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RESEARCH ARTICLE

Existence and stability analysis to the sequential coupled hybrid system of fractional differential equations with two different fractional derivatives

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ABSTRACT

In this paper, we discussed the existence, uniqueness and Ulam-type stability of solutions for sequential coupled hybrid fractional differential equations with two derivatives. The uniqueness of solutions is established by means of Banach's contraction mapping principle, while the existence of solutions is derived from Leray-Schauder's alternative fixed point theorem. Further, the Ulam-type stability of the addressed problem is studied. Finally, an example is provided to check the validity of our obtained results.

1. Introduction

Differential equations are essential for a mathematical description of Nature. Many of the general laws of Nature–in physics, chemistry, biology, economics, and engineering find their most natural expression in the language of differential equation. Differential equation (DE) allows us to study all kinds of evolutionary processes with the properties of finite-dimensionality and differentiability. Derivative of arbitrary order arises from many physical processes, such as a charge transport in amorphous semiconductors, electrochemistry and material science, where they are described by differential equations of arbitrary order, see [1–4]. Recently, many researchers have exposed attention in the field of fractional differential equations theory, which will be used to describe phenomena of real-world problems. For more details; we refer the reader to the papers [5–18]. On the other hand, hybrid differential equations have gained extensive attention from many scholars; see for example [19–21]. Hybrid differential equations and coupled hybrid systems involving fractional derivatives have also been investigated by scientific researchers; see for instance [22–28] and the references cited therein. In recent years, sequential fractional hybrid differential equations have been studied by several researchers [29–34]. On the other hand, the stability of solutions of differential equations is important in physical problems because if slight deviations from the mathematical model caused by unavoidable errors in measurement do not have a correspondingly slight effect on the solution, the mathematical equations describing the problem will not accurately predict the future outcome.

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For example, one of the difficulties in predicting population growth is the fact that it is governed by the equation $w(t) = ce^{at}$, which is an unstable solution of the equation $w'(t) = aw(t)$. Even if there are no unfavourable factors, very few inaccuracies in the initial population count (c) or breeding rate (a) will result in fairly significant errors in prediction. One of the interesting subjects in this area, is the investigation of the existence and stability of solutions, because the study of the existence of solution of the fractional differential equation(FDE) became important due to the lack of a general formula for solving nonlinear FDEs, see [29, 30, 32, 33]. Recently, Some scholars have discussed the existence, uniqueness, and different types of Ulam stability of solutions of fractional sequential hybrid differential equations [29,32,33] and the references cited therein. The classical form of hybrid differential equation [35] is given by the following differential equation

$$\begin{cases} \frac{d}{dt}\left[\frac{w(t)}{\psi(t,w(t))}\right] = \varphi(t,w(t)),\ 0 \le t \le T, \\ \qquad w(t_0) = \quad w_0,\ w_0 \in \mathbb{R}, \end{cases}$$

where $\psi \in C([0,T]\times\mathbb{R}, \mathbb{R}-\{0\})$ and $\varphi \in C([0,T]\times\mathbb{R}\to\mathbb{R})$. Many scientific researchers have studied different fractional types of the above hybrid differential equation. For example in [36], the authors have discussed the fractional hybrid differential equations involving Riemann-Liouville differential operators

$$\begin{cases} {}^{\mathtt{RL}}\mathtt{D}\left[\frac{w(t)}{\psi(t,w(t))}\right] = \varphi(t,w(t)),\ 0 \le t \le T, \\ \qquad w(0) = \quad 0, \end{cases}$$

where $0 << 1$, $\psi \in C([0,T]\times\mathbb{R}, \mathbb{R}-\{0\})$ and $\varphi \in C([0,T]\times\mathbb{R}, \mathbb{R})$.

