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RESEARCH ARTICLE

The null boundary controllability for the Mullins equation with periodic boundary conditions

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ABSTRACT

This paper studies the null controllability of the Mullins equation with a control acting on the periodic boundary. The main result proves that the system is controllable for a specific class of initial conditions and also identifies uncontrollable states. Additionally, the existence and uniqueness theorem for the solution of the backward adjoint system is provided.

1. Introduction

In this paper, we study the null controllability problem for the Mullins equation [1] with periodic boundary conditions. This equation is a linear analog of the Kuramoto-Sivashinsky equation and has the form

$$y_t + B y_{xxxx} = 0, \tag{1}$$

where B is a positive constant known as the Mullins coefficient. The Mullins equation is a linear parabolic partial differential equation that is often used to model the evolution of thin films in materials science and engineering.

The controllability problems for parabolic equations have received considerable attention in the literature (see [2–12]). However, the null controllability of fourth-order parabolic equations has been studied in a few papers. Firstly, Y.L. Guo [13] used two well-posed problems to solve the null boundary controllability problem for a fourth-order parabolic equation. Later, the null interior controllability problem for a fourth-order parabolic equation was solved by Han Yu [14] using the method based on Lebeau-Rabbino Inequality. Also, Z. Zhou [15] derived the observability inequalities for a one-dimensional linear fourth-order parabolic equation with potential using establishing global Carleman estimates and presented null controllability results for the one-dimensional fourth-order semilinear equation. More recently, S. Guerrero and K. Kassab obtained the null controllability results for the higher dimensional fourth-order parabolic equation in [16]. These studies have mostly focused on the case of Dirichlet boundary conditions. This paper, however, explores the null controllability problem with periodic boundary conditions. There have been some works on null controllability for different types of systems using periodic boundary conditions. For example, Imanuvilov considered the controllability problem for the Boussinesq system with periodic boundary conditions [17], Beauchard and Zuazua studied the null controllability problem of the Kolmogorov equation under periodic boundary conditions [18], and Chowdhury and Mitra proved that the linearized compressible Navier-Stokes equations with periodic boundary conditions are null controllable [19]. More recently, Oner obtained null controllability results for a heat equation with periodic boundary conditions [20]. However, to the best of our knowledge, no work on the controllability problem for fourth-order parabolic equations with periodic boundary conditions has been published in the literature. This observation motivated us to consider this problem.

In addition, the above-aforementioned studies generally preferred the Carleman method to solve this problem and this method is quite technical. Here, we used duality and the moment method. The moment method was developed by Fattorini and Russell (see [3, 21]), and it allows obtaining the solution of the problem using the spectral properties of the system.

The main contributions of this article are as follows. First of all, the existence and uniqueness of the solution of the adjoint system have been proven. Then, with periodic boundary conditions, it is shown that the system is not always controllable for every initial condition, and a class containing controllable initial conditions is determined. Finally, for this admissible initial data class, the null boundary controllability problem of the Mullins equation with periodic boundary conditions has been solved by using the moment method.

The paper is organized as follows. In Section 2, we define the problem and give some initial results by using duality between controllability and observability. Subsequently, in Section 3, we provide some spectral results to reduce the null controllability problem to a moment problem. In Section 4, we focus on the null boundary controllability problem for the Mullins equation with periodic boundary conditions. Since the null controllability of the system is not always possible, we first determine the restricted initial data class and then show that the system is null controllable for this initial data class. Finally, in Section 4, we indicate the conclusion.

2. Problem Formulation

In the present work, we consider the null controllability of the following system:

$$\begin{cases} u_t + u_{xxxx} + cu = 0, & \text{in } D \\ u(\pi,t) - u(-\pi,t) = v(t), & \text{in } (0,T) \\ u_x(\pi,t) - u_x(-\pi,t) = 0, & \text{in } (0,T) \\ u_{xx}(\pi,t) - u_{xx}(-\pi,t) = 0, & \text{in } (0,T) \\ u_{xxx}(\pi,t) - u_{xxx}(-\pi,t) = 0, & \text{in } (0,T) \\ u(x,0) = u^0(x), & \text{in } \Omega \end{cases} \tag{2}$$

where $D = \Omega \times (0,T)$, $\Omega = (-\pi,\pi)$, $u^0(x) \in L^2(\Omega)$, $v(t) \in L^2(0,T)$, and c is any positive number. The system we are considering is not always controllable. Therefore, we will first identify the uncontrollable cases and then determine the conditions under which the system is controllable. Let us call this class $\mathcal{F}$, which will be determined later. Now, we can define the null controllability.

