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RESEARCH ARTICLE

# A study on the approximate controllability results of fractional stochastic integro-differential inclusion systems via sectorial operators

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ABSTRACT

The study deals with the findings of the outcome of the approximate controllability results of inclusion type fractional stochastic system in Banach space with the order of the fractional system $\varrho \in (1, 2)$. At first, we implement Bohnenblust-Karlin's fixed point technique to deduce the required conditions on which the fractional system with inital conditions is approximately controllable, and there by, we postulate the sufficient conditions for extending the obtained results to the system with nonlocal conditions.



## 1. Introduction

The recent development in the area of fractional theory plays an important role in mathematics. It is well understood by physical interpretation of differential equations that many of the realistic systems are better modeled by fractional order derivatives than integer order. Hence, there has been a huge growth in the fractional research field. The progress of this particular theory has a wide range of application in electro-magnetic, viscoelasticity, image processing, signal processing, control theory, diffusion, porous media, fluid flow and other fields. For more noteworthy contributions of fractional field the readers are referred to the books [1–5] and the research papers [6–14]. Moreover fractional integro-differential equations are used in various scientific domains such as control theory, medicine, biology and ecology etc. In the following research articles the above discussed concepts are well explained [6, 10, 11, 15, 16].

Inclusion type differential equation establishes a relation of the type $\dot{x} \in F(x)$ in such a way that the map $F$ assigns any point $x \in R^n$ to a set $F(x) \subset R^n$. To put in simple terms, the generalization of the differential function $\dot{x} = F(x)$ is termed as differential inclusion. In 1995, El-Sayed and Ibrahim extended the theory of integer order differential inclusion to fractional order [17]. Differential inclusion of fractional order acts as a key technique in analyzing differential equation with discontinuous right hand side which basically arises while modelling dynamical system which involves friction and impact problem. A sectorial operator is a type of linear operator that maps functions from one Banach space to another. It is a type of operator that is widely used in the study of partial differential equations and their associated boundary value problems. Sectorial

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operators play an important role in the analysis of differential calculus, especially in the study of well-posedness and stability of boundary value problems. They are also used in the theory of semigroups of operators and in the study of evolution equations. In [18] Kazufumi Ito et. al. analyzed the various secorial properties of Caputo derivative of order $\varrho \in (1,2)$. In [19] JinRong Wang et. al. investigated the existence of piece wise mild solutions of nonlocal impusive fractional differential inclusions with fractional sectorial operator on Banach spaces. The readers can refer to [12, 20–24] for present qualitative research topics in differential equations of inclusion type. In [13, 14, 25, 26] the authors studied the existence and solvability of mild solution for various fractional order systems with sectorial operator of the type $(P, \eta, \varrho, \gamma)$.

In general, while dealing with complicated differential systems such as growth modeling, economics, biology and quantum field theory the random noise or stochastic perturbation is unavoidable. Therefore there are numerous ongoing research in analyzing the existence and uniqueness of stochastic control models using various fixed point methods. The concept of stochastic fractional control system has been well developed with the help of different kinds of fixed point approaches in [6, 13, 27, 28]. The weaker notion of control theory is called as approximate controllability. This type of controllable system ensures that the system is steered to any random small neighborhood of the final state. Recently, the approximate controllability of control systems defined by impulsive functional inclusions and neutral integro-differential systems are well discussed in the research publications [6, 8, 10, 12, 29–32].

Very recently the autors in [10] investigated the following existence results for Caputo fractional mixed Volterra Fredholm-type integro differential inclusions of order $\varrho \in (1,2)$ with sectorial operators. Further in the past few years the application of nonlocal condition in fractional differential equations has emerged as a magnificient area of investication since it describes the evolution of the system in an efficient way. Therfore we extend out theoritical result of the Caputo fractional stochastic integro-differential inclusions system to nonlocal conditions with sectorial operators.

$$
\begin{aligned}
{}^{C}D_{\zeta}^{\varrho} z(\zeta) \in & A z(\zeta) + G\Big(\zeta, z(\zeta), \int_0^{\zeta} f(\zeta,\nu,z(\nu)),\\
&\int_0^T f(\zeta,\nu,z(\nu))d\nu\Big),\ \zeta \in V=[0,T],\\
& z(0)=z_0,\ z'(0)=z_1.
\end{aligned}
$$

Being motivated by the above works, in this paper we establish the sufficient conditions for the approximate controllability of Caputo fractional stochastic integro-differential inclusions with sectorial operators of the form

$$
\begin{aligned}
{}^{C}D_{\zeta}^{\varrho} z(\zeta) \in A z(\zeta) + G\Big(\zeta, z(\zeta), \int_0^T f(\zeta,\nu,z(\nu))d\nu\Big)&\\
\frac{dW(\zeta)}{d\zeta} + \mathbb{B}x(\zeta),\ \zeta \in V=[0,T],& \qquad (1)\\
z(0)=z_0,\ z'(0)=z_1.&
\end{aligned}
$$

where $\varrho \in (1,2)$, the sectorial operator $A$ is a mapping from $D(A) \subset \mathcal{X}$ to $\mathcal{X}$ of type $(P,\eta,\varrho,\gamma)$ in Banach space $\mathcal{X}$. $W(\zeta)$ be a standard cylindrical Wiener process in $\mathcal{X}$ defined on a stochastic space $\big(\Omega, \Im, \{\Im_\zeta\}_{\zeta\geqslant 0}, \mathbb{P}\big)$. The nonempty, closed, convex and bounded multivalued function $G: V\times\mathcal{X}\times\mathcal{X} \to 2^{\mathcal{X}}\setminus\{\emptyset\}$ and $f$ be a mapping from $V\times V\times\mathcal{X}$ into $\mathcal{X}$, $x \in L^2(V,\mathcal{H})$, where $\mathcal{H}$ stand for Banach space. In addition, the linear operator $\mathbb{B}:\mathcal{H}\to\mathcal{X}$ is bounded.

The article contains the following parts:

Part 2 : Consists of the preliminaries and definitions.

Part 3 : The controllability results for the chosen fractional inclusion systems (1) are derived by using fixed point technique.

Part 4 : The outcome of approximate controllability results derived for system (1) is extended to fractional nonlocal system.

Part 5: Appropriate illustrations for the obtained results have been established.

Part 6: Conclusion and future works of the presented system are discussed.

## 2. Preliminaries

Consider the Hilbert spaces $\mathcal{X}$, $\mathbb{K}$ and the complete probability space (CPS) $\big(\Omega,\Im,\mathbb{P}\big)$ outfitted with a normal filtration $\{\Im_\zeta, \zeta\in V\}$ satisfies the regular conditions ($\Im_\zeta$ is a increasing right continuous family such that $\Im_\zeta \subseteq \Im$, $\Im_0$ contains all $\mathbb{P}$-null set). Let $E(.)$ denotes the expectation with respect to the measure $\mathbb{P}$. Let $\{e_j\}_{j=1}^{\infty}$ be a complete orthonormal basis of $\mathbb{K}$. Suppose that $W=(W_\zeta)_{\zeta\geqslant 0}$ is a cylindrical $\mathbb{K}$-valued Wiener process defined on the CPS $\big(\Omega,\Im,\mathbb{P}\big)$ with covariance operator $Q \geqslant 0$, such that $Trace(Q) = \sum_{j=1}^{\infty}\lambda_j = \lambda < \infty$, and $Qe_j = \lambda_j e_j$. Then $W(\zeta) = \sum_{j=1}^{\infty}\sqrt{\lambda_j}W_j(\zeta)$, with $W_j(\zeta)$, $j=1$ to $\infty$

are mutually independent one-dimensional standard Wiener processes. We consider that $\Im_\zeta = T\{W(s) : 0 \leqslant s \leqslant \zeta\}$ is the sigma algebra generated by $W$ and $\Im_\zeta = \Im$. Let $L(\mathbb{K}, \mathcal{X})$ be a bounded linear operator space with the usual operator norm $\|\cdot\|$. For $\varphi \in L(\mathbb{K}, \mathcal{X})$ we define $\|\varphi\|^2 = Trace\big(\varphi Q \varphi^*\big) = \sum_{j=1}^{\infty} \|\sqrt{\lambda_j}\varphi e_j\|^2$. If $\|\varphi\|^2 < \infty$ then $\varphi$ is called $Q-$Hilbert-Schmidt operator and the space of such operators is denoted by $L_Q(\mathbb{K}, \mathcal{X})$. The completion $L_Q(\mathbb{K}, \mathcal{X})$ of $L(\mathbb{K}, \mathcal{X})$ w.r.t the topology induced by the norm $\|.\|_Q$ where $\|\varphi\|_Q^2 = (\varphi, \varphi)$ is a Hilbert space with the above norm topology.