In [37], the authors studied the existence and uniqueness of solutions of coupled hybrid fractional differential equations described by

$$\begin{cases} {}^{\mathtt{C}}\mathtt{D}\left[\frac{w(t)}{\psi_1(t,w(t),z(t))}\right] = \varphi(t,w(t),z(t)), \\ {}^{\mathtt{C}}\mathtt{D}\left[\frac{z(t)}{\psi_2(t,w(t),z(t))}\right] = \phi(t,w(t),z(t)), \\ w(0) = w(1) = 0, \quad z(0) = z(1) = 0, \end{cases}$$

where $t \in [0,1], 1 <\le 2, 1 <\le 2$, $\psi_j \in C([0,1]\times\mathbb{R}, \mathbb{R}-\{0\}), j = 1,2$ and $\varphi, \phi \in C([0,1]\times\mathbb{R}, \mathbb{R})$. The existence and uniqueness results were obtained by applying Leray-Schauder's alternative criterion and Banach's contraction mapping principle.

Motivated by above-mentioned works, in this paper, we discuss the existence, uniqueness and Ulam-Hyers-Rassias stability of solution for sequential coupled fractional hybrid system of the following form

$$\begin{cases} {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\frac{w(t)}{\psi_1(t,w(t),z(t))}\right]\right] \\ \qquad = \sum_{i=1}^{k} \varphi_i(t,w(t),z(t)), \\ {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\frac{z(t)}{\psi_2(t,w(t),z(t))}\right]\right] \\ \qquad = \sum_{i=1}^{k} \phi_i(t,w(t),z(t)), \\ w(0) = w(1) = 0,\ z(0) = z(1) = 0, \end{cases} \tag{1}$$

where $0 \le t \le 1, 0 <, < 1, + > 1, 0 <, < 1, + > 1$, ${}^{\mathtt{RL}}\mathtt{D}, \in \{,\}$ and ${}^{\mathtt{C}}\mathtt{D}, \in \{,\}$ are the Riemann-Liouville and Caputo fractional derivatives respectively, $\psi_j : [0,1]\times\mathbb{R}^2 \to \mathbb{R}-\{0\}, j = 1,2$ and $\varphi_i, \phi_i : [0,1]\times\mathbb{R}^2 \to \mathbb{R}, 1 \le i \le k$, are continuous functions.

We impose the following hypotheses throughout the paper:

(H_1) For each $i = 1,2,...,k$, the functions $\varphi_i, \phi_i : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist constants $\pi_i > 0, \vartheta_i > 0$ such that for all $t \in [0,1]$ and $(w_1,z_1),(w_2,z_2) \in \mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist constants $\pi_i > 0, \vartheta_i > 0$ such that for all $t \in [0,1]$ and $(w_1,z_1),(w_2,z_2) \in \mathbb{R}^2$,

$$|\varphi_i(t,w_1,z_1) - \varphi_i(t,w_2,z_2)| \le \pi_i(|w_1 - z_1| + |w_2 - z_2|),$$

and

$$|\phi_i(t,w_1,z_1) - \phi_i(t,w_2,z_2)| \le \vartheta_i(|w_1 - z_1| + |w_2 - z_2|)$$

for $i = 1,2,\cdots,k$,

(H_2) For all $j = 1,2$, the functions $\psi_j : [0,1]\times\mathbb{R}^2 \to \mathbb{R}-\{0\}$ are continuous and there exist constants $\Pi_j > 0$ such that

$\psi_1(t,w,z) \le \Pi_1$ and $\psi_2(t,w,z) \le \Pi_2$,

for each $t \in [0,1]$ and $(w,z) \in \mathbb{R}^2$.

