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RESEARCH ARTICLE

The effect of fractional order mathematical modelling for examination of academic achievement in schools with stochastic behaviors

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ABSTRACT

Academic achievement is very important, as it enables students to be well-equipped for professional and social life and shapes their future. In the case of a possible academic failure, students generally face many cognitive, social, psychological, and behavioral problems. This problem experienced by the students has been addressed with the mathematical model in this study. This mathematical model will be handled with the help of the fractional operator, and the existence, uniqueness, and positivity of the solutions to the model equation system will be examined. In addition, local and global stability analyses of the equilibrium points of the model are planned. Numerical simulations are performed with different values of fractional orders and densities of randomness. This study is very important in terms of its original and multidisciplinary approach to a subject in the field of social sciences.

1. Introduction

Academic achievement is an almost compulsory process that occurs as a result of social progress. Because the professions that emerge as a result of the division of labor require very broad and comprehensive knowledge, as well as technical expertise and new perspectives, It has become a necessity for individuals who want to have a job and a profession to enroll in long and comprehensive education programs and succeed in teaching processes in order to acquire knowledge and skills related to that profession. The concept of success is expressed as reaching the desired result, reaching the intended goal, and achieving the desired [1]. Academic achievement, on the other hand, is defined as the skills or the expression of learned knowledge that are developed in the lessons taught at school and determined by grades appreciated by teachers, test scores, or both [2]. In addition to the grades deemed appropriate by the teachers, the recently developed standard achievement tests are also preferred as a criterion for measuring academic achievement [3]. Some criteria are taken into account when determining whether a student is academically accomplished. Some of these criteria are the general goals and desired behaviors determined for the education level of the student, the overall success of the student's class or group, all the topics that need to be learned, norms of achievement developed at the country or local level, opinions of teachers and relevant experts, the student's own level of ability, the student's level of success at entry to the education program, the student's socioeconomic level, and current conditions and opportunities [4]. Another important academic achievement criterion is grade point average (GPA). According to York, Gibson, and

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Rankin, GPA is one of the best indicators to reflect the academic achievement of students [5]. The GPA is a summary of all the effort put forth by the student in a given time period. GPA is not based on a single course but can be defined as the numerical expression of the achievements obtained from different courses in which various tasks are given. From this point of view, while determining the main actors of this research, the grade point averages of the students were taken into account. The students were handled in three different groups: those with achievement above GPA, those with average achievement, and those with below average achievement. With the help of the mathematical model developed within the scope of this research, it is aimed to determine the main factors that play a role in determining the academic achievement levels of students, to what extent they are effective, and to determine their relations with each other, to determine the level of academic failure in schools, and to obtain important findings about how to prevent failure. The tendency of students toward academic achievement is not only related to the individual characteristics of students but also to the fact that many factors such as family and social-cultural environment play a role. Studies have shown that students who achieve academically owe their success primarily to themselves, while family and school are shown as auxiliary factors [6]. In this context, it can be said that all three factors are important in studies aimed at increasing academic achievement. In the meta-analysis study conducted by Sarer, self-efficacy perception, student motivation, and self-esteem came to the fore as factors related to academic achievement with students [7]. Self-efficacy refers to an individual's personal judgment of his or her ability to perform in any field [8]. An individual's self-efficacy belief affects whether he or she can successfully perform a given job [9]. In this context, it can be said that a positive self-efficacy belief motivates the individual to be successful, encourages the unknown and difficult tasks to be overcome, and encourages them to make an effort [10]. Studies have shown that there is a highly significant relationship between students' self-efficacy and academic achievement [11], they revealed that self-efficacy and motivation are important predictors of academic achievement [12]. As students succeed in the academic field, their self-efficacy will increase, and they will be more motivated for academic achievement [13]. Another individual characteristic that affects the academic achievement of students is motivation. Academic achievement motivation can be defined as doing an action skillfully, accomplishing it perfectly, overcoming obstacles, doing better than others, resisting failure, and striving to accomplish a task [14]. In this context, it can be said that high motivation for academic achievement will lead to high academic achievement. The social cognitive approach emphasizes that motivation can change with the influence of the social environment. From this point of view, success motivation is not a fixed feature for the student; it can be said that it varies in relation to class, school, social environment, family, and the context of the subject. The last factor related to academic achievement is self-esteem. Rosenberg conceptualized self-esteem as a positive or negative attitude towards the self, which is derived from the sum of self-evaluation across different domains [15]. When the literature is examined, it is understood that there is a positive relationship between self-esteem and academic achievement [16]. In this context, it can be said that positive self-esteem will increase students' academic achievement. Another important factor affecting the academic achievement of students is family. In his meta-analysis study, Sarer determined the family-related factors affecting the academic achievement of students as parents' attitudes and behaviors, participation in education, the educational status of parents, and the socioeconomic level of the family [7]. The child is born into a family environment. It should not be overlooked that this environment has a significant impact on the child's social adaptation and personality development, as well as on academic achievement [17]. The family is the first institution where the child starts school. The child forms his/her perspective on education for the first time here [18]. If a home is a place where the child's basic needs are met and he lives in peace and security, this positive atmosphere will contribute to the child's self-confidence in school [19]. In addition to a healthy and orderly family environment, the parents' interest in and inclinations toward the academic field will positively affect the child's interest in academic activities and his or her desire to achieve success. The education level of the parents is another factor that shapes the academic achievement of the child. It is known that as the education level of the parents increases, the attitudes of the parents change positively [20], which enables parents to act more consciously about their children's educational lives. In addition, the socioeconomic level of the family is shown as one of the most important factors affecting the child's ability, interests, and attitude toward education, and thus his success and harmony at school [21]. Studies have shown that as the socioeconomic status of the family increases, students' success [22] and their motivation may increase [23]. The last important factor that affects the academic achievement of students is school. In his meta-analysis study, Sarer found some school-related factors that are determinants of academic achievement: school culture, teacher behavior, and the leadership of the school principal [7]. Students spend most of their daily lives at school. For this reason, it is inevitable that the structure of the school and the attitudes of teachers will have significant effects on students' behavior and academic achievement [17]. The existence of a positive school climate not only facilitates students' academic achievement and learning but also contributes to their healthy social and emotional development [24]. The dominant culture in a school has an impact on the behavior of everyone working in that school and on students. A collaborative or positive school culture causes students to be more committed to the school's goals, and as a result, academic achievement rises. Otherwise, academic achievement is expected to be low [25]. On the other hand, it was found that the supportive behaviors of the teachers increased the success of the students. It is known that children who perceive the school environment as safe and supportive have higher school success [26]. In addition to the observations and inspections he makes, the decisions he makes, and the high expectations he creates for teaching, the school principal can significantly affect the academic achievement of the students with his leadership behaviors, such as providing the necessary resources for quality education, evaluating and developing teachers, and leading the formation of a learning-centered school climate [27]. It can be seen that all these variables, which are effective on the academic achievement of the students, interact with each other and determine which group the student will be in in terms of academic achievement. Because of the interaction between the variables, academic achievement in this study was handled with the mathematical model developed through the metapopulation model. Mathematical models can be helpful in explaining a system, examining the effects of different components, and predicting behavior. Mathematical models can be used in the social sciences (economics, psychology, sociology, political science, etc.) as well as the natural sciences (physics, biology, earth science, meteorology, etc.) and engineering disciplines (computer science, artificial intelligence, etc.) [28–33]. In the literature review, it was understood that a comprehensive mathematical model for academic achievement, which is an extremely important concept for social sciences and students, has not yet been developed and that the limited number of studies [34] are still at the initial level. Based on this deficiency in the literature, our study aimed to develop a realistic mathematical model for academic achievement. The idea seems efficient if we model problems with crossover behaviors. Because of this, in this paper, we aim to modify a metapopulation model with the concept of stochastic situations.

2. Preliminaries

In this section, we give some important definitions of non-integer fractional derivatives and their useful properties [35–37].