Definition 1. *System* (2) *is null controllable at time T if for every initial condition $u^0 \in \mathcal{F}$, there exists a control $v(t) \in L^2(0,T)$ such that $u(x,T) = 0$ for all $x \in \Omega$.*

Now, we can present a lemma that will be used in the proof of our main result.

Lemma 1. *The system* (2) *is null controllable in time $T > 0$ if and only if for any $u^0 \in \mathcal{F}$ there exists $v(t) \in L^2(0,T)$ such that*

$$\int_{-\pi}^{\pi} u^0(x)\varphi(x,0)dx + \int_0^T v(t)\varphi_{xxx}(\pi,t)dt = 0 \tag{3}$$

holds for any $\varphi^0 \in L^2(\Omega)$, where $\varphi(x,t)$ is a solution of the backward adjoint problem given in as follows.

$$\begin{aligned} \varphi_t - \varphi_{xxxx} - c\varphi &= 0, \ \textit{in } D && (4a)\\ \varphi(\pi,t) - \varphi(-\pi,t) &= 0, \ \textit{in } (0,T) && (4b)\\ \varphi_x(\pi,t) - \varphi_x(-\pi,t) &= 0, \ \textit{in } (0,T) && (4c)\\ \varphi_{xx}(\pi,t) - \varphi_{xx}(-\pi,t) &= 0, \ \textit{in } (0,T) && (4d)\\ \varphi_{xxx}(\pi,t) - \varphi_{xxx}(-\pi,t) &= 0, \ \textit{in } (0,T) && (4e)\\ \varphi(x,T) &= \varphi^0(x), \ \textit{in } \Omega && (4f) \end{aligned}$$

Proof. Let v be an arbitrary element of $L^2(0,T)$, and let φ be the solution of (4). By multiplying (2) by φ and integrating the resulting expression over D using integration by parts, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_{-\pi}^{\pi} (u_t + u_{xxxx} + cu)\varphi dxdt \\ &= \int_0^T \int_{-\pi}^{\pi} u(-\varphi_t + \varphi_{xxxx} + c\varphi)dxdt \\ &+ \int_0^1 u\varphi\Big|_0^T dx \\ &+ \int_0^T [\varphi u_{xxx} - \varphi_x u_{xx} + \varphi_{xx}u_x - u\varphi_{xxx}]\Big|_{-\pi}^{\pi} dt. \end{aligned}$$

Using the given initial condition and boundary conditions, we have

$$\int_{-\pi}^{\pi} u(x,T)\varphi^0(x)dx - \int_{-\pi}^{\pi} u^0(x)\varphi(x,0)dx - \int_0^T v(t)\varphi_{xxx}(\pi,t)dt = 0. \tag{5}$$

If equation (3) holds, then it follows that $\int_{-\pi}^{\pi} u(x,T)\varphi^0(x)dx = 0$ for all $\varphi^0(x) \in L^2(\Omega)$ which means that $u(x,T) = 0$ for all $x \in \Omega$. As a result, system (2) is null-controllable. On the contrary, suppose that system (2) is null controllable at time T , that is, for every initial condition $u^0 \in \mathcal{F}$, there exists a control $v(t) \in L^2(0,T)$ such that $u(x,T) = 0$ for all $x \in \Omega$. Substituting $u(x,T) = 0$ into (5), we conclude that (3) holds. □

Above Lemma shows that the system (2) is controllable if and only if equation (3) holds. Therefore, we need to find a solution for system (4). In the following section, we will first prove the existence and uniqueness of the solution for system (4).

3. Fourier Series representation of adjoint system

To find solution of system in equation (4), we will apply the method of separation of variables by letting $\varphi(x,t) = X(x)T(t)$. This gives us:

$$\begin{cases} X''''(x) = (\lambda - c)X, & -\pi < x < \pi \\ X(\pi) - X(-\pi) = 0, & \text{in } (0,T) \\ X_x(\pi) - X_x(-\pi) = 0, & \text{in } (0,T) \\ X_{xx}(\pi) - X_{xx}(-\pi) = 0, & \text{in } (0,T) \\ X_{xxx}(\pi) - X_{xxx}(-\pi) = 0, & \text{in } (0,T) \end{cases}$$