The Banach space $L_2\big(\Omega, \Im_T, \mathcal{X}\big)$ is the collection of all square-integrable, strongly measurable, $\Im_\zeta-$ adapted, $\mathcal{X}-$ valued random variables. Also take

$$C\big(V, L_2\big(\Omega, \Im_T, \mathcal{X}\big)\big) = \big\{z : V \to L_2\big(\Omega, \Im_T, \mathcal{X}\big) \mid z \text{ is continuous and } \sup_{\zeta \in V} E\|z(\zeta)\|^2 < \infty\big\}$$

be a Banach space. Finally, we define the set

$$\mathcal{C} = \big\{z \in C\big(V, L_2(\Omega, \Im_T, \mathcal{X})\big) \mid z \text{ is measurable, } \Im_\zeta - \text{adapted } \mathcal{X} \text{ valued functions}\big\}$$

be a closed subspace of $C\big(V, L_2(\Omega, \Im_T, \mathcal{X})\big)$ with norm $\|z\| = \sup_{\zeta \in V} E\|z(\zeta)\|^2$, $E$ determine the integration w.r.t the probability measure.

**Definition 1.** *[3] The Riemann-Liouville fractional integral of order $\beta$ having the lower limit 0 for a function $g$ mapping $[0, \infty)$ into $\mathbb{R}^+$ is defined as*

$$I^\beta g(\zeta) = \frac{1}{\Gamma(\beta)} \int_0^\zeta \frac{g(\nu)}{(\zeta - \nu)^{1-\beta}} d\nu, \quad \zeta > 0,\ \beta \in \mathbb{R}^+.$$

**Definition 2.** *[3] The Riemann-Liouville fractional derivative of order $\beta$ employing the lower limit 0 for a function $g$ is defined as*

$$^{RL}D^\beta g(\zeta) = \frac{1}{\Gamma(j - \beta)} \frac{d^j}{d\zeta^j} \int_0^\zeta g^{(j)}(\nu)(\zeta - \nu)^{j-\beta-1} d\nu, \quad \zeta > 0,\ j - 1 < \beta < j,\ \beta \in \mathbb{R}^+,\ j \in \mathbb{N}.$$

**Definition 3.** *[3] Caputo fractional derivative of order $\beta$ employing the lower limit 0 for a function $g$ is defined as*

$$^C D^\beta g(\zeta) = {}^L D^\beta \left(g(\zeta) - \sum_{i=0}^{j-1} \frac{g^{(i)}(0)}{i!}\zeta^i\right), \quad \zeta > 0,\ j - 1 < \beta < j,\ \beta \in \mathbb{R}^+,\ j \in \mathbb{N}.$$

**Definition 4.** *[14] The closed and linear operator $A$ mapping from $D$ into $\mathcal{X}$ is called sectorial operator of type $(P, \eta, \varrho, \gamma)$ provided that there exist $\gamma$ belongs to $\mathbb{R}$, $\eta$ belongs to $(0, \frac{\pi}{2})$ and $P > 0$ such that the $\varrho$-resolvent of $A$ exists outside the sector*

$$\gamma + \mathcal{S}_\eta = \{\eta + \mu^\varrho : \mu \text{ in } C(V, \mathcal{X}),\ |Arg(-\mu^\varrho)| < \eta\}$$

$$\|(\mu^\varrho I - A)^{-1}\| \leq \frac{P}{|\mu^\varrho - \gamma|},\ \mu^\varrho \notin \gamma + \mathcal{S}_\eta.$$

Further, throughout the paper we assume that $A$ is a sectorial operator of type $(P, \eta, \varrho, \gamma)$, hence it is easy to establish that $A$ stands for infinitesimal generator of a $\varrho$-resolvent family $\{\mathcal{W}_\varrho(\zeta)\}_{\zeta \geq 0}$ which belongs to Banach space, where

$$\mathcal{W}_\varrho(\zeta) = \frac{1}{2\pi i} \int_c e^{\mu\varrho} \mathscr{R}\left(\mu^\varrho, A\right) d\mu.$$

**Definition 5.** *[14] Let $G : V \times \Omega \to L(\mathbb{K}, \mathcal{X})$ be the strongly measurable mapping such that $\int_0^T E\|G(\zeta)\|^p_{L(\mathbb{K},\mathcal{X})} d\zeta < \infty$ then $E\| \int_0^\zeta G(\nu) dW(\nu)\|^p \leqslant L_g \int_0^\zeta E\|G(\nu)\|^p_{L(\mathbb{K},\mathcal{X})} d\nu$, for all $\zeta \in J$ and $p \geqslant 2$, where $L_g$ is a constant.*

**Definition 6.** *[14] A function $z$ belongs to $C(V, \mathcal{X})$ is called mild solution of (1) provided that it fulfills the operator equation*

$$z(\zeta) = \mathcal{K}_\varrho(\zeta) z_0 + \mathcal{Q}_\varrho(\zeta) z_1 + \int_0^\zeta \mathcal{W}_\varrho(\zeta - \nu) g(\nu) dW(\nu) + \int_0^\zeta \mathcal{W}_\varrho(\zeta - \nu) \mathbb{B} x(\nu) d\nu.$$

*In the above*

$$\mathcal{K}_\varrho(\zeta) = \frac{1}{2\pi i} \int_c e^{\mu\varrho} \mu^{\varrho-1} \mathscr{R}(\mu^\varrho, A) d\mu,\ \mathcal{Q}_\varrho(\zeta) = \frac{1}{2\pi i} \int_c e^{\mu\varrho} \mu^{\varrho-2} \mathscr{R}(\mu^\varrho, A) d\mu,$$

$$\mathcal{W}_\varrho(\zeta) = \frac{1}{2\pi i} \int_c e^{\mu\varrho} \mathscr{R}(\mu^\varrho, A) d\mu,$$

*with c being a suitable path such that $\mu^\varrho \notin \gamma + \mathcal{S}_\eta$ for $\varphi$ belongs to $C$.*

**Theorem 1.** *[14, 33] If $A$ is a sectorial operator then the following hold on $\|\mathcal{K}_\varrho(\zeta)\|$ :*

*(i) Take $\gamma \geq 0$ and $0 < \chi < \pi$, then*

$$\|\mathcal{K}_\varrho(\zeta)\| \leq \frac{M_1(\eta, \chi) P e^{[M_1(\eta,\chi)(1+\gamma\zeta^\varrho)]^{\frac{1}{2}} \left[\left(1+\frac{\sin\chi}{\sin(\chi-\eta)}\right)^{\frac{1}{\varrho}} - 1\right]}}{\pi \sin^{1+\frac{1}{\varrho}} \eta} (1 + \eta\zeta^\varrho) + \frac{\Gamma(\varrho) P}{\pi(1 + \gamma\zeta^\varrho)|\cos\frac{\pi-\chi}{\varrho}|^\varrho \sin\eta \sin\chi},$$

*where, $\zeta > 0, M_1(\eta, \chi) = \max\left\{\frac{\sin\chi}{\sin(\chi - \eta)}, 1\right\}$.*

(*ii*) *Take* $\gamma < 0$ *and* $0 < \chi < \pi$ *then for* $\zeta > 0$

$$\|\mathcal{K}_\varrho(\zeta)\| \leq \frac{1}{1+|\gamma|\zeta^\varrho} \left(\frac{eP[(\sin\chi+1)^{\frac{1}{\varrho}}-1]}{\pi|\cos\chi|^{1+\frac{1}{\varrho}}} + \frac{\Gamma(\varrho)P}{\pi|\cos\chi|\,|\cos\frac{\pi-\chi}{\varrho}|^\varrho}\right).$$

