $z^\top \pi(q_i,i) > 0$, then evidently, the inequalities (15) hold true for a sufficiently large $\bar\theta > 0$.

Suppose that $z^\top \pi(q_i,i) = 0$. Since $z \in \mathcal{SOC}(i)$, we can distinguish the following three cases:

1) $z = \mathbf{0}$, 2) $z \in \operatorname{int} \mathcal{SOC}(i)$, and 3) $z \in \operatorname{bd}^+ \mathcal{SOC}(i)$.

In case 1), relations (15) are trivially satisfied with any $\bar\theta > 0$.

In case 2), the equality $z^\top \pi(q_i,i) = 0$ implies $\pi(q_i,i) = \mathbf{0}$ that contradicts the assumption $\pi(q_i,i) \ne \mathbf{0}$. Therefore, this case is impossible.

In case 3), the equality $z^\top \pi(q_i,i) = 0$ and the inequality $\pi(q_i,i) \ne \mathbf{0}$ imply $z = \alpha_i(z)\mathcal{R}_i \pi(q_i,i)$ with some $\alpha_i(z) > 0$. Hence, taking into account the latter relations and Condition A2), we obtain the following inequalities:

$$z^\top \pi(k,i) = \alpha_i(z)\pi(k,i)^\top \mathcal{R}_i \pi(q_i,i) \ge 0,$$

for all $k = q_i+1,\dots,k_0$. These inequalities together with equalities (16) and $z^\top \pi(q_i,i) = 0$, ensure that relations (15) are satisfied for any $\bar\theta > 0$. □

For a given $x \in X$, introduce an index set

$$I_a(x) := \{i \in I : ||z_*(i,x)|| = z_0(i,x)\}.$$

Theorem 4. *[Optimality Criterion 2] A vector $x^0 \in X$ is an optimal solution of the problem* (**SOCP**) *iff there exist an integer number k_0, $0 \le k_0 \le |I_a(x^0)|$, and the sets of vectors*

$$\{\pi(k,i) \in \mathbb{R}^{m_i+1}, k = 0,1,\dots,k_0\}, i \in I_a(x^0), \tag{17}$$

satisfying Condition (A) for all $i \in I_a(x^0)$, such that

$$\begin{aligned} &\sum_{i \in I_a(x^0)} A_i^\top \pi(k,i) = \mathbf{0}\ \forall k = 0,\dots,k_0-1; \\ &\sum_{i \in I_a(x^0)} A_i^\top \pi(k_0,i) = -b, \end{aligned} \tag{18}$$

and

$$\begin{aligned} &z(i,x^0)^\top \pi(k,i) = 0 \\ &\forall k = 0,1,\dots,k_0,\ \forall i \in I_a(x^0). \end{aligned} \tag{19}$$

Proof. *Sufficiency.* Suppose that there exists a set of vectors (17) satisfying Condition (A) and relations (18) and (19). Then it follows from (19) that

$$\begin{aligned} 0 &= \sum_{i \in I_a(x^0)} z(i,x^0)^\top \pi(k,i) \\ &= \sum_{i \in I_a(x^0)} [A_i x^0 + c(i)]^\top \pi(k,i) \\ &= \sum_{i \in I_a(x^0)} c(i)^\top \pi(k,i) + {x^0}^\top \sum_{i \in I_a(x^0)} A_i^\top \pi(k,i), \end{aligned}$$

for all $k = 0,1,\dots,k_0$. From these equalities and (18), we obtain

$$\begin{aligned} &\sum_{i \in I_a(x^0)} c(i)^\top \pi(k,i) = 0\ \forall k = 0,\dots,k_0-1, \\ &\sum_{i \in I_a(x^0)} c(i)^\top \pi(k_0,i) = b^\top x^0. \end{aligned} \tag{20}$$

It follows from Proposition 2 that for any $x \in X$, there exists $\bar\theta = \bar\theta(x) > 0$ such that

$$\sum_{k=0}^{k_0} \bar\theta^{k_0-k} z(i,x)^\top \pi(k,i) \ge 0 \quad \forall i \in I_a(x^0). \tag{21}$$

For $x \in X$, taking into account (18) - (20), let us calculate

$$\begin{aligned} b^\top x &= -\sum_{i\in I_a(x^0)} x^\top A_i^\top \pi(k_0,i) \\ &= -\sum_{i\in I_a(x^0)} x^\top A_i^\top \sum_{k=0}^{k_0} \bar{\theta}^{k_0-k}\pi(k,i) \\ &= -\sum_{i\in I_a(x^0)} z(i,x)^\top \sum_{k=0}^{k_0} \bar{\theta}^{k_0-k}\pi(k,i) \\ &+ \sum_{i\in I_a(x^0)} c(i)^\top \sum_{k=0}^{k_0} \bar{\theta}^{k_0-k}\pi(k,i) \\ = &- \sum_{i\in I_a(x^0)} \sum_{k=0}^{k_0} \bar{\theta}^{k_0-k} z(i,x)^\top \pi(k,i) + b^\top x^0. \end{aligned}$$

These relations together with (21), permit one to conclude that $b^\top x \leq b^\top x^0$ for all $x \in X$. Hence $x^0 \in X$ is an optimal solution of the problem (**SOCP**).

Necessity. Let $x^0 \in X$ be an optimal solution to the problem (**SOCP**). Let us construct a set of vectors (17) satisfying the Condition (A) and relations (18), (19). We will do this iteratively by performing the following iterations.

Iteration # 0. Consider the problem

$$\textbf{P-0}: \quad \max \mu, \quad \text{s.t. } A_i x + c(i) - e_0(i)\mu = z(i), \quad z(i) \in \mathcal{SOC}(i) \ \forall i \in I,$$

where $e_0(i) = (1,0,\ldots,0)^\top \in \mathbb{R}^{m_i+1}$, $i \in I$.

The constraints of this problem satisfy the Slater condition. In fact, for any $x \in X$, the vector $(x,\, \mu = -1,\ z(i),\, i \in I)^\top$ with $z(i) = (z_0(i) = z_0(i,x)+1, z_*(i) = z_*(i,x)), i \in I$, is a feasible solution of the problem (**P-0**) satisfying the strict inequalities

$$||z_*(i)|| < z_0(i) \ \forall i \in I.$$

If this problem admits a feasible solution $(\bar{x},\, \bar{\mu},\, \bar{z}(i), i \in I)$ with $\bar{\mu} > 0$, then set $k_0 = 0$ and go to the Final Step.