Definition 1. *The Gamma function $\Gamma(x)$ is defined by the integral expression given as*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \tag{1}$$

which converges in the right half of the complex plane $Re(x) > 0$.

Definition 2. *Riemann-Liouville definition of fractional order differ-integral:*

$${}_a D_t^v f(t) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dt^n} \int_a^t (t-\tau)^{n-v-1} f(\tau)\, d\tau, \tag{2}$$

where

$$n-1 < v \leqslant n, n \in \mathbb{N} \tag{3}$$

and $v \in \mathbb{R}$ ($\mathbb{R}$ is the set of real numbers) is a fractional order of the differ-integral of the function $f(t)$.

Definition 3. *Caputo's definition of fractional order differ-integral:*

$${}_a^C D_t^v f(t) = \frac{1}{\Gamma(v-n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{v+1-n}} d\tau, \tag{4}$$

where $n-1 < v \leqslant n, n \in \mathbb{N}$, $v \in \mathbb{R}$ is a fractional order of the differ-integral of the function $f(t)$.

Definition 4. *Let f be continuous not necessary differentiable in $[t_1, T]$. Thus, the piecewise Riemann-Liouville derivative is presented as*

$${}_0^{PRL} D_t^v f(t) = \begin{cases} f'(t), & if\, 0 \le t \le t_1 \\ {}_{t_1}^{RL} D_t^v f(t), & if\, t_1 \le t \le T \end{cases}, \tag{5}$$

where ${}_0^{PRL} D_t^v$ presents classical derivative on $0 \le t \le t_1$ and Riemann-Liouville fractional derivative on $t_1 \le t \le T$.

Definition 5. *Let f be continuous and $\upsilon > 0$ then a piecewise integral of f is given as*

$$ {}^{PPL}J_t^{\upsilon} f(t) = \begin{cases} \int_0^{t_1} f(\tau)d\tau, & \text{if } 0 \le t \le t_1 \\ \frac{1}{\Gamma(\upsilon)} \int_{t_1}^{t} (t-\tau)^{\upsilon-1} f(\tau)d\tau, & \text{if } t_1 \le t \le T \end{cases} \tag{6} $$

where ${}^{PPL}J_t^{\upsilon} f(t)$ presents classical integral on $0 \le t \le t_1$ and the integral with power-law kernel on $t_1 \le t \le T$.

3. Model derivation

In this paper, we considered and studied an academic achievement model with given standard incidence with takes the following form [34]:

$$ \begin{aligned} \frac{dP}{dt} &= \mu - \beta PK - \mu P + \alpha I, \\ \frac{dK}{dt} &= \beta PK - \mu K - \delta(1-\gamma) K, \\ \frac{dI}{dt} &= \delta(1-\gamma) K - (\mu+\alpha) I, \end{aligned} \tag{7} $$

where P, K, I denotes the numbers of students with above average achievement (aac) at any time t, students with average achievement (ac), students with below-average achievement (bac) and $N = P + K + I$ is the number of total population of individuals.

$$ P(t_0) = P_0,\ K(t_0) = K_0 \text{ and } I(t_0) = I_0. \tag{8} $$

The parameter β denotes the rate of students exposed to negative teacher attitudes; μ denotes rate of students with academic motivation; γ is rate of students with high self-efficacy, δ denotes the rate of students with low self-esteem and α denotes the rate of students with positive family attitudes. The parameters involved in the system (7) are all positive constans.

Fractional calculus, which means fractional derivatives and fractional integrals is of increasing interest among researchers. It is known that fractional operators describe the system behavior more accurately and efficiently than integer-order derivatives. Because of the great advantage of memory properties, let us modify the above system by replacing the integer-order time derivative by the Caputo fractional derivative below:

$$ \begin{aligned} {}_0^C D_t^{\alpha} P(t) &= \mu - \beta PK - \mu P + \alpha I, \\ {}_0^C D_t^{\alpha} K(t) &= \beta PK - \mu K - \delta(1-\gamma) K, \\ {}_0^C D_t^{\alpha} I(t) &= \delta(1-\gamma) K - (\mu+\alpha) I, \end{aligned} \tag{9} $$

with the initial conditions

$$ P(t_0) = P_0, K(t_0) = K_0 \text{ and } I(t_0) = I_0. \tag{10} $$

3.1. Positiveness and boundness of solutions

In this section, to show the positivity of the solutions of system for $\forall t \geqslant 0$, we define the norm

$$ \|f\|_{\infty} = \sup_{t\in[0,T]} |f(t)|. \tag{11} $$

Let us write the system and start with the second equation

$$ \begin{aligned} \frac{dP(t)}{dt} &= \mu - \beta P(t) K(t) - \mu P(t) + \alpha I(t), \forall t \geqslant 0, \\ &\ge -(\beta K(t) + \mu) P(t), \forall t \geqslant 0, \\ &\ge -\left(\beta \sup_{t\in[0,T]} |K(t)| + \mu\right) P(t), \forall t \geqslant 0, \\ &\ge -(\beta \|K\|_{\infty} + \mu) P(t), \forall t \geqslant 0. \end{aligned} \tag{12} $$

Then this provides that

$$ P(t) \ge P_0 e^{-(\beta\|K\|_{\infty}+\mu)t}, \forall t \geqslant 0. \tag{13} $$

Secondly for the function $I(t)$, we obtain

$$ \begin{aligned} \frac{dI(t)}{dt} &= \delta(1-\gamma) K(t) - (\mu+\alpha) I(t), \forall t \geqslant 0, \\ &\ge -(\mu+\alpha) I(t), \forall t \geqslant 0. \end{aligned} \tag{14} $$

So this provides that

$$ I(t) \ge I_0 e^{-(\mu+\alpha)t}, \forall t \geqslant 0. \tag{15} $$

Finally we assume that $P(t) K(t)$ are nonnegative then for the function $K(t)$, we obtain

$$ \begin{aligned} \frac{dK(t)}{dt} &= \beta P(t) K(t) - \mu K(t) - \delta(1-\gamma) K(t), \forall t \geqslant 0, \\ &\ge -(\mu + \delta(1-\gamma)) K(t), \forall t \geqslant 0. \end{aligned} $$

This provides that

$$ K(t) \ge K_0 e^{-(\mu+\delta(1-\gamma))t}, \forall t \geqslant 0. \tag{16} $$

Now let us check for the following total population size is given by

$$\begin{aligned} N'(t) &= P'(t) + K'(t) + I'(t) \\ &= \mu - \beta P(t) K(t) - \mu P(t) + \alpha I(t) \\ &+ \beta P(t) K(t) - \mu K(t) - \delta(1-\gamma) K(t) \\ &+ \delta(1-\gamma) K(t) - (\mu+\alpha) I(t) \\ &= \mu - \mu (P(t) + K(t) + I(t)) \\ &= \mu - \mu N(t) \end{aligned} \tag{17}$$

Integrating over $[0, t]$ then we get,

$$N(t) = 1 - e^{-\mu t} + N(0) e^{-\mu t}, \tag{18}$$
$$\lim_{t\to\infty} N(t) = 1.$$

So the model has the following feasible region

$$\Gamma = \left\{ P(t), K(t), I(t) \in R^3_+ : N(t) \leq 1 \right\}. \tag{19}$$

3.2. Equilibrium points of system

In this subsection, we find equilibrium points by solving the equations obtained by equating time derivatives in the system to zero.

$$\begin{aligned} P'(t) &= \mu - \beta P(t) K(t) - \mu P(t) + \alpha I(t) = 0, \\ K'(t) &= \beta P(t) K(t) - \mu K(t) - \delta(1-\gamma) K(t) = 0, \\ I'(t) &= \delta(1-\gamma) K(t) - (\mu+\alpha) I(t) = 0. \end{aligned}$$

