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RESEARCH ARTICLE

M-truncated soliton solutions of the fractional (4+1)-dimensional Fokas equation

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ABSTRACT

This article aims to examine M-truncated soliton solutions of the fractional $(4+1)$-dimensional Fokas equation (FE), which is a generalization of the Kadomtsev-Petviashvili (KP) and Davey-Stewartson (DS) equations. The fractional $(4+1)$-dimensional Fokas equation with the M-truncated derivative is also studied first time in this study. The generalized projective Riccati equations method (GPREM) is successfully implemented. In the application of the presented method, a suitable fractional wave transformation is chosen to convert the proposed model into a nonlinear ordinary differential equation. Then, a linear equation system is acquired utilizing the GPREM, the system is solved, and the suitable solution sets are obtained. Dark and singular soliton solutions are successfully derived. Under the selection of appropriate values of the parameters, 2D, 3D, and contour plots are also displayed for some solutions.

1. Introduction

A variety of real-life problems have been modeled using the nonlinear partial differential equations (NLPDEs) with integer or fractional order. Moreover, NLPDEs with integer or fractional order have main roles in area of quantum mechanics, fluid dynamics, nonlinear optics, plasma physics as well as biology, chemistry, and finance. Because of its wide application, investigation of the analytical and soliton solutions of the NLPDEs with integer or fractional order has been very popular among authors over the past few decades. Numerous techniques consisting of the analytical and numerical methods have been improved to gain the soliton and analytical solutions of the PDEs such as the combined improved Kudryashov-new extended auxiliary sub equation method [1], the enhanced modified extended tanh-expansion approach [2,3], the sine-Gordon equation approach [4], F-expansion method [5], the tanh–coth function, the modified kudryashov expansion and rational sine–cosine approaches [6], the Riccati equation method [7], the $tan(\Theta/2)$ expansion approach [8], the Jacobi elliptic functions methodology [9], the generalized Bernoulli sub-ODE scheme [10], the extended $(\frac{G'}{G^2})$-expansion scheme [11], Nucci's reduction method [12], the new Kudryashov method [13–15], the sub-equation method based on Riccati equation [16], and the modified Sardar sub-equation method [17].

Fractional differential equations have been utilized the modeling many phenomena in a variety of branches of engineering and science [18,19]. Thus, various and substantial definitions of fractional derivatives types have been enhanced in the literature such as: Grunwald– Letnikov, Riemann–Liouville, Caputo [20], Caputo–Fabrizio [21], Atangana–Baleanu [22,23], the conformable fractional derivative [24], and the M-truncated derivative [11].

The $(4+1)$-dimensional Fokas equation is expressed by the following structure [25]:

$$4\vartheta_{tx} - \vartheta_{xxxy} + \vartheta_{xyyy} + 12\vartheta_x\vartheta_y + 12\vartheta\vartheta_{xy} - 6\vartheta_{zs} = 0, \quad (1)$$

in which $\vartheta(x, y, z, s, t)$. The model was firstly presented by A. S. Fokas [25] and the FE is the extension form of Davey–Stewartson and Kadomtsev–Petviashvili equations to some higher-dimensional nonlinear wave equations [25]. So, the $(4+1)$-dimensional FE is taken into account as a higher dimensional integrable model in mathematical physics. The $(4+1)$-dimensional FE models the finite-amplitude wave packet in fluid dynamics.

Up to now, soliton and analytical solutions of the FE have been investigated by utilizing various significant approaches. In [26], Yinghui He discussed the analytical solutions using the extended F-expansion scheme. Hirota's bilinear methodology was used to examine the FE in [27]. Kim and Sakthival obtained some traveling wave solutions of the (4+1)-dimensional FE by applying the $(\frac{G'}{G})$-expansion scheme in [28]. In [29], the Sardar subequation scheme and new extended hyperbolic function approach were utilized to build the soliton solutions of the (4+1)-dimensional fractional-order FE. Bo and Sheng used the generalized F-expansion method to construct some exact solutions with arbitrary functions in [30]. Wazwaz implemented the simplified Hirota's approach to gain a variety of soliton solutions of the presented model in [31]. In [32], the (4+1)-dimensional FE was studied using the modified simple equation and the extended simplest equation schemes by Al-Amr and El-Ganaini. Baskonus et.al. constructed various soliton solutions by performing sine-Gordon expansion method in [33].