which is self adjoint in $L^2(\Omega)$. Now, we will find a basis for $L^2(\Omega)$ formed by the eigenfunctions of this auxiliary problem. The eigenvalues and normalized eigenfunctions of this auxiliary spectral problem are $\lambda_n = n^4 + c,\ n = 0, 1, \dots$ and

$$X_0(x) = \frac{1}{\sqrt{2\pi}},$$
$$X_{2n-1}(x) = \frac{\cos(nx)}{\sqrt{\pi}},$$
$$X_{2n}(x) = \frac{\sin(nx)}{\sqrt{\pi}}$$

for $n = 1, 2, \dots$ Then, the solution of (4) can be expressed as a Fourier series expansion as follows:

$$\varphi(x,t) = \frac{\beta_0 e^{-\lambda_0(T-t)}}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \frac{e^{-\lambda_n(T-t)}[\beta_{2n-1}\cos(nx) + \beta_{2n}\sin(nx)]}{\sqrt{\pi}} \tag{6}$$

where $\beta_n = (\varphi(x,T), X_n(x))$ for $n = 0, 1, 2, \dots$

To prove the existence and uniqueness of the solution of system (4), we need an auxiliary result that will be presented in the following lemma.

Lemma 2. *Assume that function $\varphi^0(x) \in C^4[-\pi,\pi]$ satisfies the following conditions:*

$$\varphi^0(\pi) - \varphi^0(-\pi) = 0,$$
$$\varphi^0_x(\pi) - \varphi^0_x(-\pi) = 0,$$
$$\varphi^0_{xx}(\pi) - \varphi^0_{xx}(-\pi) = 0,$$
$$\varphi^0_{xxx}(\pi) - \varphi^0_{xxx}(-\pi) = 0.$$

Then, the following inequality holds.

$$\sum_{n=1}^{\infty} n^3(|\beta_{2n-1}| + |\beta_{2n}|) \le 2C\|(\varphi^0)''''\|_{L_2(-\pi,\pi)} \tag{7}$$

where $\beta_{2n-1} = (\varphi^0, X_{2n-1})$, $\beta_{2n} = (\varphi^0, X_{2n})$, and $C = \dfrac{\pi}{\sqrt{6}}$.

Proof. Let $\varphi^0(x) \in C^4[-\pi,\pi]$ satisfy the assumption of lemma. From equation (6), it is seen that

$\beta_{2n-1} = (\varphi^0(x), X_{2n-1})$ and $\beta_{2n} = (\varphi^0(x), X_{2n})$.

Then, we have

$$n^3(\beta_{2n-1}+\beta_{2n}) = \frac{1}{n}(\varphi^0, n^4X_{2n-1}) + \frac{1}{n}(\varphi^0, n^4X_{2n}).$$

Since

$$X''''_{2n-1} = n^4X_{2n-1} \text{ and } X''''_{2n} = n^4X_{2n},$$

we can rewrite the equation as follows.

$$n^3(\beta_{2n-1} + \beta_{2n}) = \frac{1}{n}(\varphi^0, X''''_{2n-1}) + \frac{1}{n}(\varphi^0, X''''_{2n}).$$

Applying integration by part, we obtain

$$= \frac{1}{n}((\varphi^0)'''', X_{2n-1}) + \frac{1}{n}((\varphi^0)'''', X_{2n}).$$

Using this equation, we get

$$\sum_{n=1}^{\infty} n^3(|\beta_{2n-1}| + |\beta_{2n}|) = \sum_{n=1}^{\infty} \frac{1}{n}|(\varphi^0)'''', X_n)|$$

By using Cauchy -Schwartz and Bessel inequalities, we obtain

$$\sum_{n=1}^{\infty} n^3(|\beta_{2n-1}| + |\beta_{2n}|) = \sum_{n=1}^{\infty} \frac{1}{n}|((\varphi^0)'''', X_n)|$$
$$\le \Big(\sum_{n=1}^{\infty} \frac{1}{n^2}\Big)^{\frac{1}{2}} \Big(\sum_{n=1}^{\infty} \big[|((\varphi^0)'''', X_n)|^2\Big)^{\frac{1}{2}}$$
$$\le C\|(\varphi^0)''''\|_{L_2(-\pi,\pi)}$$

with $C = \big(\sum_{n=1}^{\infty} \frac{1}{n^2}\big)^{\frac{1}{2}} = \dfrac{\pi}{\sqrt{6}}$. □

Now, we can prove the existence and uniqueness of the solution.