Then we write

$$\begin{aligned} \mu - \beta P^* K^* - \mu P^* + \alpha I^* &= 0, \\ \beta P^* K^* - \mu K^* - \delta(1-\gamma) K^* &= 0, \\ \delta(1-\gamma) K^* - (\mu+\alpha) I^* &= 0. \end{aligned} \tag{20}$$

From the last equality, we get

$$\delta(1-\gamma) K^* = (\mu+\alpha) I^*, \tag{21}$$
$$K^* = \frac{(\mu+\alpha)}{\delta(1-\gamma)} I^*$$

and from the second equality we get

$$\beta P^* K^* = \mu K^* + \delta(1-\gamma) K^*, \tag{22}$$
$$P^* = \frac{\mu + \delta(1-\gamma)}{\beta}.$$

If we put them in the first equation we will get following

$$\mu - (\beta K^* + \mu) P^* + \alpha I^* = 0,$$
$$\mu - \left(\beta \frac{(\mu+\alpha)}{\delta(1-\gamma)} I^* + \mu\right) \left(\frac{\mu + \delta(1-\gamma)}{\beta}\right) + \alpha I^* = 0,$$
$$\mu - \left\{(\mu+\alpha) I^* \left(\frac{\mu}{\delta(1-\gamma)} + 1\right) + \frac{\mu^2 + \delta\mu(1-\gamma)}{\beta}\right\} + \alpha I^* = 0,$$

from the last equality

$$I^* = \frac{\mu - \frac{\mu^2 + \delta\mu(1-\gamma)}{\beta}}{(\mu+\alpha)\left(\frac{\mu}{\delta(1-\gamma)} + 1\right) - \alpha}. \tag{23}$$

So we have success equilibrium point $E^* = (P^*, K^*, I^*)$ given as:

$$E^* = \left(\frac{\mu + \delta(1-\gamma)}{\beta}, \frac{(\mu+\alpha)}{\delta(1-\gamma)} I^*, \frac{\mu - \frac{\mu^2 + \delta\mu(1-\gamma)}{\beta}}{(\mu+\alpha)\left(\frac{\mu}{\delta(1-\gamma)} + 1\right) - \alpha} \right), \tag{24}$$

and success free equilibrium point $E^0 = \left(P^0, K^0, I^0\right) = (1, 0, 0)$ given as

$$\begin{aligned} \mu - \beta P^* K^* - \mu P^* + \alpha I^* &= 0, \\ \beta P^* K^* - \mu K^* - \delta(1-\gamma) K^* &= 0, \\ \delta(1-\gamma) K^* - (\mu+\alpha) I^* &= 0, \end{aligned} \tag{25}$$

$$\begin{aligned} K^* &= 0 \; and \; I^* = 0, \\ \mu - \mu P^* &= 0, \\ \mu &= \mu P^*, \\ P^* &= 1. \end{aligned} \tag{26}$$

3.3. Reproductive number for model

Here we will discuss the reproductive number of the academic achievement model by considering the next generation matrix method. Remembering the system;

$$\begin{aligned} \frac{dP(t)}{dt} &= \mu - \beta P(t) K(t) - \mu P(t) + \alpha I(t), \\ \frac{dK(t)}{dt} &= \beta P(t) K(t) - \mu K(t) - \delta(1-\gamma) K(t), \\ \frac{dI(t)}{dt} &= \delta(1-\gamma) K(t) - (\mu+\alpha) I(t), \end{aligned} \tag{27}$$

and divide the system into two parts.

We call f with the nonlinear part of system and V is called with linear part of system as below:

$$\begin{bmatrix} P' \\ K' \\ I' \end{bmatrix} = f - V. \tag{28}$$

So we have the following matrices

$$f = \begin{bmatrix} -\beta P(t) K(t) \\ \beta P(t) K(t) \\ 0 \end{bmatrix}, V = \begin{bmatrix} -\mu + \mu P(t) - \alpha I(t) \\ \mu K(t) + \delta(1-\gamma) K(t) \\ -\delta(1-\gamma) K(t) + (\mu+\alpha) I(t) \end{bmatrix}, \tag{29}$$

From the above matrices, we will obtain F and V which are partial derivatives of f and V.

$$F=\begin{bmatrix} -\beta K & -\beta P & 0 \\ \beta K & \beta P & 0 \\ 0 & 0 & 0 \end{bmatrix}, V=\begin{bmatrix} \mu & 0 & -\alpha \\ 0 & \mu+\delta(1-\gamma) & 0 \\ 0 & -\delta(1-\gamma) & \mu+\alpha \end{bmatrix} \tag{30}$$

and

$$V^{-1}=\begin{bmatrix} \frac{1}{\mu} & \frac{\alpha\delta(1-\gamma)}{\mu^3+\mu^2\alpha+\mu^2\delta(1-\gamma)+\mu\alpha\delta(1-\gamma)} & \frac{\alpha}{\mu^2+\mu\alpha} \\ 0 & \frac{1}{\mu+\delta(1-\gamma)} & 0 \\ 0 & \frac{\delta(1-\gamma)}{\mu^2+\mu\alpha+\mu\delta(1-\gamma)+\alpha\delta(1-\gamma)} & \frac{1}{\mu+\alpha} \end{bmatrix}, \tag{31}$$

$$F(E_0)=\begin{bmatrix} 0 & -\beta & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{32}$$

$$FV^{-1}(E_0)=\begin{bmatrix} 0 & \frac{-\beta}{\mu+\delta(1-\gamma)} & 0 \\ 0 & \frac{\beta}{\mu+\delta(1-\gamma)} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{33}$$

$$R_0=\max\{\lambda_i\}_{i=1,2,3} \tag{34}$$

where λ_i are obtained from

$$\left|FV^{-1}-\lambda I\right|=0. \tag{35}$$

So we have the following reproductive number which is important for us while deciding the analysis

$$R_0=\frac{\beta}{\mu+\delta(1-\gamma)}. \tag{36}$$

3.4. Strength number

The concept of strength number (A_0) has been suggested and will be used in this section [38]. The component F_A is obtained with deriving the nonlinear part of the model classes. In our model there are two nonlinear classes given by

$$\frac{dP(t)}{dt}=\mu-\beta P(t)K(t)-\mu P(t)+\alpha I(t), \tag{37}$$
$$\frac{dK(t)}{dt}=\beta P(t)K(t)-\mu K(t)-\delta(1-\gamma)K(t).$$

Again here we use nonlinear parts for $\frac{dP(t)}{dt}$ and $\frac{dK(t)}{dt}$ classes

$$\frac{dP(t)}{dt}=-\beta P(t)K(t), \tag{38}$$
$$\frac{dK(t)}{dt}=\beta P(t)K(t).$$

Then

$$\frac{\partial}{\partial P}=-\beta K(t), \tag{39}$$
$$\frac{\partial}{\partial K}=-\beta P(t),$$

and

$$\frac{\partial^2}{\partial P^2}=0, \tag{40}$$
$$\frac{\partial^2}{\partial K^2}=0.$$

In this case, we can have the following

$$F_A=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{41}$$

Then

$$\det(F_AV^{-1}-\lambda I)=0, \tag{42}$$

leads to

$$A_0=0. \tag{43}$$

$A_0=0$ means there is no strength. Also, there are more conclusions when strength is zero. Students will get motivation for having good marks and expecting a good future, therefore, the number of incompetent students.

4. Stability analysis of equilibrium points

In this section, a detailed analysis of equilibrium points is presented. To do this we search for local and global stability.

4.1. Local stability analysis

There exist two equilibrium points of the model that are found by solving the equations obtained by equating time derivatives in the system to zero.Then we have $E^0=(1,0,0)$ and $E^*=(P^*,K^*,I^*)$.