In this paper, we intend to achieve the soliton solutions of the space-time fractional $(4+1)$-dimensional FE involving M-truncated derivative in the form:

$$\begin{aligned} 4D_{M,t}^{\alpha,\gamma}D_{M,x}^{\alpha,\gamma}\vartheta - D_{M,x}^{3\alpha,\gamma}D_{M,y}^{\alpha,\gamma}\vartheta + D_{M,x}^{\alpha,\gamma}D_{M,y}^{3\alpha,\gamma}\vartheta \\ +12D_{M,x}^{\alpha,\gamma}\vartheta D_{M,y}^{\alpha,\gamma}\vartheta + 12\vartheta D_{M,x}^{\alpha,\gamma}D_{M,y}^{\alpha,\gamma}\vartheta \\ -6D_{M,z}^{\alpha,\gamma}D_{M,s}^{\alpha,\gamma}\vartheta = 0. \end{aligned} \tag{2}$$

Herein, $D_{M,x}^{\alpha,\gamma}\vartheta(x,y,z,s,t)$ represents the M-truncated derivative of ϑ with respect to x, $0<\alpha\le 1$. The space-time fractional form including the M-truncated derivative of $(4+1)$-dimensional FE has been examined utilizing the GPREM for the first time in this study.

The remain of this study is arranged as follows: The definition and properties of the M-truncated derivative are expressed in section 2. Mathematical analysis of the presented model is offered in section 3. We also submit the description and enforcement of GPREM in section 4. To observe the physical explanations of the derived results, we present the graphical potraits in section 5. Finally, we give the conclusion in the last section.

2. The M-truncated derivative

Definition 1. *The truncated Mittag-Leffler function [11] is identified as:*

$$_iE_\gamma(c) = \sum_{m=0}^{i} \frac{c^m}{\Gamma(m\gamma+1)},$$

for $\gamma>0$, and $c\in$ C.

Definition 2. *Presume that $\delta:[0,\infty)\to$ R, the M-truncated derivative of δ with order α is defined by [11]*

$$D_M^{\alpha,\gamma}(\delta(x)) = \lim_{\varepsilon\to 0}\frac{\delta(x\ {}_iE_\gamma(\varepsilon x^{-\alpha})) - \delta(x)}{\varepsilon},$$

where $x>0$ and $\alpha\in(0,1)$.

Theorem 1. *Consider if $0<\alpha\le 1, \gamma>0$ and considering $\delta(t)$ and $\theta(t)$ are differentiable of α's order at $x>0$, then*

(1) $D_M^{\alpha,\gamma}(a\delta(x)+b\theta(x)) = aD_M^{\alpha,\gamma}(\delta(x)) + bD_M^{\alpha,\gamma}(\theta(x))$, *for all* $a,b\in$ R,
(2) $D_M^{\alpha,\gamma}(\delta(x)\theta(x)) = \theta(x)D_M^{\alpha,\beta}(\delta(x)) + \delta(x)D_M^{\alpha,\gamma}(\theta(x))$,
(3) $D_M^{\alpha,\gamma}\left(\frac{\delta(x)}{\theta(x)}\right) = \frac{\theta(x)D_M^{\alpha,\gamma}(\delta(x)) - \delta(x)D_M^{\alpha,\gamma}(\theta(x))}{\theta^2(x)}$,
(4) $D_M^{\alpha,\gamma}\delta(x) = \frac{x^{1-\alpha}}{\Gamma(\gamma+1)}\frac{d\delta}{dx}$.

The truncated M-fractional derivative is an extension structure of the conformable fractional derivative.

3. Nonlinear ordinary differential form of the fractional $(4+1)$-dimensional FE

In order to gain the NODE form of Eq. (2), we should firstly determine wave transformation with M-truncated derivative as follows:

$$\begin{aligned} \vartheta(x,y,z,s,t) &= V(\zeta), \\ \zeta &= \frac{\Gamma(1+\gamma)(\rho_1x^\alpha+\rho_2y^\alpha+\rho_3z^\alpha+\rho_4s^\alpha+\rho_5t^\alpha)}{\alpha}. \end{aligned} \tag{3}$$

Herein, $\rho_1,\rho_2,\rho_3,\rho_4$ and ρ_5 are nonzero real numbers. Using the wave transformation in Eq.(3), Eq. (2) transform into the following NODE:

$$\begin{aligned} (\rho_1{\rho_2}^3 - {\rho_1}^3\rho_2)V^{(iv)} + (4\rho_1\rho_5 - 6\rho_3\rho_4)V'' \\ +\ 12\rho_1\rho_2(VV')' = 0. \end{aligned} \tag{4}$$

Integrating Eq.(4) twice with respect to ζ and presuming the integration constants to zero, we achieve the following equation:

$$\begin{aligned} (\rho_1{\rho_2}^3 - {\rho_1}^3\rho_2)V'' + (4\rho_1\rho_5 - 6\rho_3\rho_4)V \\ +12\rho_1\rho_2V^2 = 0. \end{aligned} \tag{5}$$