**Theorem 2.** *[14, 33] If A is a sectorial operator then the following hold for* $\|\mathcal{W}_\varrho(\zeta)\|$ *and* $\|\mathcal{Q}_\varrho(\zeta)\|$:

(*i*) *Take* $\gamma \geq 0$ *and* $0 < \chi < \pi$ *then* $\zeta > 0$

$$\|\mathcal{W}_\varrho(\zeta)\| \leq \frac{P\left[\left(1+\frac{\sin\chi}{\sin(\chi-\eta)}\right)^{\frac{1}{\varrho}}-1\right]}{\pi\sin\eta} \times (1+\eta\zeta^\varrho)^{\frac{1}{\varrho}}\zeta^{\varrho-1}e^{[M_1(\eta,\chi)(1+\gamma\zeta^\varrho)]^{\frac{1}{\varrho}}} + \frac{P\zeta^{\varrho-1}}{\pi(1+\gamma\zeta^\varrho)|\cos\frac{\pi-\chi}{\varrho}|^\varrho\sin\eta\sin\chi},$$

$$\|\mathcal{Q}_\varrho(\zeta)\| \leq \frac{P\left[\left(1+\frac{\sin\chi}{\sin(\chi-\eta)}\right)^{\frac{1}{\varrho}}-1\right]M_1(\eta,\chi)}{\pi\sin\eta^{\frac{\varrho+2}{\varrho}}} (1+\eta\zeta^\varrho)^{\frac{\varrho-1}{\varrho}}\zeta^{\varrho-1}e^{[M_1(\eta,\chi)(1+\gamma\zeta^\varrho)]^{\frac{1}{\varrho}}} + \frac{P\varrho\Gamma(\varrho)}{\pi(1+\gamma\zeta^\varrho)|\cos\frac{\pi-\chi}{\varrho}|^\varrho\sin\eta\sin\chi},$$

*where* $M_1(\eta,\chi) = \max\left\{1, \frac{\sin\eta}{\sin(\chi-\eta)}\right\}$.

(*ii*) *Take* $\gamma < 0$ *and* $0 < \chi < \pi$ *then*

$$\|\mathcal{W}_\varrho(\zeta)\| \leq \frac{\zeta^{\varrho-1}}{1+|\gamma|\zeta^\varrho} \left(\frac{eP[(\sin\chi+1)^{\frac{1}{\varrho}}-1]}{\pi|\cos\chi|} + \frac{P}{\pi|\cos\frac{\pi-\chi}{\varrho}|\,|\cos\chi|}\right).$$

$$\|\mathcal{Q}_\varrho(\zeta)\| \leq \left(\frac{eP[(\sin\chi+1)^{\frac{1}{\varrho}}-1]t}{\pi|\cos\chi|^{1+\frac{2}{\varrho}}} + \frac{\varrho\Gamma(\varrho)P}{\pi|\cos\frac{\pi-\chi}{\varrho}|\,|\cos\chi|}\right) \frac{1}{1+|\gamma|\zeta^\varrho}, \quad for \ \ \zeta > 0.$$

Let $(\mathcal{X}, d)$ be a metric space. The following expressions are used in this article:

- $\mathcal{N}(\mathcal{X}) = \{H \in \mathcal{P}(\mathcal{X}) : H \neq \emptyset\}$,
- $\mathcal{N}_{cl}(\mathcal{X}) = \{H \in \mathcal{N}(\mathcal{X}) : H \text{ closed}\}$,
- $\mathcal{N}_b(\mathcal{X}) = \{H \in \mathcal{N}(\mathcal{X}) : H \text{ bounded}\}$,
- $\mathcal{N}_{cp}(\mathcal{X}) = \{H \in \mathcal{N}(\mathcal{X}) : H \text{ compact}\}$,
- $\mathcal{N}_c(\mathcal{X}) = \{H \in \mathcal{N}(\mathcal{X}) : H \text{ convex}\}$.

For the multivalued map $\mathscr{K} : \mathcal{C} \to 2^{\mathcal{C}} \setminus \{\emptyset\}$ the following definition holds. Additional information on multivalued maps can be found in the books [34].

**Definition 7.** *[35] If for all* $z \in \mathcal{C}$, $\mathscr{K}(z)$ *is closed(convex) then the map* $\mathscr{K}$ *is closed(convex). For every bounded set* $C$ *of* $\mathcal{C}$, $\mathscr{K}(C) = \bigcup_{z\in\mathcal{C}}\mathscr{K}(z)$ *is bounded in* $\mathcal{C}$ *then* $\mathscr{K}$ *is bounded on bounded sets.*

**Definition 8.** *[35]* $\mathscr{K}$ *is known as upper semi continuous (u.s.c) on* $\mathcal{C}$ *if the following conditions holds:*

(*i*) *For all* $z_0 \in \mathcal{C}$ *the set* $\mathscr{K}(z_0) \neq \phi$ *and it is closed.*
(*ii*) *For all open set* $C \in \mathcal{C}$ *such that* $C \supset \mathscr{K}(z_0)$ *then there exist an open neighborhood* $\mathscr{K}(\mathcal{W}) \subseteq C$.

**Definition 9.** *[35] If* $\mathscr{K}(C)$ *is a relatively compact, for all bounded subset* $C$ *of* $\mathcal{C}$ *then* $\mathscr{K}$ *is completely continuous.*

**Definition 10.** *[35] If the completely continuous map* $\mathscr{K}$ *has a nonempty values then* $\mathscr{K}$ *is u.s.c if and only if* $\mathscr{K}$ *has a closed graph i.e.,* $z^k \to z^*$, $u^k \to u^*$, $u^k$ *such that* $\mathscr{K}z^k$ *signify* $u^* \in \mathscr{K}z^*$. *Moreover , if there exists* $z \in Y$ *such that* $z \in \mathscr{K}(z)$ *then* $\mathscr{K}$ *has a fixed point.*

An u.s.c function $\mathscr{K} : \mathcal{X} \to \mathcal{X}$ is said to be condensing if for all bounded subset $\mathscr{C} \subseteq \mathcal{X}$ having $\iota(\mathscr{C}) \neq 0$, where $\iota$ stands for the Kuratowski measure of non compactness, we get

$$\iota(\mathscr{K}(\mathscr{C})) < \iota(\mathscr{C}).$$

**Definition 11.** *[35]* $G$ *mapping from* $V \times \mathcal{X} \times \mathcal{X}$ *into* $\mathcal{N}_{b,cl,cp}(L(\mathbb{K}, \mathcal{X}))$ *is called* $L^1$*-Caratheodory provided that*

(*i*) $\zeta \to G(\zeta, z, x, y)$ *is measurable for all* $z, x, y$ *belongs to* $\mathcal{X}$.
(*ii*) $(z, x, y) \to G(\zeta, z, x, y)$ *is u.s.c for all* $\zeta$ *belongs to* $V$.
(*iii*) *For all* $p > 0$, *there exist* $j_p$ *belongs to* $L^1(V, \mathbb{R}^+)$ *such that*

$$E\|G(\zeta, z, x, y)\|^2 \leq \sup\{E\|g\|^2 : g \in G(\zeta, z, x, y)\} \leq j_p(\zeta), \ \ for \ all \ \ \zeta \in V.$$

For further information on multivalued functions refer the books [34]. Detail analysis in multivalued maps are presended in this work. The following are two suitable operators and their underlying assumptions:

$$\Gamma_0^T = \int_0^T \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_\varrho^*(\zeta-\nu)d\nu : \mathcal{X} \to \mathcal{X},$$

$$\mathscr{R}(\hbar, \Gamma_0^T) = (\hbar I + \Gamma_0^T)^{-1} : \mathcal{X} \to \mathcal{X}.$$

In the above $\mathcal{W}_{\varrho}^{*}(\zeta-\nu)$ and $\mathbb{B}^{*}$ stands for adjoints of $\mathcal{W}_{\varrho}(\zeta-\nu)$ and $\mathbb{B}$ respectively and Clearly $\Gamma_0^T$ is the bounded linear operator.

