RESEARCH ARTICLE

The solvability of the optimal control problem for a nonlinear Schrödinger equation

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ABSTRACT

In this paper, we analyze the solvability of the optimal control problem for a nonlinear Schrödinger equation. A Lions-type functional is considered as the objective functional. First, it is shown that the optimal control problem has at least one solution. Later, the Frechet differentiability of the objective functional is proved and a formula is obtained for its gradient. Finally, a necessary optimality condition is derived.

ARTICLE INFO

Article History:
Received 15 February 2023
Accepted 30 March 2023
Available 29 July 2023

Keywords:
Optimal control problem
Schrödinger equation
Frechet differentiability
Optimality conditions

AMS Classification 2010:
49J20; 35J10; 49K20

1. Introduction

The nonlinear Schrödinger equation (NLSE) describes the behavior of wave packets in weakly nonlinear media. It is an adaptable model to many disciplines in applied sciences such as dynamical systems, materials science, nonlinear optics, fluid dynamics, astrophysics, particle physics, and nonlinear transmission networks. NLSE represents the evolution of optical waves in a nonlinear fiber, various biological systems, and the price of options in economics [1].

In the present paper, we consider a specific case of the following Schrödinger equation

\[ \varepsilon \frac{\partial u}{\partial \tau} + R_2(\varsigma, \tau, u) \frac{\partial^2 u}{\partial \varsigma^2} + R_1(\varsigma, \tau, u) \frac{\partial u}{\partial \varsigma} + R_0(\varsigma, \tau, u) u = 0, \quad (1) \]

where \( \varepsilon = \text{const.} \), \( u(\varsigma, \tau) \) is the wave’s complex amplitude. The coefficients \( R_j(\varsigma, \tau, u) \) for \( j = 0, 1, 2 \) describe the variation of the medium. If the functions \( R_j \) depend on \( u(\varsigma, \tau) \), it shows that the medium has the nonlinear properties [2]. Linear and nonlinear Schrödinger equations are obtained from equation (1) with respect to the characteristics of the coefficients \( R_j(\varsigma, \tau, u) \) for \( j = 0, 1, 2 \) and \( \varepsilon = i \).

Optimal control problems (OCPs) arise in many branches of science. They have numerous applications in optics, medical imaging, geophysics, system identification, communication theory, astronomy, medicine [3–11]. As it is known, in the OCPs, there is an objective functional, a controlled system, and a set of admissible controls. The objective functionals can be diversely chosen with regard to our purpose such as final, boundary or Lions-type functional [12]. In the studies [13–22], the objective functional is considered as a final functional and the controlled system is generally stated by the Schrödinger equation. In [23–27], the OCPs with Lions functional has been studied and the controlled system is stated by linear or nonlinear Schrödinger equations. Also, in [28–30], the
OCPs for systems whose state is expressed by the Schrödinger equation with the boundary functional has been studied. In this paper, we consider an OCP with Lions functional for NLSE derived from (1). It is proved that the OCP has a unique solution and the objective functional is Frechet differentiable. Also, by proving the continuity of the gradient of the objective functional, a necessary optimality condition is obtained.

Differently from the previous studies, in this paper, we analyze the solution of the OCP for NLSE derived from (1). It is proved in this paper, we consider an OCP with Lions objective functional is Frechet differentiable. Also, the objective functional for NLSE derived from (1). It is proved in this paper, we consider an OCP with Lions objective functional is Frechet differentiable. Also, the objective functional for NLSE derived from (1). It is proved in this paper, we consider an OCP with Lions objective functional is Frechet differentiable. Also, the objective functional for NLSE derived from (1). It is proved in this paper, we consider an OCP with Lions objective functional is Frechet differentiable. Also, the objective functional for NLSE derived from (1). It is proved in this paper, we consider an OCP with Lions objective functional is Frechet differentiable. Also, the objective functional for NLSE derived from (1). It is proved in this paper, we consider an OCP with Lions objective functional is Frechet differentiable. Also, the objective functional for NLSE derived from (1). It is proved that the OCP has a unique solution and the objective functional is Frechet differentiable. Also, by proving the continuity of the gradient of the objective functional, a necessary optimality condition is obtained.