Otherwise, for any $x \in X$, the vector

$$(x,\, \mu = 0,\, z(i) = z(i,x), i \in I) \tag{22}$$

is an optimal solution of the problem (**P-0**). Since the constraints of this problem satisfy the Slater condition, applying the classical KKT optimality conditions to its optimal solution (22), we conclude that there exist vectors

$$y^0(i) = \begin{pmatrix} y_0^0(i) \\ y_*^0(i) \end{pmatrix} \in \mathbb{R}^{m_i+1}, \quad y_*^0(i) \in \mathbb{R}^{m_i},\ i \in I, \tag{23}$$

such that the following relations hold true for any $x \in X$:

$$\sum_{i\in I} A_i^\top y^0(i) = \mathbf{0},\ \sum_{i\in I} y_0^0(i) = 1, \tag{24}$$

$$z(i,x)^\top y^0(i) = 0,\ y^0(i) \in \mathcal{SOC}(i) \ \forall i \in I. \tag{25}$$

Consider the index set

$$\Delta I_1 := \{i \in I : y_0^0(i) > 0\}.$$

It follows from (24) that $\Delta I_1 \neq \emptyset$. Let us show that

$$||z_*(i,x)|| = z_0(i,x) \ \forall i \in \Delta I_1,\ \forall x \in X, \tag{26}$$

and consequently, the indices in ΔI_1 are immobile.

Suppose the contrary: there exist $i_0 \in \Delta I_1$ and $\bar{x} \in X$ such that $||z_*(i_0,\bar{x})|| < z_0(i_0,\bar{x})$. Then from the equality in (25) with $i = i_0$ and the conditions $z(i_0,\bar{x}) \in \mathcal{SOC}(i_0)$, $y^0(i_0) \in \mathcal{SOC}(i_0)$, we can conclude that $y^0(i_0) = \mathbf{0}$. But this contradicts the inequality $y_0^0(i_0) > 0$ that is fulfilled by construction. Hence equalities (26) are satisfied. Remind here that relations (24), (25) are valid for all $x \in X$.

Let us show that for all $i \in \Delta I_1$ and $x \in X$, the following is true:

$$\exists \alpha_i(x) \geq 0 \textit{ such that } z(i,x) = \alpha_i(x)\mathcal{R}_i y^0(i). \tag{27}$$

Let $i \in \Delta I_1$ and $x \in X$. If $y^0(i) \in \operatorname{int}\mathcal{SOC}(i)$, then it follows from the equality in (25) and the condition $z(i,x) \in \mathcal{SOC}(i)$, that $z(i,x) = \mathbf{0}$. Hence, in this case, relations (27) are satisfied with $\alpha_i = 0$. If $y^0(i) \in \operatorname{bd}^+\mathcal{SOC}(i)$, then it follows from (25) and the inclusion $z(i,x) \in \mathcal{SOC}(i)$, that $z(i,x) = \alpha_i(x)\mathcal{R}_i y^0(i)$ with some $\alpha_i(x) \geq 0$. Consequently, the equality in (27) holds true in this case as well. Taking into account that $y^0(i) \neq \mathbf{0}$ for $i \in \Delta I_1$, we conclude that relations (27) are proved.

It follows from (27) that for an immobile index $i \in \Delta I_1$ and for a feasible solution $x \in X$, the non-linear condition

$$z(i,x) \in \mathcal{SOC}(i) \iff ||z_*(i,x)|| \leq z_0(i,x)$$

can be replaced by $(m_i + 1)$ linear equalities $z(i,x) = \alpha_i \mathcal{R}_i y^0(i)$ with one additional variable $\alpha_i \geq 0$. Based on this, it is easy to see that $X = X_0$, where

$$\begin{aligned} X_0 :=& \{x \in \mathbb{R}^n : z(i,x) \in \mathcal{SOC}(i), i \in I \setminus I_1; \\ & z(i,x) = \alpha_i \bar{\gamma}(i) \text{ with some } \alpha_i \geq 0,\, i \in I_1\}, \\ I_1 :=& \Delta I_1, \\ \bar{\gamma}(i) :=& \mathcal{R}_i y^0(i) \in \mathcal{SOC}(i) \ \forall i \in \Delta I_1. \end{aligned} \tag{28}$$

In fact, if $x \in X_0$, then it is evident that $z(i,x) \in \mathcal{SOC}(i)$ for all $i \in I$. Hence, $x \in X$, and consequently, $X_0 \subset X$. Now suppose that $x \in X$. Then it follows from (27) and (28) that $x \in X_0$

and hence, $X \subset X_0$. The equality $X = X_0$ is proved.

The set of vectors (23) constructed above satisfies the conditions

$$\begin{aligned} &y^0(i) = \mathbf{0}\ \forall i \in I \setminus I_1;\\ &y^0(i) \in \mathcal{SOC}(i),\ y^0(i) \neq \mathbf{0}\ \forall i \in I_1. \end{aligned} \tag{29}$$

Go to the next Iteration #1 with the data (28).

Iteration # k ($k \geq 1$). At the beginning of this iteration, we have the following set and vectors:

$$I_k = \Delta I_1 \cup \cdots \cup \Delta I_k,\ \bar{\gamma}(i) = \mathcal{R}_i y^s(i) \in \mathcal{SOC}(i),$$
$$\bar{\gamma}_0(i) \neq 0,\ i \in \Delta I_{s+1},\ s = 0, 1, \ldots, k-1.$$

Consider the problem

$$\begin{aligned} \textbf{P-k}: \quad & \max \mu,\\ \text{s.t. } & A_i x + c(i) - e_0(i)\mu = z(i)\ \forall i \in I \setminus I_k,\\ & A_i x + c(i) = \alpha_i \bar{\gamma}(i)\ \forall i \in I_k,\\ & z(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_k, \alpha_i \geq 0\ \forall i \in I_k. \end{aligned}$$

The constraints of this problem satisfy the generalized Slater condition (see [31]).