Theorem 1. *The academic achievement free equilibrium point E^0 of system is locally asymptotically stable if and only if $R_0<1$.*

Proof. Let us consider the right sides of equations by solving functions $F_i: 1\leq i\leq 3$.

$$F_1(t,P(t))=\mu-\beta P(t)K(t)-\mu P(t)+\alpha I(t),$$
$$F_2(t,K(t))=\beta P(t)K(t)-\mu K(t)-\delta(1-\gamma)K(t),$$
$$F_3(t,I(t))=\delta(1-\gamma)K(t)-(\mu+\alpha)I(t). \tag{44}$$

The Jacobian matrix of the system is given by

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial P} & \frac{\partial F_1}{\partial K} & \frac{\partial F_1}{\partial I} \\ \frac{\partial F_2}{\partial P} & \frac{\partial F_2}{\partial K} & \frac{\partial F_2}{\partial I} \\ \frac{\partial F_2}{\partial P} & \frac{\partial F_3}{\partial K} & \frac{\partial F_3}{\partial I} \end{bmatrix} \tag{45}$$
$$= \begin{bmatrix} -\beta K-\mu & -\beta P & \alpha \\ \beta K & \beta P-\mu-\delta(1-\gamma) & 0 \\ 0 & \delta(1-\gamma) & -(\mu+\alpha) \end{bmatrix}$$

at $E^0 = (1,0,0)$,

$$J = \begin{bmatrix} -\mu & -\beta & \alpha \\ \beta K & \beta-\mu-\delta(1-\gamma) & 0 \\ 0 & \delta(1-\gamma) & -(\mu+\alpha) \end{bmatrix}. \tag{46}$$

If we solve the associated characteristic equation we will get the following eigenvalues;
The academic achievement free equilibrium point is asymptotically stable if all of the eigenvalues λ_1,λ_2,λ_3 of $J(E^0)$ satisfy the condition

$$|\arg \lambda_i| > \frac{\upsilon\pi}{2}, i = 1,2,3. \tag{47}$$

These eigenvalues can be determined by solving the characteristic equation $\det\left(J\left(E^0\right) - \lambda I\right) = 0$ which leads to the following equation;

$$\det\left(J\left(E^0\right) - \lambda I\right)$$
$$= \begin{vmatrix} -\mu-\lambda & -\beta & \alpha \\ \beta K & \beta-\mu-\delta(1-\gamma)-\lambda & 0 \\ 0 & \delta(1-\gamma) & -(\mu+\alpha)-\lambda \end{vmatrix}, \tag{48}$$
$$= (-\mu-\lambda)(\beta-\mu-\delta(1-\gamma)-\lambda)(-(\mu+\alpha)-\lambda)$$

$$\lambda_1 = \mu,\ \lambda_2 = \beta-\mu-\delta(1-\gamma) \text{ and } \lambda_3 = -(\mu+\alpha). \tag{49}$$

Here λ_1, λ_3 are negative. For λ_2 following must be satisfied;

$$\lambda_2 = \beta-\mu-\delta(1-\gamma) < 0, \tag{50}$$
$$\beta < \mu+\delta(1-\gamma),$$
$$R_0 = \frac{\beta}{\mu+\delta(1-\gamma)} < 1.$$

So the proof is completed. □

Theorem 2. *The academic achievement equilibrium point $E^* = (P^*, K^*, I^*)$ of system is locally asymptotic stable if and only if $R_0 > 1$.*

Proof. The Jacobian matrix $J(P^*, K^*, I^*)$ for the system given in (7) is.

$$J = \begin{bmatrix} -\beta K^*-\mu & -\beta P^* & \alpha \\ \beta K^* & \beta P^*-\mu-\delta(1-\gamma) & 0 \\ 0 & \delta(1-\gamma) & -(\mu+\alpha) \end{bmatrix}. \tag{51}$$

We now discuss the asymptotics stability of the $E^* = (P^*, K^*, I^*)$ equilibrium the system given before:

$$P^* = \frac{\mu+\delta(1-\gamma)}{\beta}, \tag{52}$$
$$K^* = \frac{(\mu+\alpha)}{\delta(1-\gamma)} I^*,$$
$$I^* = \frac{\mu - \frac{\mu^2+\delta\mu(1-\gamma)}{\beta}}{(\mu+\alpha)\left(\frac{\mu}{\delta(1-\gamma)}+1\right)-\alpha}.$$

The characteristic equation of the system is obtained via the determination of

$$L(\lambda) = \det(J-\lambda I) = 0. \tag{53}$$

The characteristic roots are obtained by solving the following equation

$$L(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0. \tag{54}$$

where

$$a_1 = \beta K^* + \mu - \beta P^* + \mu\delta(1-\gamma),$$
$$a_2 = (\mu+\alpha)(\beta K^* + \mu - \beta P^* + \mu\delta(1-\gamma))$$
$$+ \beta K^*\mu\delta(1-\gamma) - \mu\beta P^* + \mu^2\delta(1-\gamma),$$
$$a_3 = (\mu+\alpha)\left(\beta K^*\mu\delta(1-\gamma) - \mu\beta P^* + \mu^2\delta(1-\gamma)\right)$$
$$- \alpha\beta K^*\delta(1-\gamma).$$

For $a_1, a_2, a_3 > 0$ and $a_1a_2 - a_3 > 0$, so by Routh-Hurwitz Criterion, all characteristics roots have negative real parts [39]. Therefore academic achievement equilibrium point is asymptotic stable. □

4.2. Global stability of equilibrium point

In this section, we present the global stability of (PKI) model named by the academic achievement model. Let us consider the model again;

$$P'(t) = \mu - \beta P(t)K(t) - \mu P(t) + \alpha I(t) = 0,$$
$$K'(t) = \beta P(t)K(t) - \mu K(t) - \delta(1-\gamma)K(t) = 0,$$
$$I'(t) = \delta(1-\gamma)K(t) - (\mu+\alpha)I(t) = 0. \tag{55}$$

Theorem 3. *If $R_0 \geq 1$, the point $C^*(P^*, K^*, I^*)$ is global asymptotically stable.*

Proof. Here we show the proof of the theorem by using the Lyapunov function. We start with defining the Lyapunov function associated the system as below:

$$L(C^*(P^*, K^*, I^*)) = \left(P - P^* + P^*\log\frac{P^*}{P}\right)$$
$$+ \left(K - K^* + K^*\log\frac{K^*}{K}\right) + \left(I - I^* + I^*\log\frac{I^*}{I}\right).$$

By the derivative of the Lyapunov function with respect to t, we get

$$\begin{aligned} L'(t) &= \left(\frac{P-P^*}{P}\right)P' + \left(\frac{K-K^*}{K}\right)K' + \left(\frac{I-I^*}{I}\right)I', \\ &= \left(1-\frac{P^*}{P}\right)(\mu - \beta PK - \mu P + \alpha I) \\ &+ \left(1-\frac{K^*}{K}\right)(\beta PK - \mu K - \delta(1-\gamma)K) \\ &+ \left(1-\frac{I^*}{I}\right)(\delta(1-\gamma)K - (\mu+\alpha)I) \\ &= 0. \end{aligned} \tag{56}$$

Then we write;

$$\begin{aligned} L'(t) &= \mu - \beta PK - \mu P + \alpha I - \frac{P^*}{P}\mu + \beta P^* K + \mu P^* - \frac{P^*}{P}\alpha I \\ &+ \beta PK - \mu K - \delta(1-\gamma)K - \beta PK^* + \mu K^* + \delta(1-\gamma)K^* \\ &+ \delta(1-\gamma)K - (\mu+\alpha)I - \frac{I^*}{I}\delta(1-\gamma)K + (\mu+\alpha)I^*. \end{aligned}$$

Let us write above also two part (positive and negative terms) below;

$$L'(t) = \phi_1 - \phi_2, \tag{57}$$

where

$$\begin{aligned} \phi_1 &= \mu + \alpha I + \beta P^* K + \mu P^*, \\ &+ \beta PK + \mu K^* + \delta(1-\gamma)K^*, \\ &+ \delta(1-\gamma)K + (\mu+\alpha)I^*, \end{aligned} \tag{58}$$

and

$$\begin{aligned} \phi_2 &= \beta PK + \mu P + \frac{P^*}{P}\mu + \frac{P^*}{P}\alpha I, \\ &+ \mu K + \delta(1-\gamma)K + \beta PK^*, \\ &+ (\mu+\alpha)I + \frac{I^*}{I}\delta(1-\gamma)K. \end{aligned} \tag{59}$$

Therefore if

$$\begin{aligned} &\phi_1 - \phi_2 > 0 \text{ then } L'(t) > 0, \\ &\phi_1 - \phi_2 = 0 \text{ then } L'(t) = 0, \\ &\phi_1 - \phi_2 < 0 \text{ then } L'(t) < 0. \end{aligned} \tag{60}$$

□

5. Existence and uniqueness

In this section, we present a detailed analysis of the existence and uniqueness of the system of equations. To achieve this, the following theorem is to be verified [38].