2. The statement of optimal control problem

The OCP is the problem of finding the minimum of the objective functional

\[ J_0(p) = \|u_1 - u_2\|^2_{L^2(\Omega)} + \alpha \|p - w\|^2_{L^2(I)} \] (2)

subject to

\[ i \frac{\partial u_1}{\partial \tau} + a_0 \frac{\partial^2 u_1}{\partial \xi^2} + i a_1(\xi) \frac{\partial u_1}{\partial \xi} - a_2(\xi) u_1 + p(\xi) u_1 + i a_3 |u_1|^2 u_1 = f_1, \quad u_1(\xi, 0) = \vartheta_1(\xi), \quad \xi \in I, \]
\[ u_1(0, \tau) = u_1(1, \tau) = 0, \quad \tau \in (0, T) \] (4)

and

\[ i \frac{\partial u_2}{\partial \tau} + a_0 \frac{\partial^2 u_2}{\partial \xi^2} + i a_1(\xi) \frac{\partial u_2}{\partial \xi} - a_2(\xi) u_2 + p(\xi) u_2 + i a_3 |u_2|^2 u_2 = f_2, \]
\[ u_2(\xi, 0) = \vartheta_2(\xi), \quad \xi \in I, \]
\[ \frac{\partial u_2}{\partial \xi}(0, \tau) = \frac{\partial u_2}{\partial \xi}(1, \tau) = 0, \quad \tau \in (0, T) \] (6)

on admissible controls set

\[ P = \{ p \in L^2(I) : |p(\xi)| \leq b_0 \text{ for almost all } \xi \in I \}, \]

where \( \xi \in I = (0, t), \ \tau \in Q = [0, T], \ i = \sqrt{-1}. \)

Let \( \Omega = I \times (0, T), \ \Omega_\tau = I \times (0, \tau), \ \Omega_\tau = I \times (\tau, T) \) and \( a_0, a_3, b_0 > 0 \) are given real numbers, \( a_1(\xi), \ a_2(\xi), \ \vartheta_1, \ \vartheta_2, \ f_1, \ f_2 \) are functions which satisfy the conditions, respectively

\[
|a_1(\xi)| \leq \mu_1, \quad \left| \frac{\partial a_1(\xi)}{\partial \xi} \right| \leq \mu_2 \text{ for almost all } \xi \in I,
\]
\[
a_1(0) = a_1(1) = 0, \quad \mu_1, \mu_2 = \text{const.} > 0, \tag{7}
\]
\[
0 < \mu_3 \leq a_2(\xi) \leq \mu_4 \text{ for almost all } \xi \in I,
\]
\[
\mu_3, \mu_4 = \text{const.} > 0,
\]
\[
\vartheta_1 \in W^2_2(I), \ \vartheta_2 \in W^2_2(I), \ \frac{\partial \vartheta_2(0)}{\partial \xi} = \frac{\partial \vartheta_2(I)}{\partial \xi} = 0,
\]
\[
f_r \in W^{0,1}_0(O) \text{ for } r = 1, 2,
\]

where \( W^m_2(I), W^m_2(\Omega), W^m_2(O) \) for \( m \geq 0, s \geq 1 \) are Sobolev spaces. These Sobolev spaces are in detail explained in [31]. Also, \( \alpha \geq 0 \) is a Tikhonov regularization parameter 32 and \( w \in L^2(O) \) is a given element.