4. A brief sketch of the GPREM and its application

4.1. Outline of the GPREM

According to the GPREM [14], the solution of the Eq.(5) has the following structure:

$$V(\zeta)=A_0+\sum_{k=1}^{M}\kappa^{k-1}(\zeta)\left[A_k\kappa(\zeta)+B_k\sigma(\zeta)\right], \quad (6)$$

in which A_0, A_k and B_k $(1,2,...,M)$ are real contants to be computed, M is the balance number, and $\kappa(\zeta)$ and $\sigma(\zeta)$ satisfy the following equations:

$$\kappa'(\zeta)=\varepsilon\kappa(\zeta)\sigma(\zeta), \quad (7)$$

$$\sigma'(\zeta)=\lambda+\varepsilon\sigma^2(\zeta)-\chi\kappa(\zeta), \quad \varepsilon=\mp1, \quad (8)$$

in which

$$\sigma^2(\zeta)=-\varepsilon\left(\lambda-2\chi\kappa(\zeta)+\frac{\chi^2+c}{\lambda}\kappa^2(\zeta)\right), \quad (9)$$

$c=\mp1, \lambda$ and χ are nonzero real constants. Assuming as $\lambda=\chi=0$, Eq.(5) has the following solution structure:

$$V(\zeta)=\sum_{k=1}^{M}A_k\sigma^k(\zeta), \quad (10)$$

in which $\sigma(\zeta)$ satisfies the following relation:

$$\sigma'(\zeta)=\sigma^2(\zeta). \quad (11)$$

Utilizing the Eqs.(7) and (8), the following solution functions are gained:

Case 1: If $\varepsilon=-1, c=-1$ and $\lambda>0$, we get,

$$\kappa_1(\zeta)=\frac{\lambda\,\mathrm{sech}(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{sech}(\sqrt{\lambda}\zeta)+1}, \quad \sigma_1(\zeta)=\frac{\sqrt{\lambda}\tanh(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{sech}(\sqrt{\lambda}\zeta)+1}. \quad (12)$$

Case 2: If $\varepsilon=-1, c=1$ and $\lambda>0$, we get,

$$\kappa_2(\zeta)=\frac{\lambda\,\mathrm{csch}(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{csch}(\sqrt{\lambda}\zeta)+1}, \quad \sigma_2(\zeta)=\frac{\sqrt{\lambda}\coth(\sqrt{\lambda}\zeta)}{\chi\,\mathrm{csch}(\sqrt{\lambda}\zeta)+1}. \quad (13)$$

Case 3: If $\varepsilon=1, c=-1$ and $\lambda>0$,we get,

$$\kappa_3(\zeta)=\frac{\lambda\sec(\sqrt{\lambda}\zeta)}{\chi\sec(\sqrt{\lambda}\zeta)+1}, \quad \sigma_3(\zeta)=\frac{\sqrt{\lambda}\tan(\sqrt{\lambda}\zeta)}{\chi\sec(\sqrt{\lambda}\zeta)+1},$$
$$\kappa_4(\zeta)=\frac{\lambda\csc(\sqrt{\lambda}\zeta)}{\chi\csc(\sqrt{\lambda}\zeta)+1}, \quad \sigma_4(\zeta)=-\frac{\sqrt{\lambda}\cot(\sqrt{\lambda}\zeta)}{\chi\csc(\sqrt{\lambda}\zeta)+1}. \quad (14)$$

Case 4: If $\lambda=\chi=0$,

$$\kappa_5(\zeta)=\frac{C}{\zeta}, \quad \sigma_5(\zeta)=\frac{1}{\varepsilon\zeta}. \quad (15)$$

Herein, C is a nonzero constant. Insert Eq. (6) and its derivatives to Eq. (5) and taking into account Eqs. (7)-(9), we acquire a polynomial consisting of $\kappa^k(\zeta)\sigma^l(\zeta), (k,l=0,1.2.3,...,M)$. If we collect the coefficients of $\kappa^k(\zeta)\sigma^l(\zeta)$ involving the same power and equal each coefficient to zero, we gain a system of algebraic equations which consist of $A_0, A_k, B_k, \chi, \lambda, \rho_1, \rho_2, \rho_3, \rho_4$, and ρ_5. Solving this system, then inserting these parameter values into Eq. (5), afterwards utilizing the solutions Eqs.(12)-(15) and Eq. (3), we achieve the solutions to Eq. (2).

4.2. Implementation of GPREM to the fractional $(4+1)$-dimensional FE

Considering Eq. (5) and applying the balance rule, we get $M=2$. Eq. (6) is rewritten in the following structure:

$$V(\zeta)=A_0+A_1\kappa(\zeta)+B_1\sigma(\zeta)+A_2\kappa(\zeta)+B_2\sigma(\zeta), \quad (16)$$

in which A_0, A_1, A_2, B_1, and B_2 are constants. Inserting Eq. (16) into Eq.(5) taking into account Eqs. (7)-(9), we get a system of algebraic equations. Considering the coefficients of $\kappa^k(\zeta)\sigma^l(\zeta)$ as zero, then solving the system, we derive the solution functions as:

Case 1: If $\varepsilon=-1, c=-1$ and $\lambda>0$, we get the following results:

Result$_1$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{(\rho_1{}^2-\rho_2{}^2)\rho_1\rho_2},\ A_0=0, A_1=-\frac{1}{4}\chi\left(\rho_1{}^2-\rho_2{}^2\right),\\ A_2=\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=-\frac{(\rho_1{}^2-\rho_2{}^2)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)(\rho_1{}^2-\rho_2{}^2)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \quad (17)$$

Result$_2$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{(\rho_1{}^2-\rho_2{}^2)\rho_1\rho_2},\ A_0=0, A_1=-\frac{1}{4}\chi\left(\rho_1{}^2-\rho_2{}^2\right),\\ A_2=\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=\frac{(\rho_1{}^2-\rho_2{}^2)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)(\rho_1{}^2-\rho_2{}^2)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \quad (18)$$

and subsequently, we get the following solution functions:

$$\vartheta_{1,1}(x,y,z,s,t)=\left(\frac{(3\chi\rho_3\rho_4-2\chi\rho_1\rho_5)cosh(\sqrt{\lambda}\zeta)}{2\rho_1\rho_2(\chi+cosh(\sqrt{\lambda}\zeta))}\right)-\left(\frac{\sqrt{\lambda}\omega_1\sinh(\frac{1}{\alpha}(\sqrt{\lambda}\zeta))-3\rho_3\rho_4+2\rho_1\rho_5}{2\rho_1\rho_2(\chi+cosh(\sqrt{\lambda}\zeta))^2}\right), \quad (19)$$

$$\vartheta_{1,2}(x,y,z,s,t)=\left(\frac{(3\chi\rho_3\rho_4-2\chi\rho_1\rho_5)cosh(\sqrt{\lambda}\zeta)}{2\rho_1\rho_2(\chi+cosh(\sqrt{\lambda}\zeta))}\right)+\left(\frac{\sqrt{\lambda}\omega_1\sinh(\frac{1}{\alpha}(\sqrt{\lambda}\zeta))+3\rho_3\rho_4-2\rho_1\rho_5}{2\rho_1\rho_2(\chi+(cosh(\sqrt{2}\lambda\zeta)))^2}\right), \quad (20)$$

in which

$$\omega_1=\sqrt{\rho_1\rho_2\left(\rho_1{}^2-\rho_2{}^2\right)\left(\chi^2-1\right)\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{2}}$$

and

$$\zeta=\frac{\Gamma(1+\gamma)\left(\rho_1x^\alpha+\rho_2y^\alpha+\rho_3z^\alpha+\rho_4s^\alpha+\rho_5t^\alpha\right)}{\alpha}.$$

Case 2: If $\varepsilon=-1, c=1$ and $\lambda>0$,we get the following results:

Result$_3$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{(\rho_1{}^2-\rho_2{}^2)\rho_1\rho_2},\ A_0=0, A_1=-\frac{1}{4}\chi\left(\rho_1{}^2-\rho_2{}^2\right),\\ A_2=\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2+1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=-\frac{\left(\rho_1{}^2-\rho_2{}^2\right)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)(\rho_1{}^2-\rho_2{}^2)(\chi^2+1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{21}$$

Result$_4$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{(\rho_1{}^2-\rho_2{}^2)\rho_1\rho_2},\ A_0=0, A_1=-\frac{1}{4}\chi\left(\rho_1{}^2-\rho_2{}^2\right),\\ A_2=\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2+1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=\frac{\left(\rho_1{}^2-\rho_2{}^2\right)\sqrt{2}\sqrt{(2\rho_1\rho_5-3\rho_3\rho_4)(\rho_1{}^2-\rho_2{}^2)(\chi^2+1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{22}$$

and subsequently, we get the following solution functions:

$$\vartheta_{2,1}(x,y,z,s,t)=\left(\frac{-\sqrt{\lambda}\omega_2 cosh\left(\sqrt{\lambda}\zeta\right)+(3\rho_3\rho_4-2\rho_1\rho_5)\left(\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}{2\rho_1\rho_2\left(\left(cosh\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+2\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}\right), \tag{23}$$

$$\vartheta_{2,2}(x,y,z,s,t)=\left(\frac{\sqrt{\lambda}\omega_2 cosh\left(\sqrt{\lambda}\zeta\right)+(3\rho_3\rho_4-2\rho_1\rho_5)\left(\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}{2\rho_1\rho_2\left(\left(cosh\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+2\chi\sinh\left(\sqrt{\lambda}\zeta\right)-1\right)}\right), \tag{24}$$

in which

$$\omega_2=-\sqrt{\rho_1\rho_2\left(\rho_1{}^2-\rho_2{}^2\right)\left(\chi^2+1\right)\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{2}}$$

and

$$\zeta=\frac{\Gamma(1+\gamma)\left(\rho_1x^\alpha+\rho_2y^\alpha+\rho_3z^\alpha+\rho_4s^\alpha+\rho_5t^\alpha\right)}{\alpha}.$$

Case 3: If $\varepsilon=1, c=-1$ and $\lambda>0$,we get the following results:

Result$_5$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{(\rho_1{}^2-\rho_2{}^2)\rho_1\rho_2},\ A_0=-\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{6\rho_1\rho_2},\\ A_1=\frac{1}{4}\chi\left(\rho_1{}^2-\rho_2{}^2\right),\\ A_2=-\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=-\frac{\left(\rho_1{}^2-\rho_2{}^2\right)\sqrt{2}\sqrt{-(2\rho_1\rho_5-3\rho_3\rho_4)\left(\rho_1{}^2-\rho_2{}^2\right)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{25}$$

Result$_6$:

$$\left\{\begin{array}{l}\lambda=\frac{2(2\rho_1\rho_5-3\rho_3\rho_4)}{(\rho_1{}^2-\rho_2{}^2)\rho_1\rho_2},\ A_0=-\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{6\rho_1\rho_2},\\ A_1=\frac{1}{4}\chi\left(\rho_1{}^2-\rho_2{}^2\right),\\ A_2=-\frac{(\rho_1-\rho_2)^2(\rho_1+\rho_2)^2(\chi^2-1)\rho_1\rho_2}{16\rho_1\rho_5-24\rho_3\rho_4}, B_1=0,\\ B_2=\frac{\left(\rho_1{}^2-\rho_2{}^2\right)\sqrt{2}\sqrt{-(2\rho_1\rho_5-3\rho_3\rho_4)\left(\rho_1{}^2-\rho_2{}^2\right)(\chi^2-1)\rho_1\rho_2}}{16\rho_1\rho_5-24\rho_3\rho_4},\end{array}\right\} \tag{26}$$

and subsequently, we get the following solution function

$$\vartheta_{3,1}(x,y,z,s,t)=\left(\frac{3\sqrt{\lambda}\sqrt{-\omega_1}\sin\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2-\chi\cos\left(\sqrt{\lambda}\zeta\right)+\chi^2-3\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right), \tag{27}$$

$$\vartheta_{3,2}(x,y,z,s,t)=\left(\frac{-3\sqrt{\lambda}\sqrt{-\omega_1}\sin\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2-\chi\cos\left(\sqrt{\lambda}\zeta\right)+\chi^2-3\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right), \tag{28}$$

$$\vartheta_{4,1}(x,y,z,s,t)=\left(\frac{3\sqrt{\lambda}\sqrt{-\omega_1}\cos\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi\sin\left(\sqrt{\lambda}\zeta\right)-\chi^2+2\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(2\chi\sin\left(\sqrt{\lambda}\zeta\right)-2\left(cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+1\right)}\right), \tag{29}$$

$$\vartheta_{4,2}(x,y,z,s,t)=\left(\frac{-3\sqrt{\lambda}\sqrt{-\omega_1}\cos\left(\sqrt{\lambda}\zeta\right)}{6\rho_1\rho_2\left(\chi+cos\left(\sqrt{\lambda}\zeta\right)\right)^2}\right)+\left(\frac{\left(\left(\cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi\sin\left(\sqrt{\lambda}\zeta\right)-\chi^2+2\right)(3\rho_3\rho_4-2\rho_1\rho_5)}{6\rho_1\rho_2\left(2\chi\sin\left(\sqrt{\lambda}\zeta\right)-2\left(cos\left(\sqrt{\lambda}\zeta\right)\right)^2+\chi^2+1\right)}\right), \tag{30}$$

in which

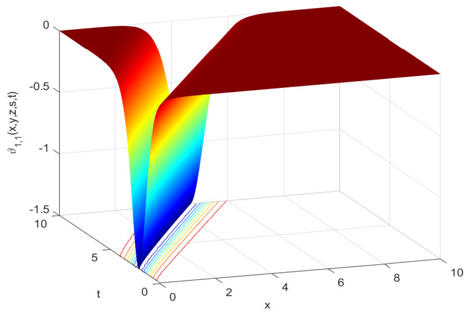
$$\omega_1=\sqrt{\rho_1\rho_2\left(\rho_1{}^2-\rho_2{}^2\right)\left(\chi^2-1\right)\frac{(2\rho_1\rho_5-3\rho_3\rho_4)}{2}}$$

and $\zeta=\frac{\Gamma(1+\gamma)(\rho_1x^\alpha+\rho_2y^\alpha+\rho_3z^\alpha+\rho_4s^\alpha+\rho_5t^\alpha)}{\alpha}$.