If the problem (**P-k**) admits a feasible solution $(\bar{x},\ \bar{\mu},\ \bar{z}(i), i \in I \setminus I_k,\ \bar{\alpha}_i, i \in I_k)$ with $\bar{\mu} > 0$, then set $k_0 = k$ and go to the Final Step.

Otherwise, for any $x \in X$, the vector

$$\begin{aligned} &(x,\ \mu = 0,\ z(i,x), i \in I \setminus I_k;\\ &\alpha_i(x) = z_0(i,x)/\bar{\gamma}_0(i),\ i \in I_k) \end{aligned} \tag{30}$$

is an optimal solution to the problem (**P-k**). Taking into account that the constraints of this problem satisfy the generalized Slater condition and applying the KKT optimality conditions to its optimal solution (30), we conclude that there exist vectors

$$\begin{aligned} &y^k(i) = \begin{pmatrix} y_0^k(i) \\ y_*^k(i) \end{pmatrix} \in \mathbb{R}^{m_i+1},\\ &y_*^k(i) \in \mathbb{R}^{m_i},\ i \in I, \end{aligned} \tag{31}$$

such that the following relations hold true:

$$\sum_{i\in I} A_i^\top y^k(i) = \mathbf{0},\ \sum_{i \in I\setminus I_k} y_0^k(i) = 1, \tag{32}$$

$$\begin{aligned} &z(i,x)^\top y^k(i) = 0\ \forall i \in I;\\ &y^k(i) \in \mathcal{SOC}(i)\, \forall i \in I \setminus I_k;\\ &\bar{\gamma}(i)^\top y^k(i) \geq 0\, \forall i \in I_k. \end{aligned} \tag{33}$$

Consider the index set

$$\Delta I_{k+1} := \{i \in I \setminus I_k : y_0^k(i) > 0\}.$$

It follows from (32) that

$$\Delta I_{k+1} \neq \emptyset. \tag{34}$$

Similar to how it was done on the initial Iteration # 0, one can show that

$$||z_*(i,x)|| = z_0(i,x)\ \forall i \in \Delta I_{k+1},\ \forall x \in X, \tag{35}$$

$$\begin{aligned} &z(i,x) = \alpha_i(x)\mathcal{R}_i y^k(i),\\ &\alpha_i(x) \geq 0\ \forall i \in \Delta I_{k+1},\ \forall x \in X. \end{aligned} \tag{36}$$

Set

$$I_{k+1} = I_k \cup \Delta I_{k+1} = \Delta I_1 \cup \Delta I_2 \cup \cdots \cup \Delta I_{k+1},$$
$$\bar{\gamma}(i) = \mathcal{R}_i y^k(i),\ i \in\ \Delta I_{k+1}.$$

It follows from (36) that $X = X_k$, where

$$\begin{aligned} X_k := \{&x \in \mathbb{R}^n : z(i,x) \in \mathcal{SOC}(i), i \in I \setminus I_{k+1};\\ &z(i,x) = \alpha_i \bar{\gamma}(i) \text{ with some } \alpha_i \geq 0,\ i \in I_{k+1}\}. \end{aligned} \tag{37}$$

The set of vectors defined in (31)-(33), satisfies the following relations:

$$y^k(i) = \mathbf{0}\ \forall i \in I \setminus I_{k+1}, \tag{38}$$

$$y^k(i) \in \mathcal{SOC}(i),\ y_0^k(i) \neq 0\ \forall i \in \Delta I_{k+1}; \tag{39}$$

$$\begin{aligned} &y^k(i)^\top \mathcal{R}_i y^{s-1}(i) = y^k(i)^\top \bar{\gamma}(i) \geq 0\\ &\forall i \in \Delta I_s,\ s = 1, \ldots, k. \end{aligned} \tag{40}$$

Go to the next Iteration #$(k+1)$ using the set I_{k+1} and vectors $\bar{\gamma}(i),\ i \in I_{k+1},\ y^s(i),\ i \in I,$ $s = 0, 1, \ldots, k$ found above.

Final Step. It follows from condition (34) that after a finite number of iterations, we will get to the Final Step with some k_0, $0 \leq k_0 \leq |I_0|$, where I_0 is the set of immobile indices of the constraints of the problem (**SOCP**) (see (3)).

From (26) and (35) we have:

$$I_{k_0} = \Delta I_1 \cup \cdots \cup \Delta I_{k_0} \subset I_0. \tag{41}$$

By construction, a number k_0 is such that for $k = k_0$, the problem (**P-k**) has a feasible solution

$$(\bar{x},\ \bar{\mu},\ \bar{z}(i),\ i \in I \setminus I_{k_0},\ \bar{\alpha}_i,\ i \in I_{k_0})$$

with $\bar{\mu} > 0$. Hence, $\bar{x} \in X_{k_0-1} = X$, where X_{k_0-1} is defined in (37) with $k = k_0 - 1$, and

$$||z_*(i,\bar{x})|| < z_0(i,\bar{x})\ \forall i \in I \setminus I_{k_0}. \tag{42}$$

Notice that for $k_0 = 0$, the set I_{k_0} is empty.

Taking into account (41) and (42), one can conclude that $I_{k_0} = I_0$.

Consider the following problem:

$$\begin{aligned} \textbf{P-R}: \quad & \max b^\top x,\\ \text{s.t. } & A_i x + c(i) = z(i),\ z(i) \in \mathcal{SOC}(i)\ \forall i \in I \setminus I_{k_0},\\ & A_i x + c(i) = \alpha_i \bar{\gamma}(i),\ \alpha_i \geq 0\ \forall i \in I_{k_0}. \end{aligned}$$

It follows from (42) that the constraints of this problem satisfy the generalized Slater condition. Since $X = X_{k_0-1}$, the optimality of the solution x^0 in the problem (**SOCP**) implies the optimality of the solution

$$\begin{aligned} &(x^0,\ z^0(i) = z(i,x^0),\ i \in I \setminus I_{k_0},\\ &\alpha_i^0 = z_0(i,x^0)/\bar{\gamma}_0(i),\ i \in I_{k_0}) \end{aligned}$$

in the problem (**P-R**). Applying the KKT optimality conditions to the problem (**P-R**) and its optimal solution, one can conclude that there exist vectors $y^{k_0}(i),\ i \in I$, such that

$$\begin{aligned} &y^{k_0}(i) \in \mathcal{SOC}(i) \quad \forall i \in I \setminus I_{k_0};\\ &y^{k_0}(i)^\top \bar{\gamma}(i) = y^{k_0}(i)^\top \mathcal{R}_i y^{s-1}(i) \geq 0\\ &\qquad\qquad \forall i \in \Delta I_s,\ \forall\, s = 1, \dots, k_0,\\ &\sum_{i \in I} A_i^\top y^{k_0}(i) = -b,\\ &z(i, x^0)^\top y^{k_0}(i) = 0 \quad \forall i \in I. \end{aligned} \tag{43}$$