Since the solutions of (3)-(4) and (5)-(6) evidently depend on \( p \), we denote \( u_r = u_r(\xi, \tau; p) \) for \( r = 1, 2 \). We are interested in solutions of problems (3)-(4) and (5)-(6) in the following sense:

Definition 1. A function \( u_1 \in U_1 \equiv C^0(Q, W^2_2(I)) \cap C^1(Q, L^2(O)) \) is said to be a solution of problem (3)-(4), if it holds (3) for almost all \( \xi \in I \) and any \( \tau \in Q, \) (4) for almost all \( \xi \in I \) and for almost all \( \tau \in (0, T) \), respectively.

Definition 2. A function \( u_2 \in U_2 \equiv C^0(Q, W^2_2(I)) \cap C^1(Q, L^2(O)) \) is said to be a solution of problem (3)-(4), if it holds (3) for almost all \( \xi \in I \) and any \( \tau \in Q, \) (4) for almost all \( \xi \in I \) and for almost all \( \tau \in (0, T) \), respectively.

In the definitions above, for any nonnegative integer \( k, \ C^k(Q, B) \) is the Banach space of all \( B \)-valued, \( k \) times continuously differentiable functions on \( Q \) with the norm

\[
\|u\|_{C^k(Q, B)} = \sum_{m=0}^k \max_{0 \leq t \leq T} \|d^m u(t)\|_B
\]

for \( u \in C^k(Q, B) \).

By the methodology in [33], we can readily prove the theorem below:

Theorem 1. Assume that \( a_1, a_2, \vartheta_\tau, f_r \) for \( r = 1, 2 \) satisfy the conditions, respectively. Then, problems (3)-(4) and (5)-(6) for each \( p \in P \) have unique solutions \( u_1 \in U_1, \ u_2 \in U_2 \), respectively, and the functions \( u_1, u_2 \) satisfy the estimates

\[
\|u_1(\cdot, \tau)\|^2_{W^2_2(I)} + \left| \frac{\partial u_1}{\partial \tau} \right|^2_{L^2(I)} \leq c_1 \left( \|\vartheta_1\|^2_{W^2_2(I)} + \|f_1\|^2_{W^{0,1}_0(O)} + \|\vartheta_2\|^2_{W^2_2(I)} \right),
\]

where \( c_1 \) is a constant.
\[ \|u_2(\cdot, \tau)\|_{W^2_2(I)}^2 + \left\| \frac{\partial u_2}{\partial \tau} \right\|_{L_2(I)}^2 \leq c_2 \left( \|\vartheta_2\|_{W^2_2(I)} + \|f_2\|_{W^{2,1}_2(I)} + \|\delta \vartheta_2\|_{W^2_2(I)}^6 \right), \]

for any \( \tau \in Q \), where the constants \( c_1, c_2 > 0 \) are independent from \( \vartheta_1, f_1, \vartheta_2, f_2 \) and \( \tau \).

For simplicity, let’s rewrite problems (3)-(4) and (5)-(6) in the form

\[
\begin{align*}
  i \frac{\partial u_r}{\partial \tau} + a_0 \frac{\partial^2 u_r}{\partial \tau^2} + ia_1(\varsigma) \frac{\partial u_r}{\partial \varsigma} - a_2(\varsigma) u_r + (p(\varsigma) - \delta p(\varsigma)) u_r + ia_3 \left( |u_{r\delta}|^2 + |u_r|^2 \right) u_r = f_r \\
  \delta u_r(\varsigma) = 0, \varsigma \in I \\
  \delta u_r(0, \tau) = \delta u_1(l, \tau) = 0, \tau \in (0, T). 
\end{align*}
\]

(12) for any \( r = 1, 2 \), \( u_3(0, \tau) = u_1(l, \tau) = 0, \tau \in (0, T) \). \( \delta u_r(\varsigma) = 0, \varsigma \in I \)

Thus, the OCP is to minimize the objective functional (2) on \( P \) under conditions (12)-(14).

3. The solvability of optimal control problem

In this section, we show that the OCP has a unique solution on a dense subset of \( L(I) \) and it has at least one solution on \( L(I) \).