Lemma 3. *Let $\varphi^0(x)$ satisfy the conditions of Lemma (2). Then, the system (4) has a unique solution $\varphi(x,t) \in (C^{4,1}(D) \cap C^{3,0}(\bar{D}))$ of the form (6).*

Proof. Since $X_n(x)_{n\geq 0}$ are bases in $L^2(\Omega)$, $\varphi(x,t)$ can be represented by equation (6). To prove that $\varphi(x,t)$ given in (6) is solution of system (4), we need to show that the first partial derivative of $\varphi(x,t)$ with respect to t and the fourth partial derivative of $\varphi(x,t)$ with respect to x are continuous and it satisfies (4a) in Ω for $t>0$. Additionally, the function in equation (6) and its first, second, and third partial derivatives with respect to spatial variable, as well as its first partial derivative with respect to time, must be continuous at boundary points. We need to show that the series

$$\varphi_t(x,t)\sim\frac{\beta_0\lambda_0 e^{-\lambda_0(T-t)}}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\frac{\lambda_n e^{-\lambda_n(T-t)}[\beta_{2n-1}\cos(nx)+\beta_{2n}\sin(nx)]}{\sqrt{\pi}} \tag{8}$$

and

$$\varphi_{xxxx}(x,t)\sim\sum_{n=1}^{\infty}\frac{n^4 e^{-\lambda_n(T-t)}[\beta_{2n-1}\cos(nx)+\beta_{2n}\sin(nx)]}{\sqrt{\pi}}. \tag{9}$$

converge uniformly for $T-t\geq\epsilon$, where ϵ is an arbitrary positive number. The majorants of these series are

$$\sum_{n=1}^{\infty}\frac{\lambda_n e^{-\lambda_n\epsilon}(|\beta_{2n-1}|+|\beta_{2n}|)}{\sqrt{\pi}}$$

and

$$\sum_{n=1}^{\infty}\frac{n^4\lambda_n e^{-\lambda_n\epsilon}(|\beta_{2n-1}|+|\beta_{2n}|)}{\sqrt{\pi}}.$$

By using Lemma (3) and D'Alembert criterion, it is seen that these two majorant series are convergent. Therefore, the series in equations (8) and (9) are uniformly convergent for $T-t\geq\epsilon>0$. Also, we conclude from superposition principle that the function defined by (6) satisfies equation (4a) for all $T>t$ because t is arbitrary. The function in equation (6) and its first, second, and third partial derivatives with respect to spatial variable and first partial derivative with respect to time must be continuous at boundary points. Namely, the series in equation (6) must be continuous at $t=T$,

$$\varphi(x,T)=\frac{\beta_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\frac{[\beta_{2n-1}\cos(nx)+\beta_{2n}\sin(nx)]}{\sqrt{\pi}}$$

and the following functions must be continuous at boundary points $x=-\pi$ and $x=\pi$:

$$\varphi_{xxx}(x,t)\sim\sum_{n=1}^{\infty}\frac{n^3 e^{-\lambda_n(T-t)}[\beta_{2n-1}\sin(nx)-\beta_{2n}\cos(nx)]}{\sqrt{\pi}},$$

$$\varphi_{xx}(x,t)\sim\sum_{n=1}^{\infty}\frac{n^2 e^{-\lambda_n(T-t)}[-\beta_{2n-1}\cos(nx)-\beta_{2n}\sin(nx)]}{\sqrt{\pi}}.$$

$$\varphi_{x}(x,t)\sim\sum_{n=1}^{\infty}\frac{n e^{-\lambda_n(T-t)}[-\beta_{2n-1}\sin(nx)+\beta_{2n}\cos(nx)]}{\sqrt{\pi}}.$$

By using Weierstrass M-test and Lemma (2), we see that the following majorant series are uniformly convergent.

$$\sum_{n=1}^{\infty}\frac{|\beta_{2n-1}|+|\beta_{2n}|}{\sqrt{\pi}},\quad\sum_{n=1}^{\infty}\frac{n^3|\beta_{2n}|}{\sqrt{\pi}}$$

$$\sum_{n=1}^{\infty}\frac{n^2|\beta_{2n-1}|}{\sqrt{\pi}}\text{ and }\sum_{n=1}^{\infty}\frac{n|\beta_{2n}|}{\sqrt{\pi}}.$$

Therefore, the above series are continuous at the boundary points. Finally, we obtain a function $\varphi(x,t)\in(C^{4,1}(D)\cap C^{3,0}(\bar{D}))$ which is a solution of system (4) given by the Fourier series in equation (6). This solution is also unique due to the uniqueness of the Fourier representation of functions. □