From the relations above, we get

$$y^{k_0}(i) = \mathbf{0}\ \forall i \in I \setminus I_a(x^0). \tag{44}$$

Notice that by construction, we have $I_{k_0} = I_0$, and, consequently, $I_0 \subset I_a(x^0)$ for all $x \in X$. Taking into account this inclusion, (44), and (38) (with $k = 0, \dots, k_0 - 1$), we conclude that the vectors $y^k(i), i \in I,\ k = 0, 1, \dots, k_0$, constructed here, satisfy the equalities

$$y^k(i) = \mathbf{0}\ \forall i \in I \setminus I_a(x^0),\ \forall k = 0, 1, \dots, k_0.$$

It follows from the equalities above and relations (39), (40) (with $k = 0, \dots, k_0 - 1$), together with (43) that the sets of vectors

$$\{\pi(k, i) = y^k(i),\ k = 0, \dots, k_0\},\ \forall i \in I_a(x^0), \tag{45}$$

satisfy Condition (A) and relations (18)-(19). □

Remark 1. *In the theorem, it is affirmed that the integer k_0 is less than or equal to $|I_a(x^0)|$. In fact, the inequalities $k_0 \leq |I_0| \leq |I_a(x^0)|$ hold true and in the statement of the theorem, one can replace the inequality $k_0 \leq |I_a(x^0)|$ by a tighter estimate $k_0 \leq |I_0|$. However, we prefer to leave here the inequality $k_0 \leq |I_a(x^0)|$ since in a general case, one cannot expect to have any knowledge about the set I_0. Notice that if the set I_0 is known, one can use a more simple form of optimality conditions, namely Criterion 1.*

Considering the problems ($\mathbf{P}_*$) and (**P-R**), one can see that they are similar but at the same time there are some differences between them.

It was mentioned above that $I_{k_0} = I_0$. Let us introduce a subset

$$I_{00} = \{i \in I_0 :\ z_0(i, x) = 0\ \forall x \in X\}.$$

For $i \in I_0 \setminus I_{00}$, we have $\gamma(i) = \beta_i \bar{\gamma}(i)$ with $\beta_i = \gamma_0(i)/\bar{\gamma}_0(i) > 0$, *i.e.* the vectors $\gamma(i)$ and $\bar{\gamma}(i)$ coincide up to a positive nonzero factor.

For $i \in I_{00}$, we have $\gamma(i) = \mathbf{0}$ and $\bar{\gamma}(i) \neq \mathbf{0}$.

In the problem ($\mathbf{P}_*$), for $x \in X$, the corresponding variables $\alpha_i, i \in I_0 \setminus I_{00}$, are uniquely determined by the rule $\alpha_i = z_0(i, x)/\gamma_0(i),\ i \in I_0 \setminus I_{00}$, and we can choose any non-negative values for $\alpha_i,\ i \in I_{00}$.

In the problem (**P-R**), for $x \in X$, the formulas $\alpha_i = z_0(i, x)/\bar{\gamma}_0(i),\ i \in I_0$, uniquely define the corresponding variables $\alpha_i,\ i \in I_0$.

4.1. A short discussion

It was mentioned earlier that Criterion 2 proved in this section, is based on the utilization of an optimal solution to the exact extended dual problem (EEDP). In fact, the set (45) constitutes a part of an optimal solution

$$\{y^k(i),\ k = 0, \dots, k_0\}, i \in I, \tag{46}$$

to the problem (EEDP). The vectors in (46) serve as a generalization of the vectors of KKT multipliers for a given optimal solution x^0. However, unlike the vectors of KKT multipliers, which may not exist for some problems, an optimal solution to the exact extended dual problem always exists provided that the optimal value of problem (**SOCP**) is finite.

It follows from the iterative nature of the proof of Theorem 4 that testing the optimality criterion is not much more difficult than checking the KKT system. In fact, to construct generalized multipliers (46), one has to test sequentially, for $k = 0, \dots, k_0$, the classical KKT optimality conditions in the second-order programming problem (**P-k**) for the feasible solution ($\bar{x}$, $\mu = 0$, $z(i) = z(i, \bar{x})$, $i \in I$) with a fixed $\bar{x} \in X$, and one time in the second-order programming problem (**P-R**) for the feasible solution (x^0, $\alpha_i^0 = z_0(i, x^0)/\bar{\gamma}_0(i)$, $i \in I_0$).

Note here the following:

- The number k_0 satisfies the inequality $k_0 \leq |I_0|$ and hence, it is finite. One may expect the number k_0 to be less than $|I_0|$, since $|I_0| = \sum\limits_{k=1}^{k_0} |\Delta I_k|$ and, as a rule, $|\Delta I_k| > 1$ for $k = 1, \dots, k_0$.
- The constraints of all second-order problems (**P-k**), $k = 0, \dots, k_0$, and the problem (**P-R**) satisfy the Slater condition.
- For $k = 1, \dots, k_0$, the KKT system for the problem (**P-k**) is simpler than the KKT system for the problem (**P-(k-1)**), and the KKT system for the problem (**P-R**) is the simplest among them.

If $I_0 = \emptyset$, then $k_0 = 0$. It is easy to see that in this case, conditions (18), (19) coincide with the KKT conditions (1), where $y^0(i) = \pi(0, i)$ for $i \in I_a(x^0)$ and $y^0(i) = \mathbf{0}$ for $i \in I \setminus I_a(x^0)$. Hence the KKT conditions (1) are a particular case of conditions (18), (19) with $k_0 = 0$.