To begin, evaluate the below assumptions:

**(H₀)** In the strong operator topology, $\hbar\mathscr{R}(\hbar,\Gamma_0^T)\to 0$ as $\hbar\to 0^+$.

Consider the accompanying linear inclusions of fractional system

$$\begin{cases} {}^C D_{\zeta}^{\varrho} z(\zeta)\in Az(\zeta)+\mathbb{B}x(\zeta),\ \zeta\in V=[0,T],\\ z(0)=z_0,\ z'(0)=z_1,\end{cases}$$

is approximately controllable on $V$.

**Lemma 1.** *[16]. Suppose $V$ is a compact real interval and the collection of all nonempty, bounded, closed and convex subsets of $\mathcal{X}$ is called $\mathcal{N}_{b,cl,cp}(\mathcal{X})$. Consider multivalued function $G$ mapping from $V\times\mathcal{X}$ into $\mathcal{N}_{b,cl,cp}(\mathcal{X})$ is measurable to $\zeta$ for all fixed $z$ belongs to $\mathcal{X}$, upper continuous to $z$ for all $\zeta$ belongs to $V$ and for all $z$ belongs to $\mathcal{C}$,*

$$S_{G,z}=\left\{g\in L^1(V,\mathcal{X}):g(\zeta)\in G\left(\zeta,z(\zeta),\int_0^T f(\zeta,\nu,z(\nu))d\nu\right),\ \zeta\in V\right\}$$

*is nonempty. Assume that $\mathcal{M}:L^1(V,\mathcal{X})\to\mathcal{C}$ is a linear continuous function, next*

$$\mathcal{M}\circ S_G:\mathcal{C}\to\mathcal{N}_{b,cl,cp}(\mathcal{C})$$
$$z\to(\mathcal{M}\circ S_G)(z)=\mathcal{M}(S_{G,z})$$

*is a closed graph operator belongs to $\mathcal{C}\to\mathcal{C}$.*

**Lemma 2.** *[36] Suppose $H$ is a subset of $\mathcal{X}$ which is nonempty, bounded, closed and convex, assume $\mathcal{D}:H\to 2^{\mathcal{X}}\setminus\{\emptyset\}$ is u.s.c with closed, convex values such that $\mathcal{D}(H)\subset H$ where $\mathcal{D}(H)$ is compact, then $\mathcal{D}$ has a fixed point.*

## 3. Approximate controllability

The section explicitly focuses on the articulation of mild solution for the above mentioned system (1). We now present the required hypothesis for proving the main theorem:

**(H₁)** $\mathcal{K}_{\varrho}(\zeta)$, $\mathcal{Q}_{\varrho}(\zeta)$ and $\mathcal{W}_{\varrho}(\zeta)$ are compact $\varrho$-resolvent families generated by the sectorial operator $A$. For all $\zeta$ belongs to $V$, there exist $\widehat{P}>0$ such that

$$\sup_{0\le\zeta\le T}\|\mathcal{K}_{\varrho}(\zeta)\|\le\widehat{P},$$
$$\sup_{0\le\zeta\le T}\|\mathcal{Q}_{\varrho}(\zeta)\|\le\widehat{P},$$
$$\sup_{0\le\zeta\le T}\|\mathcal{W}_{\varrho}(\zeta)\|\le\widehat{P}.$$

**(H₂)** The functions $g(\zeta,s,.),h(\zeta,s,.):\mathcal{X}\longrightarrow\mathcal{X}$ are continuous for all $(\zeta,s)\in\Delta$ and for all $z\in\mathcal{X}$ the function $g(.,.,z),h(.,.,z):\Delta\longrightarrow\mathcal{X}$ are strongly measurable.

**(H₃)** The multivalued map $G:V\times\mathcal{X}\times\mathcal{X}\to\mathcal{N}_{b,cl,cp}(L(K,\mathcal{X}))$ is an $L^2-$ caratheordy function such that for all $\zeta\in V$, the function $G(\zeta,.,.):\mathcal{X}\times\mathcal{X}\to\mathcal{N}_{b,cl,cp}(L(K,\mathcal{X}))$ is u.s.c and for all $(s,z)\in\times\mathcal{X}\times\mathcal{X}$ the set

$$S_{G,z}=\left\{g\in L^2(J,L(K,\mathcal{X})):g(\zeta)\text{ in }G\left(\zeta,z(\zeta),\int_0^T f(\zeta,\nu,z(\nu))d\nu\right)\text{ for a.e }\zeta\in V\right\}$$

is nonempty.

**(H₄)** There exists a function $L_{g,p}:\mathcal{X}\to R^+$ such that.

$$\sup\left\{E\|g\|^2:g(\zeta)\in G\left(\zeta,z(\zeta),\int_0^T f(\zeta,\nu,z(\nu))d\nu\right)\right\}\le L_{g,p}(\zeta),$$

for almost every $\zeta\in V$.

**(H₅)** The function $\nu\to(\zeta-\nu)^{T-1}L_{g,p}(\zeta)\in L^1(V,\mathbb{R}^+)$ such that there exist $\varphi>0$ such that

$$\liminf_{p\to\infty}\frac{\int_0^{\zeta}\nu^{\mu-1}L_{g,p}(\nu)d\nu}{p}=\varphi<+\infty.$$

**(H₆)** If $g:C([0,T],\mathcal{X})\to\mathcal{X}$ is continues then there exists some constant $M_g$ such that $E\|g(x)\|^2\le\|x\|^2$.

Now, we can show that the system (1) is controllable approximately on the given interval. That is there exist a mild solution $z\in\mathcal{C}$ satisfies the requirements of approximate controllability, where

$$\begin{aligned} z(\zeta)&=\mathcal{K}_{\varrho}(\zeta)z_0+\mathcal{Q}_{\varrho}(\zeta)z_1\\ &\quad+\int_0^{\zeta}\mathcal{W}_{\varrho}(\zeta-\nu)g(\nu)dW(\nu)\\ &\quad+\int_0^{\zeta}\mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}x(\nu)d\nu,\ g\in S_{G,z},\end{aligned}$$

$$x(\zeta)=\mathbb{B}^{*}\mathcal{W}_{\varrho}^{*}(\zeta-\nu)\mathscr{R}(\hbar,\Gamma_0^T)q(z(\cdot)).$$

In the above

$$\begin{aligned} q(z(\cdot))&=z_T-\mathcal{K}_{\varrho}(T)z_0-\mathcal{Q}_{\varrho}(T)z_1\\ &\quad-\int_0^T\mathcal{W}_{\varrho}(\zeta-\nu)g(\nu)dW(\nu).\end{aligned}$$

**Theorem 3.** *On considering the hypothesis* **(H₀)** − **(H₆)** *are fulfilled then* (1) *contains at least one mild solution on $V$ if*

$$4\widehat{P}^2\left[1+\frac{(\widehat{P}P_{\mathbb{B}})^4}{\hbar}\right]\varphi<1$$

*with* $P_{\mathbb{B}} = \|\mathbb{B}\|$.

***Proof.*** The main aim of this theorem is to nd conditions for solvability of system (1) to be resolvable for $\hbar > 0$. Now we prove that the mapping $\Phi$ from $\mathcal{C}$ into $2^{\mathcal{C}}$ given by

$$\Phi(z) = \Big\{ z \in \mathcal{C}.\ m(\zeta) = \mathcal{K}_\varrho(\zeta)z_0 + \mathcal{Q}_\varrho(\zeta)z_1 + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)g(\nu)dW(\nu) + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x(\nu)d\nu,\ g \in S_{G,z} \Big\},$$

has a fixed point.