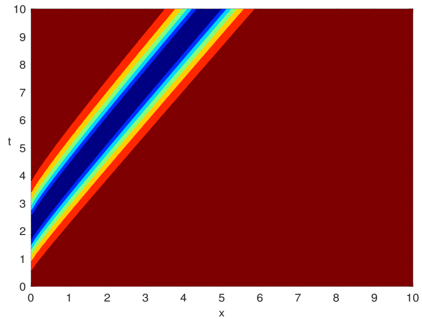
5. Results and discussion

In this section, we present some graphical portraits and physical interpretations of the resulted solutions. For appropriate variables of unknown parameters, we depict various graphs with 3D, 2D and contour plots. We get the dark and singular soliton solutions for the model. Figs. (1-2) show some of the obtained solutions. We display 3D and contour graphs of $\vartheta_{1,1}(x,y,z,s,t)$ in Eq. (19) for the parameters $\rho_1=3$, $\rho_2=-1$, $\rho_3=2, \rho_4=1$, $\rho_5=-2, \gamma=0.5$, y = 1, z = 1, $s=1, \chi=5$, and $\alpha=0.8$. Fig.1-(a) and fig.1-(b) show the dark soliton. Fig. 1-(c) also demonstrates 2D soliton profile for $t=5,7,9$. It can be seen that the amplitude and the shape of the dark soliton remain same. Moreover, as t increases, soliton goes to the right. Fig. 1-(d) is the 2D graphical portrait to demonstrate the effect of the α when α takes the values as 0.7, 0.8, 0.9 and 1.0, respectively. Soliton keeps its dark soliton view but if we pay attention to the peaks of the soliton, as if soliton moves to the right. Thus, we can say that this situation is not the various structures of the soliton resting on the fractional orders.

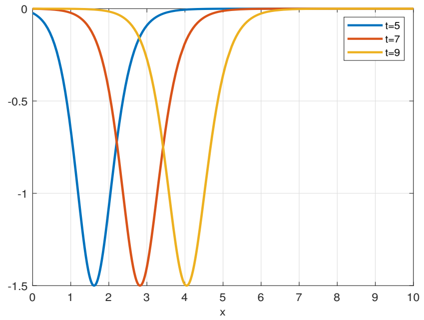
Fig.2-(a) and Fig.2-(b) are 3D and contour graphs of $\vartheta_{2,1}(x,y,z,s,t)$ given in Eq.(23) and these graphs demonstrate the singular soliton for the parameters $\rho_1=3$,$\rho_2=-1$,$\rho_3=3$,$\rho_4=1$,$\rho_5=-3$,$\gamma=0.5$, $y=1, z=1$, s = 1, $\chi=-5$, and $\alpha=1$. Fig. 2-(c) also demonstrates 2D soliton profile for $t=3,5,7$. It can be seen that the shape of the singular soliton remains the same. Moreover, while t decrases, soliton moves to the left. Fig. 2-(d) is the 2D graphical portrait to demonstrate the



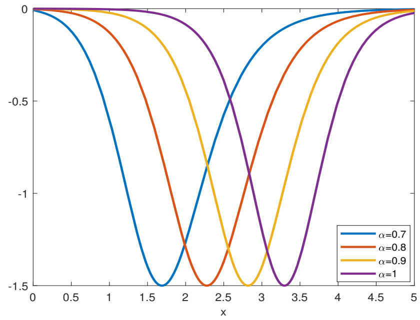
(a) 3D portrait of $\vartheta_{1,1}(x, 1, 1, 1, t)$



(b) Contour plot of $\vartheta_{1,1}(x, 1, 1, 1, t)$

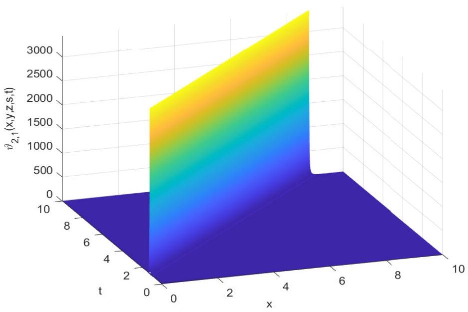


(c) 2D portrait of $\vartheta_{1,1}(x, 1, 1, 1, t)$

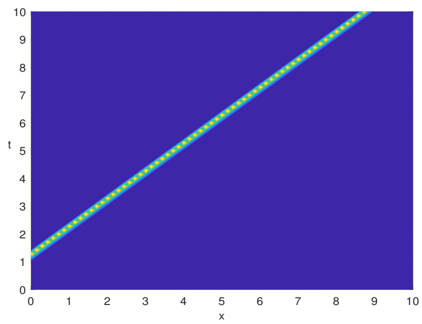


(d) 2D portraits of $\vartheta_{1,1}(x, 1, 1, 1, 7)$

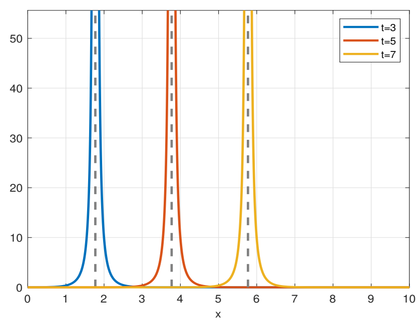
Figure 1. The dark soliton portraits of $\vartheta_{1,1}(x, 1, 1, 1, t)$ in Eq. (19) for the parameters $\rho_1 = 3, \rho_2 = -1, \rho_3 = 3, \rho_4 = 2, \rho_5 = -3, \gamma = 0.5$, $y = 1, z = 1$, s $= 1$, $\chi = 5$, and $\alpha = 0.8$.