In case $I_0 \neq \emptyset$, conditions (18), (19) are more complex than the KKT conditions, since to test them, one has to find an extended dual optimal solution. But notice that the KKT conditions are useless if, for the problem under consideration, the dual gap is positive or/and the corresponding

Lagrangian dual problem has no solution. In such situations, the KKT conditions can never be satisfied.

In contrast to the KKT conditions, Criterion 2 can always recognize the optimality of a given feasible solution, as an optimal generalized dual solution exists and there is no duality gap. This represents the main and significant advantage of conditions (18), (19) compared to the KKT conditions.

As mentioned earlier, verifying sequential optimality conditions requires finding sequences of vectors $\{x^k\}$ and $\{y^k\}$ associated with primal and dual variables, and checking certain conditions in the form of limits of functions constructed on the basis of these sequences. It is important to note that if certain CQs are not satisfied, the sequence $\{y^k\}$ may become "irregular" (or not well-defined), since $||y^k|| \to \infty$ as $k \to \infty$. This irregularity may pose challenges in numerical methods for constructing such sequences and in verifying conditions in the form of limits.

In contrast, to test the optimality Criterion 2, one needs to find a finite set (46) of concrete vectors which are "well defined" and check a finite set of equality and inequality conditions.

One drawback of our approach is the requirement to know the set I_0 in order to apply the optimality Criterion 1. This can pose a challenge, as identifying this set may take additional effort or computational resources. However, it is worth noting that if we do know this set, our optimality conditions offer advantages over traditional KKT conditions, providing a practical framework for solving optimization problems.

The second drawback of our approach is that when applying the optimality Criterion 2, we need to construct an extended (generalized) vector of Lagrange multipliers. Despite this, the criterion offers the advantage of being CQ-free.

It is known that the violation of CQs can lead to difficulties in implementation of numerical methods of the primal-dual type using the classical KKT optimality conditions. This difficulty arises from the non-existence of classical Lagrange multipliers. It can be overcome by utilizing (iteratively and in an approximate form) of some CQ-free optimality conditions, in either sequential or ordinary form. Since the optimality conditions obtained in the paper are CQ-free, they can be used for this purpose as well as the CQ-free optimality conditions in sequential form as in [18–20] *et al.*

5. Examples

Example 1. Consider the problem (**SOCP**) with the following data: $n = 6$, $I = \{1, 2, 3\}$, $m_1 = 3$, $m_2 = 3$, $m_3 = 2$,

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & -1 & 0 & 1 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 1 \\ -1 & 0 & 1 & -1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix},$$

$c(1) = (0, 6, 0, 0)^\top$; $c(2) = (0, 4, 6, -2)^\top$; $c(3) = (-4, -2, -2)^\top$, $b = (4, 2, -1, -3, 2, -5)^\top$.

Set $x^0 = (2, 1, -3, 0, 1, 1)^\top$ and calculate $z(i, x^0) = A_i x^0 + c(i)$, $i \in I$. In this example, we have:

$$z(1, x^0) = (1, 0, 1, 0)^\top, \ z(2, x^0) = (0, 0, 0, 0)^\top,$$
$$z(3, x^0) = (3, 3, 0)^\top.$$

Consequently, x^0 is a feasible solution of this problem and $I_a(x^0) = I$.

Set $k_0 = 1$, and consider the following vectors: $\pi(0, 1) = (1, 0, -1, 0)^\top$, $\pi(0, 2) = (1, 0, 0, 0)^\top$, $\pi(0, 3) = (0, 0, 0)^\top$, $\pi(1, 1) = (-2, 2, 2, 1)^\top$, $\pi(1, 2) = (3, -1, 1, -1)^\top$, $\pi(1, 3) = (3, -3, 0)^\top$.

It is easy to check that the vectors $\pi(k, i)$, $k = 0, 1$, satisfy Condition (A) for all $i \in I = I_a(x^0)$ and conditions (18), (19). Hence, according to Theorem 4 the vector x^0 is an optimal solution in the problem under consideration.

Now, suppose that in this example, the set I_0 is known: $I_0 = \{1, 2\}$. Using this information, let us test the optimality of the solution x^0 by applying Theorem 3.

Set $\tilde{x} = (1.0, 0.8, -3.4, -0.2, 1.8, 2.2)^\top$ and calculate

$$z(1, \tilde{x}) = (0.8, 0, 0.8, 0)^\top, \ z(2, \tilde{x}) = (0, 0, 0, 0)^\top,$$
$$z(3, \tilde{x}) = (3, 0.4, 0.8)^\top.$$

It is easy to see that the vector $\tilde{x}$ is a minimally active feasible solution and hence, we can choose $\gamma(i) = z(i, \tilde{x})$ for $i \in I_0$.

Set:

$$y(1) = (1, 2, -1, 1)^\top, y(2) = (-1, -1, 1, -1)^\top,$$
$$y(3) = (3, -3, 0)^\top.$$

It is easy to check that these vectors and x^0 satisfy conditions (12) and (13). Hence we have illustrated that the conditions of Theorem 3 are fulfilled as well.

Now, let us show that for the optimal solution x^0, the (classical) KKT optimality conditions formulated in Theorem 2, are not satisfied.

Suppose that in this example, for the optimal solution x^0, there exist vectors $y^0(i)$, $i \in I$, satisfying (1). Then it follows from the conditions

$$y^0(i) \in \mathcal{SOC}(i),\ z(i,x^0) \in \mathcal{SOC}(i),$$
$$y^0(i)^\top z(i,x^0) = 0 \text{ for } i = 1 \text{ and } i = 3$$

that $y^0(1) = (\alpha, 0, -\alpha, 0)^\top$, $y^0(3) = (\beta, -\beta, 0)^\top$ with some $\alpha \geq 0$ and $\beta \geq 0$.

This implies

$$A_1^\top y^0(1) = \mathbf{0},\ A_3^\top y^0(3) = \beta(-1, 0-1, 0, 0, 1)^\top.$$

Consequently,

$$\sum_{i \in I} A_i^\top y^0(i) = -b \iff$$
$$A_2^\top y^0(2) + \beta(-1, 0-1, 0, 0, 1)^\top = -b.$$

It is easy to check here that there are no $y^0(2) \in \mathbb{R}^4$ and β satisfying the latter linear system. Thus we have shown that there do not exist vectors $y^0(i)$, $i \in I$, satisfying (1).