**Step 1:** For all $\hbar > 0$, $\Phi(z)$ is convex for all $z$ belongs to $\mathcal{C}$. Let $m_1, m_2 \in \mathcal{C}$, then there exists $g_1, g_2$ belongs to $S_{G,z}$ such that $\zeta$ belongs to $V$, we obtain

$$\begin{aligned} m_i(\zeta) = {} & \mathcal{K}_\varrho(\zeta)z_0 + \mathcal{Q}_\varrho(\zeta)z_1 \\ & + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)g_i(\nu)dW(\nu) \\ & + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_\varrho^*(\zeta-\nu)\mathscr{R}(\hbar,\Gamma_0^T) \\ & \times \Big[ z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 \\ & - \int_0^T \mathcal{W}_\varrho(T-\tau)g_i(\tau)dW(\tau)\Big](\nu)d\nu,\ i = 1,2. \end{aligned}$$

Let $\kappa \in [0,1]$, then for all $\zeta$ belongs to $V$, now we have

$$\begin{aligned} & (\kappa m_1 + (1-\kappa)m_2)(\zeta) = \mathcal{K}_\varrho(\zeta)z_0 \\ & + \mathcal{Q}_\varrho(\zeta)z_1 + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)[\kappa g_1(\nu) + (1-\kappa)g_2(\nu)]d\nu \\ & + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_\varrho^*(\zeta-\nu)\mathscr{R}(\hbar,\Gamma_0^T) \\ & \times \Big[ z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 - \int_0^T \mathcal{W}_\varrho(T-\tau)[\kappa g_1(\tau) \\ & + (1-\kappa)g_2(\tau)]dW(\tau)\Big](\nu)d\nu. \end{aligned}$$

Since $S_{G,z}$ is convex, $\kappa m_1 + (1-\kappa)m_2$ belongs to $S_{G,z}$. Hence $\kappa m_1 + (1-\kappa)m_2$ belongs to $\Phi(z)$.

**Step 2:** Assume that

$$\mathcal{B}_p = \big\{ z \in \mathcal{C} | \|z\|_{\mathcal{C}} \le p \big\},\ \text{for}\, p > 0.$$

Clearly $\mathcal{B}_p$ is convex, closed and bounded subset of $\mathcal{C}$. For $\hbar > 0$, our assumption is there exist $p > 0$ such that

$$\Phi(\mathcal{B}_p) \subset \mathcal{B}_p.$$

If not, then for all $p > 0$, there exist $z^p$ belongs to $\mathcal{B}_p$, but $\Phi(z^p) \notin \mathcal{B}_p$, i.e,

$$\|\Phi(z^p)\|_{\mathcal{C}} = \sup\big\{ \|m^p\|_{\mathcal{C}} : m^p \in \Phi(z^p) \big\} > p,$$

and

$$\begin{aligned} m^p(\zeta) = {} & \mathcal{K}_\varrho(\zeta)z_0 + \mathcal{Q}_\varrho(\zeta)z_1 \\ & + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)g^p(\nu)dW(\nu) \\ & + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_\varrho^*(\zeta-\nu)\mathscr{R}(\hbar,\Gamma_0^T) \\ & \times \Big[ z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 \\ & - \int_0^T \mathcal{W}_\varrho(T-\tau)g^p(\tau)dW(\tau)\Big](\nu)d\nu, \end{aligned}$$

for some $g^p$ belongs to $S_{G,z^p}$.

By referring $(\mathbf{H_3})$, we get

$$\begin{aligned} E\|x^p(\zeta)\|^2 = {} & E\Big\|\mathbb{B}^*\mathcal{W}_\varrho^*(T-\tau)\mathscr{R}(\hbar,\Gamma_0^T) \\ & \Big[ z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 \\ & - \int_0^T \mathcal{W}_\varrho(T-\tau)g^p(\tau)dW(\tau)\Big]\Big\|^2 \\ \le {} & E\|\mathbb{B}^*\|^2 E\|\mathcal{W}_\varrho^*(T-\tau)\|^2 E\|\mathscr{R}(\hbar,\Gamma_0^T)\|^2 \\ & \times E\Big\|\Big[ z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 \\ & - \int_0^T \mathcal{W}_\varrho(T-\tau)g^p(\tau)dW(\tau)\Big]\Big\|^2 \\ \le {} & \widehat{P}^2 P_{\mathbb{B}}^2 \frac{1}{\hbar}\Big[ 4E\|z_T\|^2 \\ & + 4E\|\mathcal{K}_\varrho(T)z_0\|^2 + 4E\|\mathcal{Q}_\varrho(T)z_1\|^2 \\ & + 4L_g \int_0^T E\|\mathcal{W}_\varrho(T-\tau)g^p(\tau)\|^2 d\tau \Big] \\ \le {} & \frac{\widehat{P}^2 P_{\mathbb{B}}^2}{\hbar}\Big[ 4E\|z_T\|^2 + 4\widehat{P}^2 E\|z_0\|^2 + 4\widehat{P}^2 E\|z_1\|^2 \\ & + 4\widehat{P}^2 L_g \int_0^T \|g^p(\tau)\|^2 d\tau \Big]. \end{aligned}$$

Now for $\hbar > 0$,

$$\begin{aligned} p < {} & E\|(\Phi z^p)(\zeta)\|^2 \le 4E\|\mathcal{K}_\varrho(\zeta)z_0\|^2 \\ & + 4E\|\mathcal{Q}_\varrho(\zeta)z_1\|^2 \\ & + 4L_g \int_0^\zeta E\|\mathcal{W}_\varrho(\zeta-\nu)g^p(\nu)\|^2 d\nu \\ & + 4\int_0^\zeta E\|\mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x^p(\nu)\|^2 d \\ \le {} & 4\widehat{P}^2 E\|z_0\|^2 + 4\widehat{P}^2 E\|z_1\|^2 \\ & + 4\widehat{P}^2 L_g \int_0^\zeta E\|g^p(\nu)\|^2 d\nu \\ & + 4\widehat{P}^2 \int_0^\zeta E\|\mathbb{B}x^p(\nu)\|^2 d\nu \end{aligned}$$

$$
\begin{aligned}
&\leq 4\widehat{P}^2 E\|z_0\|^2 + 4\widehat{P}^2 E\|z_1\|^2\\
&\quad + 4\widehat{P}^2 L_g \int_0^\zeta L_{g,p}(\nu)d\nu\\
&\quad + 4\widehat{P}^2 P_{\mathbb{B}}^2 \int_0^\zeta \bigg( \frac{\widehat{P}^2 P_{\mathbb{B}}^2}{\hbar} \bigg[ E\|z_T\|^2 + \widehat{P}^2 E\|z_0\|^2\\
&\quad + \widehat{P}^2 E\|z_1\|^2 + \widehat{P}^2 L_g \int_0^T L_{g,p}(\tau)d\tau \bigg]\bigg) d\nu\\
&\leq 4\widehat{P}^2 E\|z_0\|^2 + 4\widehat{P}^2 E\|z_1\|^2\\
&\quad + 4\widehat{P}^2 L_g \int_0^\zeta L_{g,p}(\nu)d\nu\\
&\quad + 4\frac{(\widehat{P}^2 P_{\mathbb{B}}^2)^2}{\hbar} \bigg[ 4E\|z_T\|^2 + 4\widehat{P}^2 E\|z_0\|^2\\
&\quad + 4\widehat{P}^2 E\|z_1\|^2 + 4\widehat{P}^2 L_g \int_0^T L_{g,p}(\tau)d\tau \bigg].
\end{aligned}
$$

Dividing the above equation by $p$ and as $p \to \infty$ we obtain

$$4\widehat{P}^2 \bigg[ 1 + 4\frac{(\widehat{P}^2 P_{\mathbb{B}}^2)^2}{\hbar} \bigg] \varphi \geq 1,$$

which contradicts to our assumption.