Theorem 4. *Assume that there exists positive constants $\kappa_i, \overline{\kappa}_i$ such that*

(i) $\forall i \in \{1,2,3\}$,

$$\left|F_i(x_i,t) - F_i(x_i',t)\right|^2 \le \kappa_i \left|x_i - x_i'\right|^2. \tag{61}$$

(ii) $\forall (x,t) \in R^3 \times [0,T]$,

$$|F_i(x_i,t)|^2 \le \overline{\kappa}_i \left(1 + |x_i|^2\right). \tag{62}$$

We now recall our model,

$$\begin{aligned} \frac{dP(t)}{dt} &= \mu - \beta P(t)K(t) - \mu P(t) + \alpha I(t) = F_1(t,P), \\ \frac{dK(t)}{dt} &= \beta P(t)K(t) - \mu K(t) - \delta(1-\gamma)K(t) = F_2(t,K), \\ \frac{dI(t)}{dt} &= \delta(1-\gamma)K(t) - (\mu+\alpha)I(t) = F_3(t,I). \end{aligned}$$

We start with the function $F_1(t,P(t))$. Then we will show that

$$|F_1(P,t) - F_1(P_1,t)|^2 \le \kappa_1 |P - P_1|^2. \tag{63}$$

Then, we write

$$\begin{aligned} |F_1(P,t) - F_1(P_1,t)|^2 &= |\mu - \beta PK - \mu P + \alpha I \\ &- \mu + \beta P_1 K + \mu P_1 - \alpha I|^2, \\ &= |P(-\beta K - \mu) - P_1(-\beta K - \mu)|^2, \\ &= |-\beta K - \mu|^2 |P - P_1|^2, \\ &\le \left\{2\beta^2 |K|^2 + 2\mu^2\right\} |P - P_1|^2, \\ &\le \left\{2\beta^2 \sup_{t\in[0,T]} |K(t)|^2 + 2\mu^2\right\} |P - P_1|^2, \\ &\le \left\{2\beta^2 \|K\|_\infty^2 + 2\mu^2\right\} |P - P_1|^2, \\ &\le \kappa_1 |P - P_1|^2 \end{aligned}$$

where $\kappa_1 = \left\{2\beta^2 \|K\|_\infty^2 + 2\mu^2\right\}$.

Now we continue the function $F_2(t,P(t))$. Then we get

$$\begin{aligned} |F_2(K,t) - F_2(K_1,t)|^2 &= \left| \begin{array}{c} \beta PK - \mu K - \delta(1-\gamma)K \\ -\beta PK_1 + \mu K_1 + \delta(1-\gamma)K_1 \end{array} \right|^2, \\ &\le \left\{2\beta^2 |P|^2 + 2(\mu + \delta(1-\gamma))^2\right\} \\ &\quad \times |(K - K_1)|^2, \\ &\le \left\{2\beta^2 \sup_{t\in[0,T]} |P(t)|^2 + 2(\mu + \delta(1-\gamma))^2\right\} \\ &\quad \times |(K - K_1)|^2, \\ &\le \left\{2\beta^2 \|P\|_\infty^2 + 2(\mu + \delta(1-\gamma))^2\right\} \\ &\quad \times |(K - K_1)|^2, \\ &\le \kappa_2 |(K - K_1)|^2 \end{aligned}$$

where

$$\kappa_2 = \left\{2\beta^2 \|P\|_\infty^2 + 2(\mu + \delta(1-\gamma))^2\right\}. \tag{64}$$

Similary we get,

$$\begin{aligned}
&|F_3(I,t) - F_3(I_1,t)|^2 \\
&= |\delta(1-\gamma)K - (\mu+\alpha)I - \delta(1-\gamma)K + (\mu+\alpha)I_1|^2, \\
&= |-(\mu+\alpha)I + (\mu+\alpha)I_1|^2, \\
&= |-(\mu+\alpha)|^2 |(I-I_1)|^2, \\
&\le 2(\mu^2+\alpha^2)|(I-I_1)|^2, \\
&\le \kappa_3 |(I-I_1)|^2
\end{aligned}$$

where

$$\kappa_3 = 2(\mu^2+\alpha^2). \tag{65}$$

We verified the first condition for all functions. We now verify the second condition for our model.

$$\begin{aligned}
|F_1(P,t)|^2 &= |\mu - \beta PK - \mu P + \alpha I|^2, \\
&\le 2\mu^2 + 2(\beta K + \mu)^2 |P|^2 + 2\alpha^2 |I|^2, \\
&\le 2\mu^2 + 4\left(\beta^2 |K|^2 + \mu^2\right)|P|^2 + 2\alpha^2 |I|^2, \\
&\le 2\mu^2 + 4\left(\beta^2 \sup_{t\in[0,T]} |K(t)|^2 + \mu^2\right)|P|^2 \\
&+ 2\alpha^2 \sup_{t\in[0,T]} |I(t)|^2, \\
&\le \left(2\mu^2 + 2\alpha^2 \|I\|_\infty^2\right)\left(1 + \frac{2\left(\beta^2\|K\|_\infty^2 + \mu^2\right)}{\mu^2 + \alpha^2\|I\|_\infty^2}|P|^2\right), \\
&\le \overline{\kappa}_1\left(1 + |P|^2\right)
\end{aligned}$$

under the condition

$$\frac{2\left(\beta^2\|K\|_\infty^2 + \mu^2\right)}{\mu^2 + \alpha^2\|I\|_\infty^2} < 1, \tag{66}$$

and

$$\begin{aligned}
|F_2(K,t)|^2 &= |\beta PK - \mu K - \delta(1-\gamma)K|^2, \\
&\le 3\left(\beta^2|P|^2 + \mu^2 + \delta^2(1-\gamma)^2\right)\left(1+|K|^2\right), \\
&\le 3\left(\beta^2 \sup_{t\in[0,T]}|P(t)|^2 + \mu^2 + \delta^2(1-\gamma)^2\right)\left(1+|K|^2\right), \\
&\le 3\left\{\beta^2\|P\|_\infty^2 + \mu^2 + \delta^2(1-\gamma)^2\right\}\left(1+|K|^2\right), \\
&\le \overline{\kappa}_2\left(1+|K|^2\right).
\end{aligned}$$

Finally, we get

$$\begin{aligned}
|F_3(I,t)|^2 &= |\delta(1-\gamma)K - (\mu+\alpha)I|^2, \\
&\le 2\delta^2(1-\gamma)^2|K|^2 + 2(\mu+\alpha)^2|I|^2, \\
&\le 2\delta^2(1-\gamma)^2 \sup_{t\in[0,T]}|K(t)|^2 + 2(\mu+\alpha)^2|I|^2, \\
&\le 2\delta^2(1-\gamma)^2\|K\|_\infty^2\left(1 + \frac{(\mu+\alpha)^2}{\delta^2(1-\gamma)^2\|K\|_\infty^2}|I|^2\right), \\
&\le \overline{\kappa}_3\left(1+|I|^2\right)
\end{aligned}$$

under the condition

$$\frac{(\mu+\alpha)^2}{\delta^2(1-\gamma)^2\|K\|_\infty^2} < 1. \tag{67}$$

Therefore, if the condition on linear growth holds such that

$$\max\left\{\begin{array}{c} \frac{2\left(\beta^2\|K\|_\infty^2+\mu^2\right)}{\mu^2+\alpha^2\|I\|_\infty^2} \\ \frac{(\mu+\alpha)^2}{\delta^2(1-\gamma)^2\|K\|_\infty^2} \end{array}\right\} < 1, \tag{68}$$

the system of equations has a unique system of solutions. Therefore, if the condition on linear growth holds, the system has a unique solution.