Let us show that in this example the duality gap is zero. In fact, one can check directly that for all sufficiently small $\varepsilon > 0$, the vectors $y(1,\varepsilon) = (4\varepsilon + \frac{1}{\varepsilon}, 2 + \frac{3}{2}\varepsilon, -\frac{1}{\varepsilon}, 1)^\top$, $y(2,\varepsilon) = (10, -1 - \frac{5}{2}\varepsilon, 1+3\varepsilon, -1+2\varepsilon)^\top$, and $y(3,\varepsilon) = (3+\varepsilon, -3, \varepsilon)^\top$ satisfy the following conditions:

$$\sum_{i=1}^{3} A_i^\top y(i,\varepsilon) = -b,\ y(i,\varepsilon) \in \mathcal{SOC}(i)\ \forall i = 1,2,3;$$
$$\sum_{i=1}^{3} c^\top(i) y(i,\varepsilon) = 10 + 7\varepsilon.$$

Hence, these vectors form a feasible solution to the dual problem (**SOCD**) and the corresponding value of the dual cost function is equal to $10+7\varepsilon \geq b^\top x^0 = 10$. Consequently, in this example, we have the equality $val(\mathbf{SOCP}) = val(\mathbf{SOCD})$, but the dual problem has no optimal solution.

Thus in this example, despite the zero duality gap, the KKT optimality conditions do not allow to test the optimality of x^0.

Example 2. Now, we will analyze a problem (**SOCP**) with a positive duality gap. Let us consider a problem from subsection 2.2 in [27]. This problem can be formulated as problem (**SOCP**) with the following data:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

$c(1) = (0, 0, -1)^\top$, $c(2) = (0, 0)^\top$, $b = (0, -1)^\top$, $I = \{1, 2\}$, $m_1 = 2$, $m_2 = 1$, $n = 2$.

It has been shown in [27] that vector $x^0 = (0.5, 1)^\top$ is an optimal solution to the primal problem, the corresponding Lagrangian dual problem also possesses an optimal solution, but a duality gap is positive and equals to 1. In this scenario, it becomes evident that the optimality of the given optimal solution can not be verified using the KKT optimality conditions. However, we will demonstrate that the optimality conditions derived in this paper, allow us to address this issue.

First, we will apply Theorem 3. In this example, $I_0 = \{1\}$ and $\tilde{x} = (1, 1)^\top$ is a minimally active feasible solution. Consequently, we obtain: $z(1, x^0) := A_1 x^0 + c(1) = (0.5, 0.5, 0)^\top$, $\gamma(1) := A_1 \tilde{x} + c(1) = (1, 1, 0)^\top$, $z(2, x^0) := A_2 x^0 + c(2) = (0.5, 0.5)^\top$. One can easily verify that x^0 is a primal feasible solution, and it and the vectors $y(1) = (0, 0, 1)^\top$, $y(2) = (0, 0)^\top$ satisfy conditions (12), (13). Hence, due to Theorem 3 we conclude that, indeed, the vector x^0 is an optimal solution to the problem (**SOCP**) under consideration.

One can check that the conditions of Theorem 4 are satisfied with $\pi(0, 1) = (1, -1, 0)^\top$, $\pi(1, 1) = (0, 0, -1)^\top$, $\pi(0, 2) = \pi(1, 2) = (0, 0)^\top$.

6. Optimality conditions for SOCP based on a lexicographic approach

In paper [21], for convex programming problems in the form

$$\mathbf{CP}: \qquad \min f_0(x), \quad \text{s.t. } f_i(x) \leq 0,\ i \in I,$$

where $x \in \mathbb{R}^n$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $i \in I \cup \{0\}$, are given convex functions, an optimality criterion was proposed based on another approach, namely the *lexicographical separations approach.*

Like the optimality criteria 1 and 2 proved in sections 3 and 4 for the problem (**SOCP**) (Theorems 3 and 4, respectively), this criterion does not require the fulfillment of any additional conditions for the constraints of the original problem. In this section, we will apply the optimality criterion from [21] to the problem (**SOCP**) and compare the result with the criteria proven in the previous sections.

It is evident that the problem (**SOCP**) can be formulated in the form (**CP**) with the following convex functions:

$$f_0(x) := -b^\top x,$$
$$f_i(x) := ||z_*(i,x)|| - z_0(i,x),\ i \in I, \tag{47}$$

where, as before, $z(i,x) := A_i x + c(i) \in \mathbb{R}^{m_i+1}$, $z(i,x)^\top = (z_0(i,x), z_*^\top(i,x))$, $z_0(i,x) \in \mathbb{R}$, $z_*(i,x) \in \mathbb{R}^{m_i}$, $i \in I$.

Then the criterion from [21] can be reformulated as follows.

Theorem 5. *[Optimality criterion 3] A feasible solution x^0 of the problem* **(CP)** *with the functions defined by formula (47), is optimal if and only if there exist an integer number s, $0 \le s \le |I_a(x^0)|$, a vector $\lambda = (\lambda_i, i \in I)$, and an ordered partition*

$$\Delta I_0,\ \Delta I_1, \ldots, \Delta I_s, \tag{48}$$

of the index set I satisfying

(a) *the nonnegativity condition $\lambda_i \ge 0,\ i \in I$,*
(b) *the complementary slackness condition $\lambda_i f_i(x^0) = 0, i \in I$;*
(c) *the minimum conditions*

$$\sum_{i\in\Delta I_k} \lambda_i f_i(x^0) = \min_{x\in Q_k} \sum_{i\in\Delta I_k} \lambda_i f_i(x), \quad k = 0, 2, ..., s-1, \tag{49}$$

and

$$f_0(x^0) + \sum_{i\in\Delta I_s} \lambda_i f_i(x^0) = \min_{x\in Q_s} \big(f_0(x) + \sum_{i\in\Delta I_s} \lambda_i f_i(x)\big), \tag{50}$$

where $Q_0 = \mathbb{R}^n$ and

$$Q_{k+1} = \{x \in Q_k : \sum_{i\in\Delta I_k} \lambda_i f_i(x^0) = \sum_{i\in\Delta I_k} \lambda_i f_i(x)\},$$
$$k = 0, \ldots, s-1.$$

Notice that the functions $f_i(x), i \in I$, defined in (47) are convex but not smooth.