**Lemma 1.** The functional \( J_0(p) = \|u_1 - u_2\|_{L_2(\Omega)}^2 \) is continuous on \( P \).

**Proof.** Suppose \( u_r = u_r(\varsigma, \tau; p) \) and \( u_{r\delta} = u_r(\varsigma, \tau; p + \delta p) \) for \( r = 1, 2 \) are solutions of problem (12)-(14) corresponding to \( p \in P, p + \delta p \in P \) respectively, where \( \delta p \in L^\infty(I) \) is an increment of any \( p \in P \). Then, for \( r = 1, 2 \), the functions \( \delta u_r = u_r(\varsigma, \tau; p + \delta p) - u_r(\varsigma, \tau; p) \) hold the boundary value problem

\[
\begin{align*}
  i \frac{\partial \delta u_r}{\partial \tau} + a_0 \frac{\partial^2 \delta u_r}{\partial \tau^2} + ia_1(\varsigma) \frac{\partial \delta u_r}{\partial \varsigma} - a_2(\varsigma) \delta u_r + (p(\varsigma) - \delta p(\varsigma)) \delta u_r + \\
  ia_3 \left( |u_{r\delta}|^2 + |u_r|^2 \right) \delta u_r = \delta f_r \\
  \delta u_r(\varsigma) = 0, \varsigma \in I \\
  \delta u_r(0, \tau) = \delta u_1(l, \tau) = 0, \tau \in (0, T). 
\end{align*}
\]

(15) for any \( \tau \in Q \), where the positive constant \( c_3 \) does not depend on \( \delta p \) and \( \tau \).

Using formula (2) for \( \alpha = 0 \), we obtain

\[
\begin{align*}
  &\delta J_0(p) = J_0(p + \delta p) - J_0(p) = \\
  &2 \int_{\Omega} \left( (u_1 - u_2) (\delta \overline{u}_1 - \delta \overline{u}_2) \right) dx dt + \\
  &\|\delta u_1\|_{L_2(\Omega)}^2 + \|\delta u_2\|_{L_2(\Omega)}^2 \\
  &\geq \|\delta u_1\|_{L_2(\Omega)}^2 + \|\delta u_2\|_{L_2(\Omega)}^2. 
\end{align*}
\]

(21) which implies that
\[
|\delta J_0(p)| \leq 2 \|u_1\|_{L^2(\Omega)} \|\delta u_1\|_{L^2(\Omega)} + 2 \|u_2\|_{L^2(\Omega)} \|\delta u_2\|_{L^2(\Omega)} + 2 \|\delta u_1\|_{L^2(\Omega)} + 2 \|\delta u_2\|_{L^2(\Omega)}.
\]

If we use estimates (10), (11), (20) in the inequality above, we get the inequality

\[
|J_0(p + \delta p) - J_0(p)| \leq c_4 \left( \|\delta p\|_{L^\infty(\Omega)}^2 + \|\delta p\|_{L^\infty(\Omega)}^2 \right)
\]

for any \( p \in P \), where \( c_4 \) is a positive constant independent from \( \delta p \). Thus, we obtain that \( |\delta J_0(p)| \rightarrow 0 \) as \( \|\delta p\|_{L^\infty(\Omega)} \rightarrow 0 \) for any \( p \in P \), which concludes the proof.

**Theorem 2.** Let Theorem 1 be satisfied and \( w \in L^2(I) \). Then, there exists a dense subset \( V \subset L^2(I) \) such that OCP has a unique solution for any \( w \in V \) and \( \alpha > 0 \).

**Proof.** From Lemma 1, \( J_0(p) \) is a lower semicontinuous functional. Also, it is clear that \( J_0(p) \) is lower bounded. As known, \( L^2(I) \) is a uniformly convex Banach space. Furthermore, \( P \) is a closed, bounded subset of \( L^2(I) \). Therefore, based on Theorem 4 in [34] we can say that the OCP has a unique solution on a dense subset \( V \subset L^2(I) \). This completes the proof.