4. Null boundary controllability of Mullins equation

In this section, we will reduce the null controllability problem to a moment problem using the spectral properties of the problem. Since $\{X_n(x)\}_{n\geq 0}$ is a basis in $L_2(\Omega)$, any initial data $u^0\in L_2(\Omega)$ can be represented as follows.

$$u^0(x)=\frac{\eta_0}{\sqrt{2\pi}}+\sum_{n=1}^{\infty}\frac{[\eta_{2n-1}\cos(nx)+\eta_{2n}\sin(nx)]}{\sqrt{\pi}}, \tag{10}$$

where $\eta_n=(u^0(x),X_n(x))$ for $n=0,1,2,\ldots$ Substituting equations (6) and (10) into (3), we get

$$\beta_0\eta_0 e^{-\lambda_0 T}+\sum_{n=1}^{\infty}e^{-\lambda_n T}[\beta_{2n-1}\eta_{2n-1}+\beta_{2n}\eta_{2n}]-\int_0^T\frac{v(t)}{\sqrt{\pi}}\sum_{n=1}^{\infty}n^3 e^{\lambda_n(T-t)}\beta_{2n}(-1)^n dt=0 \tag{11}$$

According to Lemma 1, system (2) is null controllable in time $T>0$ if and only if for any $u^0\in\mathcal{F}$ there exists $v(t)\in L^2(0,T)$ such that (3) is satisfied. Since $\{X_n(x)\}_{n\geq 0}$ is an orthonormal basis for $L^2(\Omega)$, equation (3) is verified if and only if it is verified by $\varphi_m^0(x) = X_m(x), m = 0, 1, \ldots$. Therefore, if in particular $\varphi_m^0(x) = X_m(x)$, then $\beta_n = \delta_{m,n}$, and $\eta_0 = 0$, $\eta_{2m-1} = 0$ and

$$\int_0^T \frac{v(t)}{\sqrt{\pi}} e^{\lambda_m(T-t)} m^3 (-1)^m dt = e^{-\lambda_m T} \eta_{2m}$$

for $m = 1, 2, \ldots$. Taking $v(t) = f(T-t)$ in the last equation, we have proven the following theorem, which is the main result of this article. From above, it is clear that system (2) is not always controllable for all initial data classes. That is why we need to define the following admissible initial data classes to make the system null controllable.

$$\mathcal{F} = \{u^0(x) \in L^2(\Omega) \big| \eta_0 = 0 \text{ and } \eta_{2m-1} = 0\}.$$

Now, we are in a position to state the main theorem of this article.

Theorem 1. *The system* (2) *is null controllable in time $T > 0$ if and only if for any $u^0 \in \mathcal{F}$ with Fourier expansion*

$$u^0(x) = \sum_{n=1}^{\infty} \eta_{2n} \frac{\sin(nx)}{\sqrt{\pi}},$$

there exists a function $f \in L^2(0,T)$ such that

$$\int_0^T f(t) e^{-\lambda_m t} dt = \frac{(-1)^m \eta_{2m} \sqrt{\pi} e^{-\lambda_m T}}{m^3} \tag{12}$$

for $m = 1, 2, \cdots$.

To have a precise understanding of Theorem 1, we provide the following example.

Example 1. *It is seen that*

$$\varphi_n(x,t) = \frac{\cos(nx)}{\sqrt{\pi}} e^{-\lambda_n(T-t)}$$

is a solution of (4) *with the initial data $\frac{\cos(nx)}{\sqrt{\pi}}$ for arbitrary fixed positive integer n. Taking into consideration these values in* (5)*, we have*

$$\int_{-\pi}^{\pi} u(x,T) \frac{\cos(nx)}{\sqrt{\pi}} dx - \int_{-\pi}^{\pi} u^0(x) \frac{e^{-\lambda_n T} \cos(nx)}{\sqrt{\pi}} dx = 0$$

the second term of equation is independent of the control and non-zero unless $\eta_{2n-1} = 0$. This explains how we choose the initial data classes.