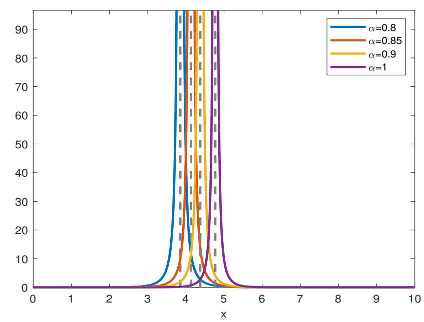
(a) 3D portrait of $\vartheta_{2,1}(x, 1, 1, 1, t)$



(b) Contour plot of $\vartheta_{2,1}(x, 1, 1, 1, t)$



(c) 2D portrait of $\vartheta_{2,1}(x, 1, 1, 1, t)$



(d) 2D portraits of $\vartheta_{2,1}(x, 1, 1, 1, 7)$ at $t = 7$

Figure 2. The singular soliton portraits of $\vartheta_{2,1}(x, 1, 1, 1, t)$ in Eq. (23) for the parameters $\rho_1 = 3, \rho_2 = -1, \rho_3 = 3, \rho_4 = 2, \rho_5 = -3, \gamma = 0.5, y = 1,\ z = 1, s = 1, \chi = -5$, and $\alpha = 1$.

effect of the α when α takes the values as 0.8, 0.85, 0.9 and 1.0, respectively. Thus, we can say that this situation is not the different structures of the soliton resting on the fractional orders.

6. Conclusion

In this study, for the first time, the generalized projective Riccati equations method was efficaciously employed to scrutinize analytical solutions for the fractional (4+1)-dimensional Fokas equation with M-truncated derivative. Some analytical solutions and singular and dark soliton solutions are acquired. 3D, contour, and 2D graphs were added to exhibit the physical illustrations of some resulted solutions. The GPREM can be successfully implemented to the different fractional forms of (4+1)-dimensional Fokas equation. Hence, the results show that the GPREM is a very effectual and profitable tool for solving such higher order NLPDEs occurring in region associated with physics, chemical, biology, and mathematics along with engineering.

References

[1] Ozisik, M., Secer, A., & Bayram, M. (2022). Dispersive optical solitons of Biswas–Arshed equation with a couple of novel approaches. *Optik*, 265, 169547.

[2] Esen, H., Ozisik, M., Secer, A., & Bayram, M. (2022). Optical soliton perturbation with Fokas–Lenells equation via enhanced modified extended tanh-expansion approach. *Optik*, 267, 169615.

[3] Ozisik, M., Bayram, M., Secer, A., Cinar, M., Yusuf, A., & Sulaiman, T. A. (2022). Optical solitons to the (1+ 2)-dimensional Chiral non-linear Schrödinger equation. *Optical and Quantum Electronics*, 54(9), 1-13.

[4] Yildirim, Y., Biswas, A., Alshehri, H. M., & Belic, M. R. (2022). Cubic–quartic optical soliton perturbation with Gerdjikov–Ivanov equation by sine-Gordon equation approach. *Optoelectronics and Advanced Materials-Rapid Communications*, 16(5-6), 236-242.

[5] Yildirim, Y., Biswas, A., & Alshehri, H. M. (2022). Cubic–quartic optical soliton perturbation with Fokas–Lenells equation having maximum intensity. *Optik*, 169336.

[6] Alquran, M. (2021). Physical properties for bidirectional wave solutions to a generalized fifth-order equation with third-order time-dispersion term. *Results in Physics*, 28, 104577.

[7] Kocak, H. (2021). Kink and anti-kink wave solutions for the generalized KdV equation with Fisher-type nonlinearity. *An International Journal of Optimization and Control: Theories & Applications (IJOCTA)*, 11(2), 123-127.

[8] Hoque, M. F., & Roshid, H. O. (2020). Optical soliton solutions of the Biswas-Arshed model by the $\tan(\ominus/2)$ expansion approach. *Physica Scripta*, 95, 075219.

[9] Al-Askar, F. M., Mohammed, W. W., Cesarano, C., & El-Morshedy, M. (2022). The influence of multiplicative noise and fractional derivative on the solutions of the stochastic fractional Hirota–Maccari system. *Axioms*, 11(8), 357.

[10] Yusuf, A., Inc, M., & Baleanu, D. (2019). Optical solitons with M-truncated and beta derivatives in nonlinear optics. *Frontiers in Physics*, 7, 126.

[11] Akram, G., Sadaf, M., & Zainab, I. (2022). Observations of fractional effects of β-derivative and M-truncated derivative for space time fractional $\Phi - 4$ equation via two analytical techniques. *Chaos, Solitons & Fractals*, 154, 111645.