**Step 3:** We check that $\{\Phi(z) : z \in \mathcal{B}_p\}$ is equicontinuous.

For all $m$ belongs to $\Phi(z)$ and $z$ belongs to $\mathcal{B}_p$, there exist $g \in S_{G,z}$ such that

$$
\begin{aligned}
m(\zeta) &= \mathcal{K}_\varrho(\zeta)z_0 + \mathcal{Q}_\varrho(\zeta)z_1 + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)g(\nu)d\nu\\
&\quad + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x(\nu)d\nu.
\end{aligned}
$$

Suppose $0 \leq \zeta_1 < \zeta_2 \leq T$. In addition,

$$
\begin{aligned}
&E\|m(\zeta_2) - m(\zeta_1)\|^2 = E\|\mathcal{K}_\varrho(\zeta_2)z_0 + \mathcal{Q}_\varrho(\zeta_2)z_1\\
&\quad + \int_0^{\zeta_2} \mathcal{W}_\varrho(\zeta_2-\nu)g(\nu)dW(\nu)\\
&\quad + \int_0^{\zeta_2} \mathcal{W}_\varrho(\zeta_2-\nu)\mathbb{B}x(\nu)d\nu - \mathcal{K}_\varrho(\zeta_1)z_0 - \mathcal{Q}_\varrho(\zeta_1)z_1\\
&\quad - \int_0^{\zeta_1} \mathcal{W}_\varrho(\zeta_1-\nu)g(\nu)dW(\nu)\\
&\quad - \int_0^{\zeta_1} \mathcal{W}_\varrho(\zeta_1-\nu)\mathbb{B}x(\nu)d\nu\|^2\\
&\leq 6E\|[\mathcal{K}_\varrho(\zeta_2) - \mathcal{K}_\varrho(\zeta_1)]z_0\|^2\\
&\quad + 6E\|[\mathcal{Q}_\varrho(\zeta_2) - \mathcal{Q}_\varrho(\zeta_1)]z_1\|^2\\
&\quad + 6L_g^2 \int_{\zeta_1}^{\zeta_2} E\|\mathcal{W}_\varrho(\zeta_2-\nu)g(\nu)\|^2 d\nu\\
&\quad + L_g^2 \int_0^{\zeta_1} E\|[\mathcal{W}_\varrho(\zeta_2-\nu) - \mathcal{W}_\varrho(\zeta_1-\nu)]g(\nu)\|^2 d\nu\\
&\quad + \int_{\zeta_1}^{\zeta_2} E\|\mathcal{W}_\varrho(\zeta_2-\nu)\mathbb{B}x(\nu)\|^2 d\nu\\
&\quad + \int_0^{\zeta_1} E\|[\mathcal{W}_\varrho(\zeta_2-\nu) - \mathcal{W}_\varrho(\zeta_1-\nu)]\mathbb{B}x(\nu)\|^2 d\nu\\
&\leq 6E\|\mathcal{K}_\varrho(\zeta_2) - \mathcal{K}_\varrho(\zeta_1)\|^2 E\|z_0\|^2\\
&\quad + 6E\|\mathcal{Q}_\varrho(\zeta_2) - \mathcal{Q}_\varrho(\zeta_1)\|^2 E\|z_1\|^2\\
&\quad + 6L_g^2\widehat{P}^2 \int_{\zeta_1}^{\zeta_2} L_{g,p}(\nu)d\nu\\
&\quad + 6L_g^2 \int_0^{\zeta_1} E\|\mathcal{W}_\varrho(\zeta_2-\nu) - \mathcal{W}_\varrho(\zeta_1-\nu)\|^2 L_{g,p}(\nu)d\nu\\
&\quad + 6\widehat{P}^2 P_{\mathbb{B}}^2 \int_{\zeta_1}^{\zeta_2} E\|x(\nu)\|^2 d\nu\\
&\quad + 6P_{\mathbb{B}}^2 \int_0^{\zeta_1} E\|\mathcal{W}_\varrho(\zeta_2-\nu) - \mathcal{W}_\varrho(\zeta_1-\nu)\|^2\\
&\quad \times E\|x(\nu)\|^2 d\nu.
\end{aligned}
$$

In the above aforementioned inequality the right hand side $\to 0$ as $\zeta_2 \to \zeta_1$ by using the continuity of functions $\zeta \to \|\mathcal{K}_\varrho(\zeta)\|, \zeta \to \|\mathcal{Q}_\varrho(\zeta)\|$ and $\zeta \to \|\mathcal{W}_\varrho(\zeta)\|$. Therefore, $\Phi(\mathcal{B}_p)$ is equicontinuous.

**Step 4:** We show that $\mathcal{H}(\zeta) = \{m(\zeta) : m \in \Phi(B_p)\}$ is relatively compact belongs in $\mathcal{X}$. For $\zeta = 0$, result is trivial, hence $\mathcal{H}(\zeta) = \{z_0\}$. For some fixed $\zeta$ belongs to $V$. Assume that $0 < \epsilon < \zeta$, $z$ belongs to $\mathcal{B}_p$ and introduce the operator $m^\epsilon$ by

$$
\begin{aligned}
m^\epsilon(\zeta) &= \mathcal{K}_\varrho(\zeta)z_0 + \mathcal{Q}_\varrho(\zeta)z_1\\
&\quad + \int_0^{\zeta-\epsilon} \mathcal{W}_\varrho(\zeta-\nu)g(\nu)dW(\nu)\\
&\quad + \int_0^{\zeta-\epsilon} \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x(\nu)d\nu.
\end{aligned}
$$

Hence $\mathcal{Q}(\epsilon)$, $\epsilon > 0$ is a compact operator, then $\mathcal{H}^\epsilon(\zeta) = \{m^\epsilon(\zeta) : m^\epsilon \in \Phi(B_p)\}$ is relatively compact belongs to $\mathcal{X}$, $0 < \epsilon < \zeta$. Further, for all $z$ belongs to $\mathcal{B}_p$, we get

$$
\begin{aligned}
E\|m(\zeta) - m^\epsilon(\zeta)\|^2 &= E\|\int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)g(\nu)dW(\nu)\\
&\quad + \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x(\nu)d\nu\\
&\quad - \int_0^{\zeta-\epsilon} \mathcal{W}_\varrho(\zeta-\nu)g(\nu)dW(\nu)\\
&\quad - \int_0^{\zeta-\epsilon} \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x(\nu)d\nu\|^2,\\
&\leq 2\int_{\zeta-\epsilon}^\zeta L_g^2 E\|\mathcal{W}_\varrho(\zeta-\nu)g(\nu)\|^2 d\nu\\
&\quad + 2\int_{\zeta-\epsilon}^\zeta E\|\mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}x(\nu)\|^2 d\nu\\
&\leq 2L_g^2\widehat{P}^2 \int_{\zeta-\epsilon}^\zeta L_{g,p}(\nu)d\nu
\end{aligned}
$$

$$+ 2\widehat{P}^2 P_{\mathbb{B}}^2 \int_{\zeta-\epsilon}^{\zeta} E\|x(\nu)\|^2 d\nu.$$

We see that $E\|m(\zeta) - m^{\epsilon}(\zeta)\|^2 \to 0$ as $\epsilon \to 0^+$. Thus there exist relatively compact set and it is arbitrarily close to $\mathcal{H}(\zeta) = \{m(\zeta) : m \in \Phi(B_r)\}$ and the set $\mathcal{H}(\zeta)$ is relatively compact in $\mathcal{X}$ for all $\zeta \in [0, T]$. At $\zeta = 0$ it is compact, hence $\mathcal{H}(\zeta)$ is relatively compact belongs to $\mathcal{X}$ for all $\zeta \in [0, T]$.