Let us compare the optimality criteria 2 and 3.

Criterion 3 looks simpler than Criterion 2, because it requires less input data for its testing. Indeed, in Criterion 3, we need to know the number s, the partition (48), and $|I|$-vector λ while in Criterion 2, we need to know the number k_0 and the set of vectors (17).

However, Criterion 2 is more constructive (since it is explicit) than Criterion 3. To apply Criterion 3, it is necessary to check whether the partition (48) and the $|I|$-vector λ satisfy conditions (49), (50). These conditions have an implicit form, since to check them, it is necessary to sequentially solve the optimization problems (49), (50) and construct (explicitly) their optimal solution sets Q_k, $k = 0, \ldots, s$. At the same time, to apply Criterion 2, one just needs to check whether the vectors in (17) satisfy conditions (18) and (19), which are **explicit** and can be easy verified.

Note that based on the explicit criterion 2, for the problem (**SOCP**), it is easy to formulate an *implicit* criterion, close in form to Criterion 3.

Theorem 6. *[Optimality criterion 4] A feasible solution $x^0 \in X$ is an optimal solution of the problem* **(SOCP)** *if and only if there exists an integer number s, $0 \le s \le |I_a(x^0)|$, a vector $\lambda = (\lambda_i, i \in I)$ and an ordered partition (48) of the index set I satisfying the following conditions:*

(a) $\lambda_i > 0,\ i \in \Delta I_k \ne \emptyset,\ k = 0, \ldots, s-1;$ $\lambda_i \ge 0,\ \lambda_i f_i(x^0) = 0,\ i \in \Delta I_s;$
(b) *the minimum conditions (49), (50), where*

$$Q_0 = \mathbb{R}^n,\ Q_{k+1} = \{x \in Q_k : f_i(x) = 0, i \in \Delta I_k\},$$
$$k = 0, \ldots, s-1.$$

The main difference between Theorems 5 and 6 is the way the sets Q_k, $k = 1, \ldots, s$, are defined.

7. Conclusions

Despite the fact that the second-order cone problems have been sufficiently studied, most of optimality conditions for these problems are formulated with some CQ. Constraint qualifications, while useful in many optimization problems, can impose restrictive assumptions on the problem structure and hinder the applicability of optimality conditions. By seeking optimality conditions that do not rely on such qualifications, researchers and practitioners can achieve a more robust and flexible framework for solving SOCPs.

The novelty of the paper consists in new optimality conditions for the second-order cone problems, namely Criteria 1 and 2. These optimality criteria are obtained using the approach based on the concept of immobile index set of the constraints of the problem and allow to detect optimality of a given feasible solution without any CQs. The absence of constraint qualifications in these criteria enhances the applicability of the theory to a broader range of optimization problems.

The findings presented in the paper enable us to conclude that the approach to optimality conditions, which is based on immobile indices and was developed in our earlier works, can be applied to the optimization of second-order cone problems.

It is worth mentioning here that there exist different formulations of exact dual problems. In the paper, we used one of them. Alternatively, it is possible to apply the same approach to other exact dual formulations and develop new optimality conditions that may have distinct properties and other ways of implementation. In the future, we will apply our approach to different classes of optimization problems.

In conclusion, it is important to recognize that all known optimality conditions for conic problems, in general, and SOCP problems, in particular, have their drawbacks and favorable properties. Nevertheless, by familiarizing oneself with a wide spectrum of optimality conditions, one can gain a more comprehensive understanding of the problem and its inherent characteristics. This empowers users to make informed decisions and select the most suitable method according to their specific requirements and preferences.

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References

[1] Nesterov, Y., & Nemirovski, A.S. (1992). Conic formulation of a convex programming problem and duality. *Optimization Methods & Software*, 1(2), 95-115. https://doi.org/10.1080/10556789208805510

[2] Glineur, F. (2001). Conic optimization: an elegant framework for convex optimization. *Belgian Journal of Operations Research, Statistics and Computer Science*, 41(1-2), 5-28.

[3] Dür, M. & Rendl, F. (2021). Conic optimization: A survey with special focus on copositive optimization and binary quadratic problems. *EURO Journal on Computational Optimization*, v.9, 100021. https://doi.org/10.1016/j.ejco.2021.100021

[4] Hanasusanto, G. A., & Kuhn, D. (2018). Conic programming reformulations of two stage distributionally robust linear programs over Wasserstein balls. *Operations Research*, 66(3), 849-869. https://doi.org/10.1287/opre.2017.1698

[5] Kocuk, B. (2021) Conic reformulations for Kullback-Leibler divergence constrained distributionally robust optimization and applications. *An International Journal of Optimization and Control: Theories and Applications*, 11(2), pp.139-151. https://doi.org/10.11121/ijocta.01.2021.001001

[6] Pataki, G. (2000). The geometry of semidefinite programming. In: H. Wolkowicz, R. Saigal, & L. Vandenberghe (eds.) *Handbook of Semidefinite Programming.* International Series in Operations Research & Management Science, vol 27. Springer, Boston, MA, 29-65. https://doi.org/10.1007/978-1-4615-4381-7_3

[7] Vandenberghe, L. & Boyd, S. (1996). Semidefinite programming. *SIAM Review*, 38(1), 49-95. https://doi.org/10.1137/1038003

[8] Anjos, M.F. & Lasserre, J.B. (2012) *Handbook on Semidefinite, Conic and Polynomial Optimization.* International Series in Operations Research & Management Science, Springer New York, NY. https://doi.org/10.1007/978-1-4614-0769-0

[9] Alizadeh, F. & Goldfarb, D. (2003). Second-order cone programming. *Mathematical Programming*, Ser. B, 95, 3-51. https://doi.org/10.1007/s10107-002-0339-5

[10] Lobo, M.S., Vandenberghe, L., Boyd, S., & Lebret, H. (1998). Applications of second order cone programming. *Linear Algebra and Its Applications*, 284, 193-228. https://doi.org/10.1016/S0024-3795(98)10032-0

[11] Luo, Z.-Q. & Sturm, J. F. (2000). Error analysis. In: H. Wolkowicz, R. Saigal, L. Vandenberghe (eds) *Handbook of Semidefinite Programming.* International Series in Operations Research & Management Science, vol 27. Springer, Boston, MA, 163-190.