[12] Hashemi, M. S. (2018). Some new exact solutions of $(2+1)$-dimensional nonlinear Heisenberg ferromagnetic spin chain with the conformable time fractional derivative. *Optical and Quantum Electronics*, 50(2), 1-11.

[13] Cinar, M., Secer, A., & Bayram, M. (2022). Analytical solutions of $(2 + 1)$-dimensional Calogero-Bogoyavlenskii-Schiff equation in fluid mechanics/plasma physics using the New Kudryashov method. *Physica Scripta*, 97(9), 094002.

[14] Esen, H., Secer, A., Ozisik, M., & Bayram, M. (2022). Dark, bright and singular optical solutions of the Kaup–Newell model with two analytical integration schemes. *Optik*, 261, 169110.

[15] Onder, I., Secer, A., Ozisik, M., & Bayram, M. (2022). On the optical soliton solutions of Kundu–Mukherjee–Naskar equation via two different analytical methods. *Optik*, 257, 168761.

[16] Akinyemi, L., Şenol, M., Az-Zo'bi, E., Veeresha, P., & Akpan, U. (2022). Novel soliton solutions of four sets of generalized (2+ 1)-dimensional Boussinesq–Kadomtsev–Petviashvili-like equations. *Modern Physics Letters B*, 36(01), 2150530.

[17] Akinyemi, L., Veeresha, P., Darvishi, M. T., Rezazadeh, H., Şenol, M., & Akpan, U. (2022). A novel approach to study generalized coupled cubic Schrödinger-Korteweg-de Vries equations. *Journal of Ocean Engineering and Science*, DOI: https://doi.org/10.1016/j.joes.2022.06.004.

[18] Veeresha, P. (2022). Analysis of the spread of infectious diseases with the effects of consciousness programs by media using three fractional operators. In Methods of Mathematical Modelling (pp. 113-135). Academic Press.

[19] Yao, S. W., Ilhan, E., Veeresha, P., & Baskonus, H. M. (2021). A powerful iterative approach for quintic complex Ginzburg–Landau equation within the frame of fractional operator. Fractals, 29(05), 2140023.

[20] Podlubny, I. (1999). *Fractional Differential Equations*. Academic Press, San Diego.

[21] Veeresha, P., Ilhan, E., & Baskonus, H. M. (2021). Fractional approach for analysis of the model describing wind-influenced projectile motion. *Physica Scripta*, 96(7), 075209.

[22] Atangana, A., & Koca, I. (2016). Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. *Chaos, Solitons & Fractals*, 89, 447-454.

[23] Baishya, C., & Veeresha, P. (2021). Laguerre polynomial-based operational matrix of integration for solving fractional differential equations with non-singular kernel. *Proceedings of the Royal Society A*, 477(2253), 20210438.

[24] Khalil, R., Al Horani, M., Yousef, A., & Sababheh, M. (2014). A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264, 65-70.

[25] Fokas, A. S. (2006). Integrable nonlinear evolution partial differential equations in $4+2$ and $3+1$ dimensions. *Physical review letters*, 96(19), 190201.

[26] He, Y. (2014). Exact solutions for $(4+1)$-dimensional nonlinear Fokas equation using extended F-expansion method and its variant. *Mathematical Problems in Engineering*, 2014.

[27] Zhang, S., Tian, C., & Qian, W. Y. (2016). Bilinearization and new multisoliton solutions for the (4+1)-dimensional Fokas equation. *Pramana*, 86(6), 1259-1267.

[28] Kim, H., & Sakthivel, R. (2012). New exact traveling wave solutions of some nonlinear higher-dimensional physical models. *Reports on Mathematical Physics*, 70(1), 39-50.

[29] Ullah, N., Asjad, M. I., Awrejcewicz, J., Muhammad, T., & Baleanu, D. (2022). On soliton solutions of fractional-order nonlinear model appears in physical sciences. *AIMS Mathematics*, 7(5), 7421-7440.

[30] Xu, B., & Zhang, S. (2019). Exact solutions with arbitrary functions of the $(4+1)$-dimensional Fokas equation. *Thermal Science*, 23(4), 2403-2411.

[31] Wazwaz, A. M. (2021). A variety of multiple-soliton solutions for the integrable (4+1)-dimensional Fokas equation. *Waves in Random and Complex Media*, 31(1), 46-56.

[32] Al-Amr, M. O., & El-Ganaini, S. (2017). New exact traveling wave solutions of the $(4+1)$-dimensional Fokas equation. *Computers & Mathematics with Applications*, 74(6), 1274-1287.

[33] Baskonus, H. M., Kumar, A., Kumar, A., & Gao, W. (2020). Deeper investigations of the $(4+1)$-dimensional Fokas and $(2+1)$-dimensional Breaking soliton equations. *International Journal of Modern Physics B*, 34(17), 2050152.

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