**Step 5:** $\Phi$ has a closed graph.

Consider $z^n \to z^*$ and $m^n \to m^*$ as $n \to \infty$. We will prove $m^* \in \Phi(z^*)$. Since $m^n \in \Phi(z^n)$, such that $g^n$ belongs to $S_{G,z^n}$ such that

$$\begin{aligned}
m^n(\zeta) &= \mathcal{K}_{\varrho}(\zeta)z_0 + \mathcal{Q}_{\varrho}(\zeta)z_1 \\
&+ \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)g^n(\nu)dW(\nu) \\
&+ \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[z_T - \mathcal{K}_{\varrho}(T)z_0 - \mathcal{Q}_{\varrho}(T)z_1 \\
&- \int_0^T \mathcal{W}_{\varrho}(T-\tau) \times g^n(\tau)dW(\tau)\Big](\nu)d\nu.
\end{aligned}$$

We need to show there exist $g^*$ belongs to $S_{G,z^*}$ such that for all $\zeta$ belongs to $V$,

$$\begin{aligned}
m^*(\zeta) &= \mathcal{K}_{\varrho}(\zeta)z_0 + \mathcal{Q}_{\varrho}(\zeta)z_1 \\
&+ \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)g^*(\nu)dW(\nu) \\
&+ \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[z_T - \mathcal{K}_{\varrho}(T)z_0 - \mathcal{Q}_{\varrho}(T)z_1 \\
&- \int_0^T \mathcal{W}_{\varrho}(T-\tau)g^*(\tau)dW(\tau)\Big](\nu)d\nu.
\end{aligned}$$

Clearly,

$$\begin{aligned}
E\Big\|&\Big(m^n(\zeta) - \mathcal{K}_{\varrho}(\zeta)z_0 - \mathcal{Q}_{\varrho}(\zeta)z_1 \\
&- \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[z_T - \mathcal{K}_{\varrho}(T)z_0 - \mathcal{Q}_{\varrho}(T)z_1 \\
&- \int_0^T \mathcal{W}_{\varrho}(T-\tau)g^n(\tau)dW(\tau)\Big](\nu)d\nu\Big) \\
&- \Big(m^*(\zeta) - \mathcal{K}_{\varrho}(\zeta)z_0 - \mathcal{Q}_{\varrho}(\zeta)z_1 \\
&- \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[z_T - \mathcal{K}_{\varrho}(T)z_0 - \mathcal{Q}_{\varrho}(T)z_1 \\
&- \int_0^T \mathcal{W}_{\varrho}(T-\tau)g^*(\tau)dW(\tau)\Big](\nu)d\nu\Big)\Big\|^2 \to 0,
\end{aligned}$$

as $n \to \infty$. Assume that $\mathcal{T} : L^1(V, \mathcal{X}) \to \mathcal{C}$,

$$\begin{aligned}
(\mathcal{T}g)(\zeta) &= \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)g(\nu)dW(\nu) \\
&+ \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[\int_0^T \mathcal{W}_{\varrho}(T-\tau)g(\tau)dW(\tau)\Big](\nu)d\nu.
\end{aligned}$$

We can conclude that the operator $\mathcal{T} \circ S_{G,z}$ is a closed graph by using Lemma 1. Then, in view of $\mathcal{T}$ we can see that

$$\begin{aligned}
&\Big(m^n(\zeta) - \mathcal{K}_{\varrho}(\zeta)z_0 - \mathcal{Q}_{\varrho}(\zeta)z_1 \\
&- \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[z_T - \mathcal{K}_{\varrho}(T)z_0 - \mathcal{Q}_{\varrho}(T)z_1 \\
&- \int_0^T \mathcal{W}_{\varrho}(T-\tau)g^*(\tau)dW(\tau)\Big](\nu)d\nu\Big) \in \mathcal{T}(S_{G,z^n}).
\end{aligned}$$

Since $g^n \to g^*$, as $n$ tends to zero, it follows that for all $\zeta$ belongs to $V$, we obtain

$$\begin{aligned}
&\Big(m^*(\zeta) - \mathcal{K}_{\varrho}(\zeta)z_0 - \mathcal{Q}_{\varrho}(\zeta)z_1 \\
&- \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_{\varrho}^*(\zeta-\nu)\mathscr{R}(\hbar, \Gamma_0^T) \\
&\times \Big[z_T - \mathcal{K}_{\varrho}(T)z_0 - \mathcal{Q}_{\varrho}(T)z_1 \\
&- \int_0^T \mathcal{W}_{\varrho}(T-\tau)g^*(\tau)dW(\tau)\Big](\nu)d\nu\Big) \\
&\in \mathcal{T}(S_{G,z^*}).
\end{aligned}$$

As a result, $\Phi$ is a closed graph.

Thus $\Phi$ is multivalued map which is completely continuous and hence as a result of the previous steps and Ascoli-Arzela theorem it is easily see that $\Phi$ is u.s.c. As a result, which has a fixed point $z(\zeta)$ on $\mathcal{B}_p$ and by referring to Lemma 2, which is the mild solution of (1). □

**Definition 12.** *The fractional integro-differential inclusions* (1) *is called approximately controllable on* $[0,T]$ *provide that* $\overline{\mathscr{R}(T,z_0)} = \mathcal{X}$, *where* $\mathscr{R}(T,z_0) = \{z_T(z_0;z) : z(\zeta)$ *in* $L^2(V,\mathcal{X})\}$ *is a mild solution of* (1).

The following assumptions are required for proving the main results.

H7 The function $G : V \times \mathcal{X} \times \mathcal{X} \to \mathcal{N}_{b,cl,cp}(L(K,\mathcal{X}))$ is uniformly bounded for all $\zeta \in V$ and $z \in \mathcal{X}$

**Theorem 4.** *Suppose* $(\mathbf{H_0}) - (\mathbf{H_7})$ *are fulfilled. Further there exist* $\mathscr{I}$ *belongs to* $L^1(V,[0,+\infty))$ *such that* $\sup_{z\in\mathcal{X}} \|G(\zeta,z(\zeta),\int_0^T f(\zeta,\nu,z(\nu))d\nu)\| \leq \mathscr{I}(\zeta)$ *for a.e.* $\zeta$ *belongs to* $V$. *In addition,* (1) *is approximately controllable.*

***Proof.*** Let $z^\alpha(.) \in B_p$ be a fixed point of the operator $\Phi$, by Theorem 3.1 any fixed point of $\Phi$ is a mild solution of 1. This means that there is $z^\alpha \in \Phi(z^\alpha)$, i.e. by the Fubini theorem there is $g^\alpha \in S_{G,z^*}$ such that for all $\zeta \in V$.

$$\begin{aligned} z^\alpha(\zeta) &= \mathcal{K}_\varrho(\zeta)z_0 - \mathcal{Q}_\varrho(\zeta)z_1 \\ &\quad - \int_0^\zeta \mathcal{W}_\varrho(\zeta-\nu)\mathbb{B}\mathbb{B}^*\mathcal{W}_\varrho^*(\zeta-\nu)\mathscr{R}(\hbar,\Gamma_0^T) \\ &\quad \times \Big[z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 \\ &\quad - \int_0^T \mathcal{W}_\varrho(T-\tau)g^*(\tau)dW(\tau)\Big](\nu)d\nu\Big). \end{aligned}$$

Define

$$\begin{aligned} P(g^\alpha) &= z_\alpha - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 \\ &\quad - \int_0^T \mathcal{W}_\varrho(T-\nu)g^\alpha(\nu)dW(\nu), \end{aligned}$$

for some $g^\alpha \in S_{G,z^*}$.
Noting that $I - \Gamma_0^T\mathscr{R}(\hbar,\Gamma_0^T) = \alpha\mathscr{R}(\hbar,\Gamma_0^T)$, i.e we get $z^\alpha(b) = z_T - \alpha\mathscr{R}(\hbar,\Gamma_0^T)P(g^\alpha)$.
By assumption (H7),

$$\begin{aligned} E\|\int_0^T g^\alpha(\nu)dW(\nu)\|^2 &\leq L_g^2\int_0^T E\|g^\alpha(\nu)\|^2 d\nu \\ &\leq L_g^2 l_r(\zeta)T \leq L_g^2 l_r T. \end{aligned}$$

Subsequently, the sequence $\{g^\alpha\}$ is uniformly bounded in $L^2(V,\mathcal{X})$. Hence we can find a subsequence of $\{g^\alpha\}$ which is still denoted by $\{g^\alpha\}$ that converges weakly to $g \in L^2(V,\mathcal{X})$
Denoting $h = z_T - \mathcal{K}_\varrho(T)z_0 - \mathcal{Q}_\varrho(T)z_1 - \int_0^T \mathcal{W}_\varrho(T-\nu)g^*(\nu)dW(\nu)$.