[12] Meng, J., Cao, P., Huang, J., Lin, H., Chen, Y., & Cao, R. (2019) Second-order cone programming formulation of discontinuous deformation analysis. *IJNME- International Journal for Numerical Methods in Engineering.* 118(5), 243-57. https://doi.org/10.1002/nme.6006

[13] Wang, D., Chen, X., Lyu, Y., & Tang, C. (2019) Geotechnical localization analysis based on Cosserat continuum theory and second-order cone programming optimized finite element method. *Computers and Geotechnics.* 114, 103-118. https://doi.org/10.1016/j.compgeo.2019.103118

[14] Zhang, Y., & Zhang, L. (2019). New constraint qualifications and optimality conditions for second order cone programs. *Set-Valued and Variational Analysis.* 27, 693-712. https://doi.org/10.1007/s11228-018-0487-2

[15] Bonnans, J.F., & Shapiro, A. (2000). *Perturbation analysis of optimization problems.* Springer, New York.

[16] Kostyukova, O. I., Tchemisova, T. V., & Dudina, O. S. (2020). Immobile indices and CQ-free optimality criteria for linear copositive programming problems. *Set-Valued and Variational Analysis.* 28, 89-107. https://doi.org/10.1007/s11228-019-00527-y

[17] Kapoor, M., Suneja, S. K., & Sharma, S. (2016). Generalized (Φ, ρ)-convexity in nonsmooth vector optimization over cones.

An International Journal of Optimization and Control: Theories & Applications, 6(1), 1-7. https://doi.org/10.11121/ijocta.01.2016.00247 https://doi.org/10.11121/ijocta.01.2016.00247

[18] Andreani, R., Martinez, J.M., Silva, P.S., & Ramos, A.(2018). Strict constraint qualifications and sequential optimality conditions for constrained optimization. *Mathematics of Operations Research online*, 43, 693-717. https://doi.org/10.1287/moor.2017.0879

[19] Jeyakumar, V. (2008). Constraint qualifications characterizing Lagrangian duality in convex optimization. *Journal of Optimization Theory and Applications*, 136, 31-41. https://doi.org/10.1007/s10957-007-9294-x

[20] Andreani, R., Fukuda, E. H., Haeser, G., Santos, D.O., & Secchin, L.D. (2024) Optimality conditions for nonlinear second-order cone programming and symmetric cone programming. *Journal of Optimization Theory and Applications*, 200, 1-13. https://doi.org/10.1007/s10957-023-02338-6

[21] Gorokhovik, V.V. (2021). Step-affine functions, halfspaces, and separation of convex sets with applications to convex optimization problems. In: *Proceedings of the Steklov Institute of Mathematics*, 313(1), S83-S99. https://doi.org/10.1134/S008154382103010X

[22] Kortanek, K. O., Yu, G., & Zhang, Q. (2021). Strong duality for standard convex programs. *Mathematical Methods of Operations Research*, 94(3), 413-436. https://doi.org/10.1007/s00186-021-00761-x

[23] Cochran, J.J., Cox, L.A., Jr., Keskinocak, P., Kharoufeh, J.P., Smith, J.C., & Solodov, M.V. (2011). Constraint qualifications. In: J.J. Cochran, L.A. Cox, P. Keskinocak,J.P. Kharoufeh, & J.C. Smith (eds.) *Wiley Encyclopedia of Operations Research and Management Science.* Wiley. https://doi.org/10.1002/9780470400531

[24] Bonnans, J.F., & Ramirez, H. (2005) Perturbation analysis of second-order cone programming problems. *Mathematical Programming*, 104(2), 205-227. https://doi.org/10.1007/s10107-005-0613-4

[25] Kruger, A.Y., Minchenko, L., Outrata, J.V. (2013) On relaxing the Mangasarian-Fromovitz constraint qualification. *Positivity*, 18, 171-189. https://doi.org/10.1007/s11117-013-0238-4

[26] Andreani, R., Haeser, G., Mito, L.M., Ramirez, H., Santos, D.O., & Silveira, T.P. (2022) Naive constant rank-type constraint qualifications for multifold second-order cone programming and semidefinite programming. *Optimization Letters*, 16, 589-610. https://doi.org/10.1007/s11590-021-01737-w

[27] Andersen, E. D., Roos, C., & Terlaky, T. (2002). Notes on duality in second order and p-order cone optimization. *Optimization*, 51(4), 627-643. https://doi.org/10.1080/0233193021000030751

[28] Kostyukova, O., & Tchemisova, T. (2015). Convex SIP problems with finitely representable compact index sets: immobile indices and the properties of the auxiliary NLP problem, *Set-Valued and Variational Analysis*, 23(3), 519-546. https://doi.org/10.1007/s11228-015-0320-0

[29] Tunçel, L., & Wolkowicz, H. (2013). Strong duality and minimal representations for cone optimization. *Computational Optimization and Applications.* 53, 619-648. https://doi.org/10.1007/s10589-012-9480-0

[30] Kostyukova, O., & Tchemisova, T. (2021). Structural properties of faces of the cone of copositive matrices. *Mathematics*, 9(21), 2698. https://doi.org/10.3390/math9212698

[31] Golshtein, E. G. (1971). *Theory of duality in mathematical programming and its applications.* Nauka. Moscow. (in Russian)

[32] Liu, M., & Pataki, G. (2018). Exact duals and short certificates of infeasibility and weak infeasibility in conic linear programming. *Mathematical Programming*, 167, 435-480. https://doi.org/10.1007/s10107-017-1136-5

[33] Ramana, M. V., Tunçel, L., & Wolkowicz, H. (1997). Strong duality for semidefinite programming. *SIAM Journal on Optimization*, 7(3), 641-662. https://doi.org/10.1137/S1052623495288350

[34] Kostyukova, O. I., & Tchemisova, T. V. (2022). On strong duality in linear copositive programming. *Journal of Global Optimization*, 83(3), 457-480. https://doi.org/10.1007/s10898-021-00995-3

[35] Kostyukova, O.I.,& Tchemisova,T.V. (2023) An exact explicit dual for the linear copositive programming problem. *Optimization Letters*, 17, 107-120. https://doi.org/10.1007/s11590-022-01870-0

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