**Theorem 3.** Let \( w \in L^2(I) \) be a given function and \( \alpha \geq 0 \). Also, assume that Theorem 1 is satisfied. Then, the OCP has at least one solution.

**Proof.** The proof of Theorem 3 is carried out as in [22].

**4. The gradient of functional and a necessary optimality condition**

In this section, we introduce the adjoint problem to investigate the differentiability of the objective functional and get a formula for its gradient. Finally, a necessary optimality condition for the OCP is derived.

By using Lagrange multiplier functions, we obtain the adjoint problem as follows:

\[
\begin{align*}
& \partial_i \eta_r + a_0 \partial_i^2 \eta_r + i \frac{\partial}{\partial s} (a_1(s) \eta_r) - a_2(s) \eta_r + p(s) \eta_r - 2a_3 \|u_r\|^2 \eta_r + i a_3 u_r^2 \eta_r = 2(-1)^r (u_1 - u_2) \quad \text{for } r = 1, 2, \\
& \eta_1(s, \tau = 0) = \eta_1(l, \tau = 0), \quad \eta_1(l, \tau = 0, \tau \in (0, T), \\
& \partial_\tau \eta_1(0, \tau = 0) = \partial_\tau \eta_1(l, \tau = 0, \tau \in (0, T),
\end{align*}
\]

where the functions \( u_r = u_r(\xi, \tau) \) are solutions of problem (12)-(14) for any \( p \in P \). It can be seen that the adjoint problem (22)-(24) includes the two boundary value problems. One of them is a Dirichlet problem with respect to \( \eta_1 \) and the other is a Neumann problem with respect to \( \eta_2 \). If we use transform \( t = T - \tau \) to the adjoint problem, we come to the conclusion that the adjoint problem is in the form of problem (12)-14. As a solution of (22)-(24), we consider two functions \( \eta_1(s, \tau) \in U_1, \eta_2(s, \tau) \in U_2 \) satisfying equation (22) for almost all \( \xi \in I \) and any \( \tau \in Q \), the condition (23) for almost all \( \xi \in I \) and the conditions (24) for almost all \( \tau \in (0, T) \), respectively. Hence, we can state the validity of the following theorem for the solution of the adjoint problem (22)-(24):

**Theorem 4.** Let the assumptions of Theorem 1 be fulfilled. Then adjoint problem (22)-(24) has a unique solution \( \eta_1 \in U_1, \eta_2 \in U_2 \) for any \( p \in P \) and the following estimates hold

\[
\begin{align*}
|\eta_1(\cdot, \tau)|^2_{W^2(I)} + \left| \frac{\partial \eta_1}{\partial \tau} \right|^2_{L^2(I)} \leq c_5 \|u_1 - u_2\|^2_{W^{0,1}(\Omega)}, \\
|\eta_2(\cdot, \tau)|^2_{W^2(I)} + \left| \frac{\partial \eta_2}{\partial \tau} \right|^2_{L^2(I)} \leq c_6 \|u_1 - u_2\|^2_{W^{0,1}(\Omega)}
\end{align*}
\]

for any \( \tau \in Q \), where the positive constants \( c_5, c_6 \) do not depend on \( \tau \).

This theorem can be easily proved by the Galerkin’s method similarly to the proof of Theorem 1.