We see that

$$\begin{aligned} &E\|P(g^\alpha) - h\|^2 \\ &= E\left\|\int_0^T \mathcal{W}_\varrho(T-\nu)[g^*(\nu) - g(\nu)]dW(\nu)\right\|^2 \\ &\leq L_g^2\int_0^T E\|\mathcal{W}_\varrho(\zeta-\nu)[g^\alpha(\nu) - g(\nu)]\|^2 d\nu \\ &\leq \sup_{0\leq t\leq T}\int_0^\zeta E\|\mathcal{W}_\varrho(\zeta-\nu)[g^\alpha(\nu) - g(\nu)]\|^2 d\nu. \end{aligned}$$

Using Ascoli-Arzela theorem, we can see that the linear operator, $\int_0^{(\cdot)}(.,-\nu)^{\mu-1}\mathcal{W}_\varrho(.-\nu)g(\nu)d\nu : L^2(V,\mathcal{X}) \to C(V,\mathcal{X})$ is compact. Therefore, we get $E\|P(g^\alpha) - h\|^2 \to 0$ as $\alpha \to 0$.
Hence,

$$\begin{aligned} &E\|z^\alpha(b) - z_T\|^2 = E\|\mathscr{R}(\alpha,\Gamma_0^T)P(g^\alpha)\|^2 \\ &\leq 2E\|\mathscr{R}(\alpha,\Gamma_0^T)(h)\|^2 + 2E\|\mathscr{R}(\alpha,\Gamma_0^T)(P(g^\alpha) - h)\|^2 \\ &\leq 2E\|\mathscr{R}(\alpha,\Gamma_0^T)(h)\|^2 + \|(P(g^\alpha) - h)\|^2 \to 0 \\ &\text{as } \alpha \to 0^+. \end{aligned}$$

This proves the approximate controllability of system (1) □

## 4. Nonlocal conditions

The idea of nonlocal initial conditions of the differential systems were inspired by physical concerns. The result pertaining to approximate controllability is extended to Hilbert space in [37]. In contrary, to local conditions Byszewski et. al [38] interrogated the abstract Cauchy with nonlocal conditions in Banach spaces. For more details on nonlocal conditions refer [13, 14, 39, 40]. Consider the fractional systems of order $\varrho \in (1,2)$ with nonlocal conditions:

$$\begin{cases} {}^C D_\zeta^\varrho z(\zeta) \in Az(\zeta) + G(\zeta, z(\zeta), \\ \int_0^T f(\zeta,\nu,z(\nu))d\nu)\frac{dW(\nu)}{d\nu} \\ +\mathbb{B}x(\zeta),\ \zeta \in V = [0,T], \\ z(0) = z_0 + w_1(z),\ z'(0) = z_1 + w_2(z). \end{cases} \tag{2}$$

In the above, $w_1, w_2$ is appropriate functions and it is mapping from $V \times \mathcal{X}$ into $\mathcal{X}$ which fulfill the subsequent condition:

$(\mathbf{H_8})$ The completely continuous functions $w_1, w_2$ belongs to $C(V,\mathcal{X})$ and there exists $c, d, e, k > 0$ such that

$$\begin{aligned} &E\|w_1(z)\|^2 \leq cE\|z\|^2 + d, \\ &E\|w_2(z)\|^2 \leq eE\|z\|^2 + k, \ \text{ for all } z \in \mathbb{Y}. \end{aligned}$$

**Definition 13.** *A function $z$ belongs to $\mathcal{C}$ is called a mild solution of* (2) *provide that*

$$z(\zeta) = \mathcal{K}_{\varrho}(\zeta)[z_0 - w_1(z)] + \mathcal{Q}_{\varrho}(\zeta)[z_1 - w_2(z)] + \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)g(\nu)dW(\nu) + \int_0^{\zeta} \mathcal{W}_{\varrho}(\zeta-\nu)\mathbb{B}x(\nu)d\nu.$$

**Theorem 5.** *Provide that* $(\mathbf{H_0})$*-*$(\mathbf{H_8})$ *are fulfilled and if*

$$\widehat{P}^2\left[1+\frac{(\widehat{P^2}P_{\mathbb{B}}^2)^2}{\hbar}\right]\varphi + \widehat{P}^2(c+e) \times \left[1+\frac{(\widehat{P}^2P_{\mathbb{B}}^2)^2}{\hbar}\right]\varphi < 1$$

*where $P_{\mathbb{B}} = \|\mathbb{B}\|$ then* (2) *has at least one mild solution on $[0,T]$ and is approximately controllable.*

***Proof.*** Since the theoretical proof of the theorem much similar to that of Theorem 3, we neglect the proof. □

## 5. Application

To illustrate our finding we consider the following fractional integro-differential system

$$\begin{cases} \frac{\partial^{\varrho}}{\partial\zeta^{r}} z(\zeta,s) \in \frac{\partial^2}{\partial s^2} z(\zeta,s) + \mathscr{J}\Big(\zeta, z(\zeta,s), \\ \int_0^{\zeta} e(\zeta,\nu,z(\nu,s))d\nu\Big)\frac{dW(\nu)}{d\nu} + w(\zeta,s), \\ z(\zeta,0) = z(\zeta,1) = 0,\ \zeta \in V, \\ z(0,s) = z_0(s), \\ z'(0,s) = z_1(s),\ s \in [0,\pi]. \end{cases} \tag{3}$$

In the above the order of fractional system $\varrho = \frac{3}{2}$, $\mathscr{J} : [0,1]\times\mathcal{X}\times\mathcal{X} \to 2^{\mathcal{X}} \setminus \{\emptyset\}$ and the continuous function $e$ mapping from $[0,1]\times[0,1]\times\mathcal{X}$ into $\mathcal{X}$. Let us consider $\mathcal{X} = \mathcal{H} = L^2([0,\pi])$ and let $W(\zeta)$ be a standard cylindrical Wiener process in $\mathcal{X}$ defined on a stochastic space $\big(\Omega, \Im, \{\Im_{\zeta}\}_{\zeta\geqslant 0}, \mathbb{P}\big)$, $D^{\varrho}z = \frac{\partial^{\varrho}z}{\partial\zeta^{\varrho}}$ is the Caputo fractional derivative of order $1 < \beta < 2$.

$$D(A) = \{z \in \mathcal{X} : z,\ z' \text{ are absolutely continuous}, z'' \in \mathcal{X},\ z(0) = z(\pi) = 0\}.$$

Now there exist a sequence $\{e_j\}_{j\geqslant 1}$ of eigenvectors of $\mathcal{A}$ such that. $\{e_j\}_{j\geqslant 0}$ is a complete orthonormal and $e_j(y) = \sqrt{\frac{2}{\pi}}\, sin\, y$. Furthermore $\mathcal{A}$ is dense in $\mathcal{X}$ and $A$ is the infinitesimal generator of a resolvent family $\{\mathcal{W}(\zeta),\ \zeta \geq 0\}$ belongs to $\mathcal{X}$, according to [14].

Put $z(\zeta) = z(\zeta,\cdot)$, $\zeta$ belongs to $[0,1]$ and $x(\zeta) = \omega(\zeta,\cdot)$. The linear bounded operator $\mathbb{B} : \mathcal{H} \to \mathcal{X}$ defined by $\mathbb{B}x(\zeta)(s) = w(\zeta,s)$. Then

$$e(\zeta,\nu,z)(s) = f(\zeta,\nu,z(s)),$$

and

$$G(\zeta,z,\wp_1)(s) = \mathscr{J}(\zeta,z(s),\wp_1(s))$$

for $\zeta,\nu$ belongs to $[0,1]$, $z,\wp_1$ belongs to $\mathcal{X}$ and $s$ belongs to $[0,\pi]$. The above mentioned fractional partial differential system (3) can be consider as the exact representation of the problem (1) with the functions our preferred choices. Then it can be easily viewed that all the requirements of the Theorem 3 satisfied and hence we can ensure the approximate controllability of (3) on $[0,T]$.

## 6. Conclusion

The findings of this research analyze the outcome results of approximate controllability of Stochastic fractional integro-differential equation considered in Banach space. Bohnenblust-Karlin's fixed point technique is used as the key factor to establish the required conditions for our chosen fractional system (1) to be controllable approximately. The above mentioned procedure to establish the approximate controllability is extended to fractional nonlocal system. In future the present work can be extended by analysing the controllability results of stochastic integro fractional differential inclusion system with impulsive conditions.

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