6. Stochastic version of model

Stochastic modeling shows many interesting outcomes that account for certain levels of randomness. Also, stochastic models give different results for a set of values. Recently, many mathematicians have developed several stochastic mathematical models with the aim to show results more variability. So in this section, we convert the deterministic academic achievement model to the following system:

$$\begin{aligned}
&dP(t) = [\mu - \beta P(t)K(t) - \mu P(t) + \alpha I(t)]\,dt + \sigma_1 P(t)dB_1(t), \\
&dK(t) = [\beta P(t)K(t) - \mu K(t) - \delta(1-\gamma)K(t)]\,dt + \sigma_2 K(t)dB_2(t), \\
&dI(t) = [\delta(1-\gamma)K(t) - (\mu+\alpha)I(t)]\,dt + \sigma_3 I(t)dB_3(t), \\
&P(0) = P_0,\ K(0) = K_0,\ \text{and } I(0) = I_0.
\end{aligned}$$

We can present a numerical solution of the model by converting the stochastic model into an integral system below with different kernels, such as power, exponential and Mittag-Leffler.

6.1. Numerical Simulation for the stochastic-deterministic model of academic achievement

In this section, we give a numerical simulation of the system of fractional stochastic differential equations. The notion of piecewise that was recently suggested is perhaps the future of modeling processes with crossover in patterns. So we have made use of the model with the piecewise differential operators and the numerical scheme where the Lagrange polynomial interpolation is used [40]. While modeling with the piecewise idea, the first part is classical, the second part is fractional, and the last part is stochastic [36]. The numerical simulation is performed for different values of fractional orders. So the stochastic-deterministic model is given as

$$\left\{\begin{array}{l}
\frac{dP(t)}{dt} = \mu - \beta P(t)K(t) - \mu P(t) + \alpha I(t), \\
\frac{dK(t)}{dt} = \beta P(t)K(t) - \mu K(t) - \delta(1-\gamma)K(t),\quad \text{if } 0 \le t \le W_1 \\
\frac{dI(t)}{dt} = \delta(1-\gamma)K(t) - (\mu+\alpha)I(t),
\end{array}\right. \tag{69}$$

$$\begin{cases} {}^{C}_{w_1}D_t^{\upsilon}P(t) = \mu - \beta P(t)K(t) - \mu P(t) + \alpha I(t), \\ {}^{C}_{w_1}D_t^{\upsilon}K(t) = \beta P(t)K(t) - \mu K(t) - \delta(1-\gamma)K(t), \\ {}^{C}_{w_1}D_t^{\upsilon}I(t) = \delta(1-\gamma)K(t) - (\mu+\alpha)I(t), \end{cases} \tag{70}$$

$$\text{if } W_1 \le t \le W_2$$
$$0 < \upsilon \le 1.$$

$$\begin{cases} dP(t) = [\mu - \beta P(t)K(t) - \mu P(t) + \alpha I(t)]\,dt + \sigma_1 P(t)dB_1(t), \\ dK(t) = [\beta P(t)K(t) - \mu K(t) - \delta(1-\gamma)K(t)]\,dt + \sigma_2 K(t)dB_2(t), \\ dI(t) = [\delta(1-\gamma)K(t) - (\mu+\alpha)I(t)]\,dt + \sigma_3 I(t)dB_3(t), \end{cases} \tag{71}$$

if $W_2 \le t \le W$. For simplicity, we consider right side of the system as

$$\begin{cases} \dot{P} = F_1(P,K,I), \\ \dot{E} = F_2(P,K,I), \\ \dot{I} = F_3(P,K,I). \end{cases} \tag{72}$$

Using the numerical scheme presented in this paper with piecewise derivative, the numerical solution of the stochastic-deterministic model is given as follows:

$$\begin{aligned} P_i^{n_1} &= P_i(0) +_{k_1=0}^{n_1} \left\{ \frac{3\Delta t}{2} F_1(t_{k_1}, P(t_{k_1})) - F_1(t_{k_1-1}, P(t_{k_1-1}))\frac{\Delta t}{2} \right\}, \quad 0 \le t \le W_1 \\ P_i^{n_2} &= P_i(W_1) + \frac{(\Delta t)^{\upsilon}}{\Gamma(\upsilon+2)}{}_{k_2=n_1}^{n_2} F_1(t_{k_2}, P(t_{k_2}))\times \\ &\quad \begin{bmatrix} (n_2-k_2+1)^{\upsilon}(n_2-k_2+2+\upsilon) \\ -(n_2-k_2)^{\upsilon}(n_2-k_2+2+2\upsilon) \end{bmatrix} \quad W_1 \le t \le W_2 \\ &\quad - \frac{(\Delta t)^{\upsilon}}{\Gamma(\upsilon+2)}{}_{k_2=n_1}^{n_2} F_1(t_{k_2-1}, P(t_{k_2-1}))\times \\ &\quad \begin{bmatrix} (n_2-k_2+1)^{\upsilon+1} \\ -(n_2-k_2)^{\upsilon}(n_2-k_2+1+\upsilon) \end{bmatrix}, \\ P_i^{n_3} &= P_i(W_2) +_{k_3=n_2}^{n_3} \left\{ \begin{array}{c} \frac{3\Delta t}{2} F_1(t_{k_3}, P(t_{k_3})) \\ -F_1(t_{k_3-1}, P(t_{k_3-1}))\frac{\Delta t}{2} \end{array} \right\} \quad W_2 \le t \le W \\ &\quad +_{k_3=n_2}^{n_3} \left\{ \frac{\sigma}{2}(P(t_{k_3+1}) + P(t_{k_3}))(B(t_{k_3+1}) - B(t_{k_3})) \right\}, \end{aligned} \tag{73}$$

$$\begin{aligned} K_i^{n_1} &= K_i(0) +_{k_1=0}^{n_1} \left\{ \frac{3\Delta t}{2} F_2(t_{k_1}, K(t_{k_1})) - F_2(t_{k_1-1}, K(t_{k_1-1}))\frac{\Delta t}{2} \right\}, \quad 0 \le t \le W_1 \\ K_i^{n_2} &= K_i(W_1) + \frac{(\Delta t)^{\upsilon}}{\Gamma(\upsilon+2)}{}_{k_2=n_1}^{n_2} F_2(t_{k_2}, K(t_{k_2}))\times \\ &\quad \begin{bmatrix} (n_2-k_2+1)^{\upsilon}(n_2-k_2+2+\upsilon) \\ -(n_2-k_2)^{\upsilon}(n_2-k_2+2+2\upsilon) \end{bmatrix}, \quad W_1 \le t \le W_2 \\ &\quad - \frac{(\Delta t)^{\upsilon}}{\Gamma(\upsilon+2)}{}_{k_2=n_1}^{n_2} F_2(t_{k_2-1}, K(t_{k_2-1}))\times \\ &\quad \begin{bmatrix} (n_2-k_2+1)^{\upsilon+1} \\ -(n_2-k_2)^{\upsilon}(n_2-k_2+1+\upsilon) \end{bmatrix} \\ K_i^{n_3} &= K_i(W_2) +_{k_3=n_2}^{n_3} \left\{ \begin{array}{c} \frac{3\Delta t}{2} F_2(t_{k_3}, K(t_{k_3})) \\ -F_2(t_{k_3-1}, K(t_{k_3-1}))\frac{\Delta t}{2} \end{array} \right\} \\ &\quad W_2 \le t \le W. \\ &\quad +_{k_3=n_2}^{n_3} \left\{ \frac{\sigma}{2}(K(t_{k_3+1}) + K(t_{k_3}))(B(t_{k_3+1}) - B(t_{k_3})) \right\} \end{aligned} \tag{74}$$