Now, let’s get the enhancement \( \delta J_\alpha(p) = J_\alpha(p + \delta p) - J_\alpha(p) \) of \( J_\alpha(p) \) for any \( p \in P \), where \( \delta p \in L^\infty(I) \) is an increment given to any \( p \in P \) such that \( p + \delta p \in P \). If we use formula (2), we achieve

\[
\delta J_\alpha(p) = \int_\Omega \delta p(\xi) Re(u_1 \overline{\eta_1}) d\xi d\tau + \int_\Omega \delta p(\xi) Re(u_2 \overline{\eta_2}) d\xi d\tau + 2 \alpha \int_0^T \int_\Omega (p - w) \delta p d\xi d\tau + R,
\]

where

\[
\begin{align*}
\eta_1(s, \tau) &= 0 \quad \text{for } r = 1, 2, \xi \in I, \\
\eta_1(0, \tau) &= \eta_1(l, \tau) = 0, \tau \in (0, T), \\
\partial_\tau \eta_1(0, \tau) &= \partial_\tau \eta_1(l, \tau) = 0, \tau \in (0, T),
\end{align*}
\]
where

\[ R = \int_{\Omega} \delta p(\zeta) Re(\delta u_1 \bar{\eta}_1) d\zeta d\tau + \]

\[ \int_{\Omega} \delta p(\zeta) Re(\delta u_2 \bar{\eta}_2(\zeta, \tau)) d\zeta d\tau + \]

\[ \| \delta u_1 \|_{L_2(\Omega)}^2 + \| \delta u_2 \|_{L_2(\Omega)}^2 - \]

\[ a_3 \int (|u_1|)^2 - |u_1|^2 Im(\delta u_1 \bar{\eta}_1) d\zeta d\tau - \]

\[ a_3 \int (|u_2|)^2 - |u_2|^2 Im(\delta u_2 \bar{\eta}_2) d\zeta d\tau - \]

\[ a_3 \int |\delta u_1|^2 Im(u_1 \bar{\eta}_1) d\zeta d\tau - \]

\[ a_3 \int |\delta u_2|^2 Im(u_2 \bar{\eta}_2) d\zeta d\tau + \alpha \| \delta p \|_{L_2(I)}^2 \]

and \( \delta u_r \equiv u_r(\zeta, \tau; p + \delta p) - u_r(\zeta, \tau; p) \) for \( r = 1, 2 \) hold problem (15) for any \( p \in P \). By Young's inequality for the term \( R \), we get

\[ |R| \leq \frac{5}{2} \| \delta u_1 \|_{L_2(\Omega)}^2 + \frac{5}{2} \| \delta u_2 \|_{L_2(\Omega)}^2 + \]

\[ \alpha \| \delta p \|_{L_2(I)}^2 + \]

\[ \frac{T}{2} \left( \max_{0 \leq \tau \leq T} \| \eta_1(\cdot, \tau) \|_{L_\infty(I)}^2 \right) \| \delta p \|_{L_2(I)}^2 + \]

\[ \frac{T}{2} \left( \max_{0 \leq \tau \leq T} \| \eta_2(\cdot, \tau) \|_{L_\infty(I)}^2 \right) \| \delta p \|_{L_2(I)}^2 + \]

\[ a_3 \int (|u_1|)^2 + |u_1|^2 |\delta u_1|^2 d\zeta d\tau + \]

\[ a_3 \int (|u_2|)^2 + |u_2|^2 |\delta u_2|^2 d\zeta d\tau + \]

\[ a_3 \int \| \eta_1(\cdot, \tau) \|_{L_\infty(I)}^2 \| \delta u_1(\cdot, \tau) \|_{L_2(I)}^2 d\tau + \]

\[ \frac{T}{2} \left( \max_{0 \leq \tau \leq T} \| \eta_2(\cdot, \tau) \|_{L_\infty(I)}^2 \right) \| \delta u_2(\cdot, \tau) \|_{L_2(I)}^2 d\tau + \]

\[ \frac{T}{2} \left( \max_{0 \leq \tau \leq T} \| \eta_1(\cdot, \tau) \|_{L_\infty(I)}^2 \right) \| \delta u_1(\cdot, \tau) \|_{L_2(0, \tau)}^2 d\tau + \]

\[ \frac{T}{2} \left( \max_{0 \leq \tau \leq T} \| \eta_2(\cdot, \tau) \|_{L_\infty(I)}^2 \right) \| \delta u_2(\cdot, \tau) \|_{L_2(0, \tau)}^2 d\tau + \]