4.1. Moment Problem

We need to find $f(t)$ that satisfies (12) to find control $v(t)$. This is a moment problem in $L^2(0,T)$ with respect to the family $\Lambda = \{e^{-\lambda_m t}\}_{m \geq 0}$. From Theorem (1), we see that controllability holds if and only if the moment problem (12) is solvable. To solve this moment problem, we can apply the general theory developed in [21] by Fattorini and Russell. Suppose that we can construct a sequence of functions $\{\Psi_n\}_{n \geq 0}$ that are biorthogonal to the set Λ in $L^2(0,T)$, such that

$$\int_0^T e^{-\lambda_m t} \Psi_n(t) dt = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n \neq m \end{cases} \tag{13}$$

for all $m, n = 0, 1, 2 \ldots$. Then, moment problems (12) have solutions by setting

$$f(t) = \sum_{n=1}^{\infty} \frac{\eta_{2n} e^{-\lambda_n T} (-1)^n \sqrt{\pi}}{n^3} \Psi_n(t)$$

Since

$$\sum_{n=0}^{\infty} \frac{1}{\lambda_n} = \sum_{n=0}^{\infty} \frac{1}{n^4 + c} < \infty, \tag{14}$$

Muntz's Theorem shows that biorthogonal sequence $\{\Psi_n\}_{n \geq 0}$ exists. In addition, the general estimations of $\|\Psi_n\|_{L^2(0,\infty)}$ was calculated by H.O. Fattoroni and D. L. Russell. They showed in [3] that if the λ_n are real and satisfy the following asymptotic relationship

$$\lambda_n = K(n + \alpha)^\zeta + o(n^{\zeta - 1}) \quad (n \to \infty)$$

where $K > 0, \zeta > 1$ and α is real, then there exists constants $\hat{K}, K_\zeta$ such that

$$\|\Psi_n(t)\|_{L^2(0,\infty)} \leq \hat{K} \exp[(K_\zeta + o(1)) \lambda_n^{1/\zeta}] \ (n \geq 1)$$

where $o(1)$ indicates a term tending to zero as n goes to infinity. The computation of the constant K_ζ is given in [21]. To relate the interval $[0, \infty]$ with the finite interval [0, T], they used results given in [22]. Since $\lambda_n = n^4 + c$, using these results it can be seen that

$$\|\Psi_n(t)\|_{L^2(0,T)} \leq K e^\rho \text{ for } n \geq 0$$

where K and ρ are some positive constants. Now, we can state the following results.

Corollary 1. *Given any $T > 0$, suppose that there exists a sequence $\{\Psi_n(t)\}_{n \geq 0}$ in $L^2(0,T)$ biorthogonal to the set Λ such that*

$$\|\Psi_n\|_{L^2(0,T)} \leq K e^{n\rho}, \quad \forall n \geq 0 \tag{15}$$

holds, where K and ρ are two positive constants. Then, system (2) *is null-controllable in time T.*

Proof. According to Theorem (1), the system (2) is null controllable in time T if for any $u^0 \in \mathcal{F}$ with Fourier expansion

$$u^0(x) = \sum_{n=1}^{\infty} \frac{\eta_{2n} \sin(nx)}{\sqrt{\pi}},$$

there exists a function $f \in L^2(0,T)$ which holds (12). Choose

$$f(t) = \sum_{n=1}^{\infty} \frac{\eta_{2n} e^{-\lambda_n T} (-1)^n \sqrt{\pi}}{n^3} \Psi_n(t) \tag{16}$$

Since $\|\Psi_n\|_{L^2(0,T)} \leq K e^{n\rho}$, for all $n \geq 0$, we deduce that

$$
\begin{aligned}
&\| \sum_{n=1}^{\infty} \frac{\eta_{2n} e^{-\lambda_n T}(-1)^n \sqrt{\pi}}{n^3} \Psi_n \|_{L^2(0,T)} \\
&\leq \sum_{n=1}^{\infty} \frac{|\eta_{2n}|\sqrt{\pi}}{n^3} e^{-\lambda_n T} \|\Psi_n\|_{L^2(0,T)} \\
&\leq K \sum_{n=1}^{\infty} \frac{|\eta_{2n}|\sqrt{\pi}}{n^3} e^{-\lambda_n T + n\rho} < \infty
\end{aligned}
$$

i.e., $f(t)$ converges in $L^2(0,T)$. Hence, (16) implies that f satisfies (12) and the proof finishes. □

5. Conclusion

In this paper, we studied the null boundary controllability of the Mullins equation with periodic boundary conditions. We demonstrated that the system is controllable on a specific admissible initial data class and solved the null boundary controllability problem by reducing it to a moment problem using the spectral properties of the system. Additionally, we established the existence and uniqueness of the solution of the system.

As a future direction, we will consider the system with nonlocal boundary conditions. In this case, the system is not self-adjoint and will require a different approach.

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