$$\begin{aligned} I_i^{n_1} &= I_i(0) +_{k_1=0}^{n_1} \left\{ \frac{3\Delta t}{2} F_3(t_{k_1}, I(t_{k_1})) - F_3(t_{k_1-1}, I(t_{k_1-1}))\frac{\Delta t}{2} \right\}, \\ &\quad 0 \le t \le W_1, \\ I_i^{n_2} &= I_i(W_1) + \frac{(\Delta t)^{\upsilon}}{\Gamma(\upsilon+2)}{}_{k_2=n_1}^{n_2} \cup_3(t_{k_2}, I(t_{k_2}))\times \\ &\quad \begin{bmatrix} (n_2-k_2+1)^{\upsilon}(n_2-k_2+2+\upsilon) \\ -(n_2-k_2)^{\upsilon}(n_2-k_2+2+2\upsilon) \end{bmatrix}, \quad W_1 \le t \le W_2 \\ &\quad - \frac{(\Delta t)^{\upsilon}}{\Gamma(\upsilon+2)}{}_{k_2=n_1}^{n_2} F_3(t_{k_2-1}, I(t_{k_2-1}))\times \\ &\quad \begin{bmatrix} (n_2-k_2+1)^{\upsilon+1} \\ -(n_2-k_2)^{\upsilon}(n_2-k_2+1+\upsilon) \end{bmatrix}, \\ I_i^{n_3} &= I_i(W_2) +_{k_3=n_2}^{n_3} \left\{ \begin{array}{c} \frac{3\Delta t}{2} F_3(t_{k_3}, I(t_{k_3})) \\ -F_3(t_{k_3-1}, I(t_{k_3-1}))\frac{\Delta t}{2} \end{array} \right\} \\ &\quad W_2 \le t \le W. \\ &\quad +_{k_3=n_2}^{n_3} \left\{ \frac{\sigma}{2}(I(t_{k_3+1}) + I(t_{k_3}))(B(t_{k_3+1}) - B(t_{k_3})) \right\}. \end{aligned} \tag{75}$$

6.2. Numerical simulations

In this section, we show the numerical simulations for the considered stochastic model with piecewise derivative. Also, for the numerical simulations of the system, we consider the values of the parameters as follows:

$$\begin{gathered} \beta = 0.001, \mu = 0.002, \\ \gamma = 0.021, \delta = 0.029, \alpha = 0.047. \end{gathered} \tag{76}$$

In the model, the densities of randomness values for Figure 1-6 are given as

$$\sigma_1 = 0.19, \ \sigma_2 = 0.3, \ \sigma_3 = 0.4.$$

Figure 1-3 is given with fractional order $\upsilon = 1$ and Figure 4-6 is given with fractional order $\upsilon = 0.6$.

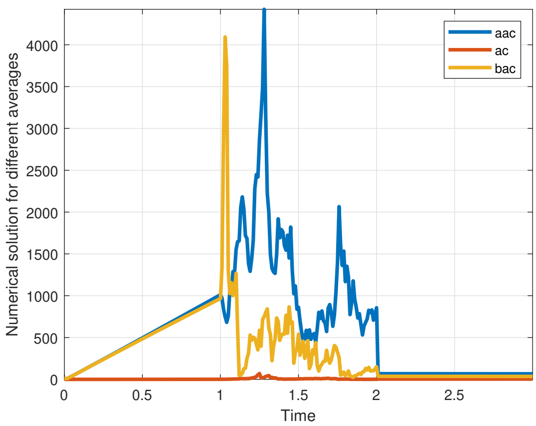

Figure 1. Numerical simulations of the system with initial conditions are given as $P(1) = 10$, $K(1) = 0$, $I(1) = 10$.

In Figure 1, It is also assumed that there are no students (ac) with average academic achievement in the school. When the simulation in Figure 1, drawn with this assumption, is examined, it is seen that when the number of students (aac) with academic achievement above the school's grade point average increases, the number of students (bac) with academic achievement below the average decreases. In this context, it has been understood that there is an inverse proportion between the number of students (aac) who are above the school's academic grade point average and the number of students (bac) who are below it. According to Figure 1, it can be said that there is an interaction between the number of students divided into groups according to their academic grade averages, there are transitions between the groups in the process, and the resulting process develops in the direction expected by the researchers.

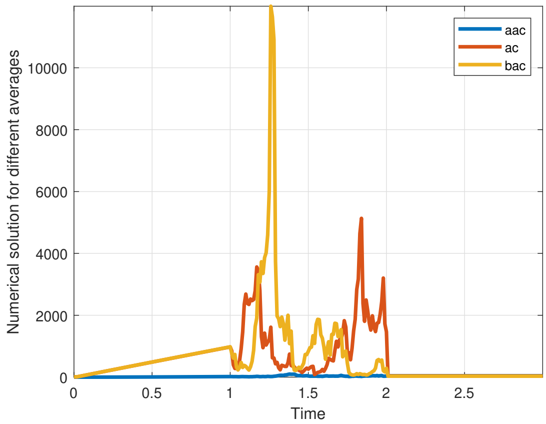

Figure 2. Numerical simulations of the system with initial conditions are given as $P(1) = 10$, $K(1) = 10$, $I(1) = 0$.

In Figure 2, it is assumed that there are no students (bac) with below average academic achievement in the school. When the simulation in Figure 2, which is drawn with this assumption, is examined, it is seen that the number of students (ac) with average academic achievement decreased along with the increase in the number of students (aac) with achievement above the school's academic grade average. In this context, it has been understood that there is an inverse proportion between the number of students (aac) who are above the school's academic grade point average and the number of students (ac) who have an average level of academic achievement. According to Figure 2, it can be said that there is an interaction between the number of students divided into groups according to their academic grade averages, there are transitions between the groups in the process, and the resulting process develops in the direction expected by the researchers.

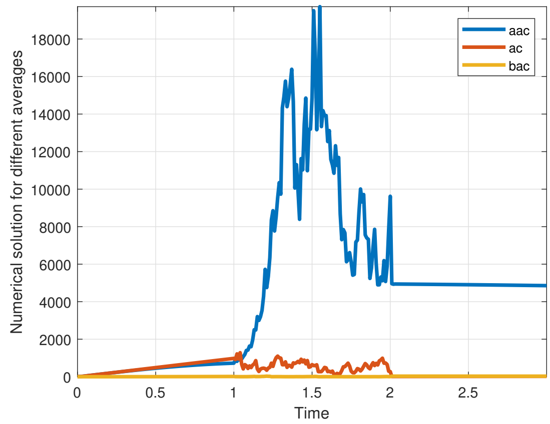

Figure 3. Numerical simulations of the system with initial conditions are given as $P(1) = 10$, $K(1) = 0$, $I(1) = 10$.

In Figure 3, It is also assumed that there are no students (ac) with average academic achievement in the school. When the simulation in Figure 3, drawn with this assumption, is examined, it is seen that when the number of students (bac) with achievement below the school's academic grade average increases, the number of students (aac) with academic achievement above the average decreases. In this context, it has been understood that there is an inverse proportion between the number of students (bac) who are below the school's academic grade point average and the number of students (aac) who are above it. According to Figure 3, it can be said that there is an interaction between the number of students divided into groups according to their academic grade averages, there are transitions between the groups in the process, and the resulting process develops in the direction expected by the researchers.