In the inequality above, if we use estimates \[ \| u(\cdot, \tau) \|_{L_\infty(I)}^2 \leq 0 \]

\[ \beta_2 \left\| \frac{\partial u(\cdot, \tau)}{\partial \zeta} \right\|_{L_2(I)} \| u(\cdot, \tau) \|_{L_2(I)}, \]

\[ \beta_2 = const. > 0 \]

for any \( \tau \in Q \), we achieve

\[ |R| \leq c_7 \| \delta p \|_{L_\infty(I)}^2 \leq c_8 \| \delta p \|_{L_\infty(I)}^2 \]

which shows that \( R = o(\| \delta p \|_{L_\infty(I)}) \), that is,

\[ \lim_{\| \delta p \|_{L_\infty(I)} \to 0} \frac{R}{\| \delta p \|_{L_\infty(I)}} = 0 \]

where the constants \( c_7, c_8 \geq 0 \) are independent from \( \delta p \) and \( \tau \). So, from (27), we can write

\[ \delta J_\alpha(p) = \int_0^T \left( \int_0^T Re(u_1 \bar{\eta}_1 + u_2 \bar{\eta}_2) d\tau \right) \delta p(\zeta) d\zeta + \int_0^T 2\alpha (p - w) \delta p(\zeta) d\zeta + o(\| \delta p \|_{L_\infty(I)}) \]

which implies that

\[ J_\alpha'(p) = \int_0^T Re(u_1 \bar{\eta}_1 + u_2 \bar{\eta}_2) d\tau + 2\alpha (p - w) \]

Consequently, the differentiability of \( J_\alpha(p) \) in the meaning of Frechet is shown and the next theorem is proved:

**Theorem 5.** Let \( w \in L_2(I) \) be a given function. Assume that the conditions of Theorem 3 are satisfied. Then, \( J_\alpha(p) \) is a differentiable functional on \( P \) and moreover, its gradient is given by formula (29).

**Lemma 2.** The functional \( J_\alpha'(p) \) is continuous on \( P \).

**Proof.** Let’s prove that \( |J_\alpha'(p + \delta p) - J_\alpha'(p)| \to 0 \) as \( \| \delta p \|_{L_\infty(I)} \to 0 \) on the set \( P \). Using formula (29), we get
\[ J'_\alpha(p + \delta p) - J'_\alpha(p) = \]
\[
\int_0^T \left[ \text{Re} \left( u_{1,\delta} \delta \eta_1 + u_{2,\delta} \delta \eta_2 \right) d\tau + \right. \\
\left. \int_0^T \text{Re} (\delta u_1 \eta_1 + \delta u_2 \eta_2) d\tau + 2\alpha \delta p(\varsigma), \right]
\]
\[
\text{where the functions } \delta \eta_r = \delta \eta_r(\varsigma, \tau) \equiv \eta_r(\varsigma, \tau; p + \delta p) - \eta_r(\varsigma, \tau; p) \text{ for } r = 1, 2 \text{ satisfy the problem}
\]
\[
\frac{ir}{2} \delta \eta_r + a_0 \frac{\partial^2 \delta \eta_r}{\partial \varsigma^2} + \frac{i}{2} \frac{\partial (a_1(\varsigma) \delta \eta_r)}{\partial \varsigma} - \\
a_2(\varsigma) \delta \eta_r + (p(\varsigma) + \delta p(\varsigma)) \delta \eta_r = -\delta \eta_r - \\
\frac{i}{2} a_3 \left( |u_{r,\delta}|^2 |\eta_\delta| \right) \delta \eta_r + \\
\frac{i}{2} a_3 \left( |u_{r,\delta}|^2 |\eta_\delta| \right) + 2(-1)^r (\delta u_1 - \delta u_2),
\]
\[
\delta \eta_r(0, \tau) = 0, \; \varsigma \in I, \; r = 1, 2,
\]
\[
\frac{\partial \delta \eta_r}{\partial \varsigma}(0, \tau) = \frac{\partial \delta \eta_r}{\partial \varsigma}(l, \tau) = 0, \; \tau \in (0, T).
\]

For this problem, as similar to obtain of inequality (20), we get the estimate
\[
\| J'_\alpha(p + \delta p) - J'_\alpha(p) \|_{L^2(I)} \leq \\
5T \| \delta \eta_1 \|_{L_\infty(\Omega)}^2 \int_0^T \| \delta \eta_2 \|_{L_2(I)}^2 d\tau + \\
5T \| \delta \eta_2 \|_{L_\infty(\Omega)}^2 \int_0^T \| \delta \eta_1 \|_{L_2(I)}^2 d\tau + \\
20\alpha^2 \| \delta p \|_{L^2(I)}^2.
\]

In inequality above, using estimates (10), (11), (20), (25), (26), (31) and inequality (28) we get
\[
\| J'_\alpha(p + \delta p) - J'_\alpha(p) \|_{L^2(I)} \leq \\
c_{10} \| \delta p \|_{L_\infty(I)}^2 \text{ for any } p \in P
\]

which implies that
\[
| J'_\alpha(p + \delta p) - J'_\alpha(p) | \to 0 \text{ as } \| \delta p \|_{L_\infty(I)} \to 0,
\]

where the constants \( c_0, c_{10} > 0 \) are independent from \( \delta p \) and \( \tau \). Thus, the proof is completed.

**Theorem 6.** Presume that the Theorem 5 and Lemma 2 hold and let \( p^* = p^*(\varsigma) \) be a solution of the OCP. Then, the inequality
\[
\int_0^l \left( \int_0^T \text{Re}(u_{r,\delta}^* \eta_{r}^* + u_{r,\delta} \eta_{r}^*) d\tau \right) (p - p^*) d\varsigma + \\
\int_0^l (2\alpha (p^* - w)) (p - p^*) d\varsigma \leq 0
\]
is valid for any \( p \in P \), where the functions \( u_{r,\delta}^* \) and \( \eta_{r}^* \), \( r = 1, 2 \) are solutions of (12)–(14) and the adjoint problem corresponding to \( p^* \in P \), respectively.

**Proof.** It is clear that the functional \( J_\alpha(p) \) is the sum of the functionals \( J_0(p) \) and \( \alpha \| p - w \|_{L_2(I)}^2 \).

Since \( \alpha \| p - w \|_{L_2(I)}^2 \) is a continuous functional on \( P \), from Lemma 1 we deduce that the functional \( J_\alpha(p) \) is continuous on the set \( P \). Also if we take
into account Lemma 2, we say that $J_\alpha(p)$ is a continuous differentiable functional on the convex set $P$. Thus, by virtue of known theorem in [35], if the functional $J_\alpha(p)$ has a minimum value at $p^* \in P$, then

\[
(J'_\alpha(p^*), p - p^*)_{L_2(I)} \geq 0 \text{ for any } p \in P
\]

which concludes the proof. \qed

5. Conclusions

In this study, we examined an optimal control problem for a system whose state is expressed by the nonlinear Schrödinger equation. We regard Lions functional as the objective functional. As it is seen from the definition of $P$, the admissible controls set contains the measurable bounded functions from $L_2(I)$. We have shown the existence and uniqueness of the solution to the optimal control problem. By means of an adjoint problem, we demonstrated that the objective functional is differentiable in the sense of Frechet. Finally, by proving that the objective functional is a continuously differentiable functional on the set of admissible controls, we derived a necessary optimality condition for the optimal control problem.

As a future direction, we will consider the optimal control problem, in which the set of admissible controls will be chosen from the wider class of functions.

References


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An International Journal of Optimization and Control: Theories & Applications (http://www.ijocta.org)

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