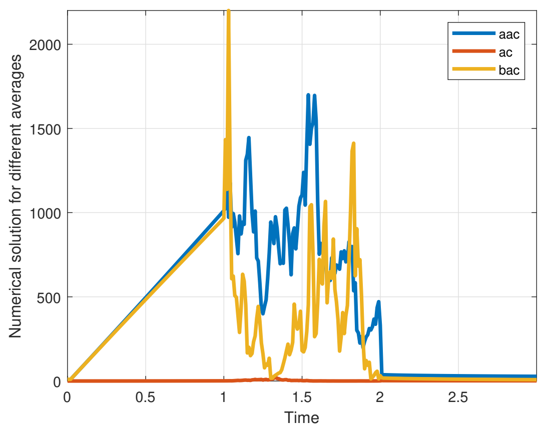

Figure 4. Numerical simulations of the system with initial conditions are given as $P(1) = 0$, $K(1) = 10$, $I(1) = 10$.

In Figure 4, it is assumed that there are no students (aac) with above average academic achievement in the school. When the simulation in Figure 4, drawn with this assumption, is examined, it is seen that when the number of students (ac) with average academic achievement increases, the number of students (bac) with below average academic achievement decreases. In this context, it has been understood that there is an inverse proportion between the number of students (ac) with average academic achievement and the number of students (bac) who are below the academic grade point average. According to Figure 4, it can be said that there is an interaction between the number of students divided into groups according to their academic grade averages, there are transitions between the groups in the process, and the resulting process develops in the direction expected by the researchers.

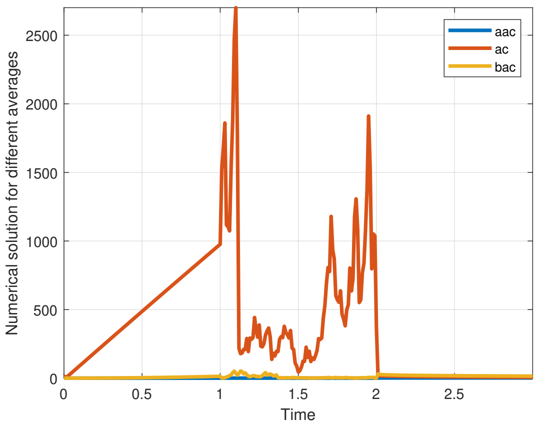

Figure 5. Numerical simulations of the system with initial conditions are given as $P(1) = 10$, $K(1) = 10$, $I(1) = 0$.

In Figure 5, it is assumed that there are no students (bac with below average academic achievement in the school. When the simulation in Figure 5, drawn with this assumption, is examined, it is seen that when the number of students (ac) with average academic achievement increases, the number of students (aac) with achievement above the academic grade point average decreases. In this context, it has been understood that there is an inverse proportion between the number of students (ac) who have an average academic achievement level at school and the number of students (aac) who are above their academic grade point average. According to Figure 5, it can be said that there is an interaction between the number of students divided into groups according to their academic grade averages, there are transitions between the groups in the process, and the resulting process develops in the direction expected by the researchers.

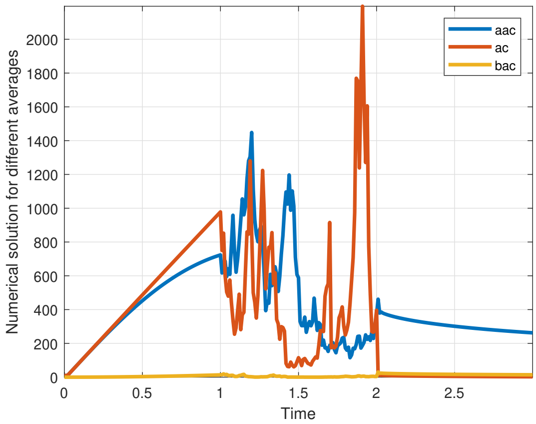

Figure 6. Numerical simulations of the system with initial conditions are given as $P(1) = 10$, $K(1) = 10$, $I(1) = 10$.

In Figure 6, It is assumed that the number of students in the groups separated according to their academic grade averages are equal to each other within the school. When the simulation in Figure 6, drawn with this assumption is examined, it is seen that there is an interaction between the number of students divided into groups according to their academic grade averages, there are transitions between the groups in the process, and over time, the students (ac) pile up into the group with average academic achievement

7. Discussion, conclusion and recommendations

Academic achievement is very important, as it enables students to be well-equipped for professional and social life and shapes their future. In the event of any academic failure, students generally face many emotional, cognitive, and behavioral problems. In this study, it was tried to calculate the academic achievement levels of the students throughout the school with the help of the mathematical model developed through the metapopulation model in order to find solutions to the possible problems that students may experience due to their academic failures. In the model developed for this purpose, academic achievement was determined by taking the GPA into account.

The students were considered three different actor groups: above-average, average, and below-average students. Individual characteristics, family, and school variables were taken into consideration as factors affecting these actors. By following this path, it was possible to calculate the academic achievement levels of the schools, in particular for the students. The developed mathematical model will make significant contributions to the determination of the effect levels of the variables that can support and harm students' academic achievement. After the relevant variables are processed, schools with low academic achievement will be able to learn from the variables which variables they need to carry out preventive and protective studies. Preventive and protective studies can be planned for the academic achievement level of the students. In these study plans, studies can be added on individual reasons (self-efficacy, self-esteem, motivation, etc.), family-related reasons (attitudes and behaviors of parents, their participation in education, education level of parents, socioeconomic level of the family, etc.), and school-related reasons (school culture, teacher behavior, school principal's leadership, etc.) included in the mathematical model developed. Although all of these variables interact with each other, they are also determinants of the probability of students being included in the academic failure risk group. In this context, it can be said that in order to minimize the problem of academic failure in schools, all these variables should be considered together and improved within the framework of a common understanding (the school and parents are in communication).

For example, both the structural characteristics of the family and the attitude of the family towards the lessons have an important place in affecting the student's motivation towards any subject or course. Therefore, the family should constantly support their child and try to keep her/his motivation high in order to be successful at school [41]. In order for parents with low educational levels, professional status, and family income levels to acquire academic predispositions; seminars, conferences, etc. informative meetings. A strong school culture causes students to be more attached to their goals and school, and as a result, academic achievement increases. School principals should be aware that they are directly influential in the creation of a strong organizational culture and the development of student success, and they should transform their institutions into learning organizations by demonstrating effective leadership behaviors. Teachers can help create a positive attitude towards lessons by taking into account the interests and needs of students, organizing various learning activities, exemplifying the application areas of the lessons in current and professional life, and emphasizing the role of lessons in the development of critical thinking and reasoning skills [7]. In addition, school counselors should organize awareness-raising seminars and plan individual and group counseling services in order to increase students' self-efficacy, self-esteem and motivation levels.

In this study, using a multidisciplinary method, academic achievement, which is a very important issue for the field of social sciences (psychology and educational sciences), is handled through a mathematical model. The fact that a subject in the field of social sciences is handled with a mathematical model apart from the computerized statistical programs (SPSS, Amos, Lisrel, Nvivo, etc.), which are frequently used in the literature, makes this study very unique in terms of method. Starting with a similar approach, the number of multidisciplinary studies can be increased by developing mathematical models about other concepts and phenomena (peer bullying, school burnout, etc.) that are important for the field of social sciences. In addition, when the literature is examined, it is striking that there are almost no studies that develop mathematical models in students. Low academic achievements will negatively affect students' chances to become successful, happy, and socially integrated individuals in their future lives. In addition, considering the potential importance of students for the society they live in, it is clear that new studies should be planned to overcome this deficiency. Of course, as with any study, this one also has some limitations. In this study, students who were above average, average, and below average (according to GPA) were taken as the actors of academic achievement. Individual, family, and school-related variables that affect these actors are emphasized. In the models to be developed later, new characters can be added among the actors of academic achievement by reducing the GPA score intervals. In addition, the variables affecting the actors include learning speed, intelligence, gender, interest, personality traits, readiness, etc. The mathematical model can be enriched by addition. It should not be forgotten! Every study that will be carried out related to academic achievement will add a different value to the literature and prepare the basis for the formation of new ideas.

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