(H_3) For each $i = 1,2,...,k$, the functions $\varphi_i, \phi_i : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous and there exist constants $\gamma_i, \omega_i, \gamma_i^{'}, \omega_i^{'} \geq 0$ and $\lambda_i > 0, \lambda_i^{'} > 0$ such that for all $t \in J$ and $w, z \in \mathbb{R}$, we have

$\varphi_i(t, w, z) \leq \lambda_i + \gamma_i |w| + \omega_i |z|,$

and

$\phi_i(t, w, z) \leq \lambda_i^{'} + \gamma_i^{'} |w| + \omega_i^{'} |z|.$

The rest of the paper is organized in the following fashion. In Section 2, we introduce some basic definitions and lemmas which are useful in our main results. In Section 3, we establish a criteria for the existence and uniqueness of solutions to the boundary value problem (1) by applying the Leray-Schauder's alternative fixed point theorem and the Banach's contraction mapping principle in a Banach space. In section 4, we study Ulam-Hyers-Rassias stability of solutions to the problem (1). Finally, as an application, we demonstrate our results with example.

2. Preliminaries

In this section, we introduce some basic definitions and lemmas which are useful for our later discussions.

Definition 1. *[38] The Riemann-Liouville fractional integral of order > 0 for a function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$\mathtt{I}^{f}(t) = \frac{1}{\Gamma()} \int_0^t (t-s)^{-1} f(s) ds,$$

provided that the right side is pointwise defined on $(0, \infty)$.

Definition 2. *[38] The Riemann-Liouville fractional derivative of order > 0 for a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined as*

$$\mathtt{D}^{f}(t) = \frac{1}{\Gamma(m-)} \left(\frac{d}{dt}\right)^m \int_0^t \frac{f(s)}{(t-s)^{-m-1}} ds,$$

where $m = [] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

Definition 3. *[38] For a function f given on the interval $[0, \infty)$, The Caputo derivative of fractional order γ for the function f continuous on $[0, \infty)$ is defined as*

$$^{\mathtt{C}}\mathtt{D}^{f}(t) = \frac{1}{\Gamma(m-)} \int_0^t (t-s)^{m--1} f^{(m)}(s) ds,$$

$m = [] + 1$.

Lemma 1. *[12] Let $, > 0$ and $h \in L^1([0,1])$. Then $\mathtt{I}\mathtt{I}h(t) = \mathtt{I}^{+}h(t)$ and $^{\mathtt{RL}}\mathtt{D}\mathtt{I}h(t) = h(t)$.*

Lemma 2. *[12] Let $>> 0$ and $h \in L^1([0,1])$. Then $^{\mathtt{RL}}\mathtt{D}\mathtt{I}h(t) = \mathtt{I}^{-}h(t)$.*

Lemma 3. *[12] Let > 0. Then for $w \in C(0,1) \cap L^1(0,1)$ and $^{\mathtt{RL}}\mathtt{D}w \in C(0,1) \cap L^1(0,1)$, we have*

$$\mathtt{I}^{\mathtt{RL}}\mathtt{D}w(t) = w(t) + c_1 t^{-1} + c_2 t^{-2} + \cdots + c_n t^{-n},$$

where $c_i \in \mathbb{R}, i = 1, 2, ..., n,\ n = [] + 1$.

Lemma 4. *[12] Let > 0. Then*

$$\mathtt{I}\left[{}^{\mathtt{C}}\mathtt{D}w(t)\right] = w(t) + c_0 + c_1 t, + c_2 t^2 + ... + c_{n-1} t^{n-1},$$

for some $c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n-1 << n$.

Lemma 5. *For $g, h \in C([0,1], \mathbb{R})$ and $\psi_j \in C(([0,1] \times \mathbb{R}^2, \mathbb{R} - \{0\}), j = 1, 2$, the boundary value problem*

$$\begin{cases} ^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{w(t)}{\psi_1(t, w(t), z(t))}\right]\right] = g(t), \\ ^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{z(t)}{\psi_2(t, w(t), z(t))}\right]\right] = h(t), \\ w(0) = w(1) = 0, \quad z(0) = z(1) = 0, \end{cases} \tag{2}$$

where $0 \leq t \leq 1$, $0 <, < 1$, $+ > 1$, $0 <, < 1$, $+ > 1$, has a unique solution

$$w(t) = \psi_1(t, w(t), z(t)) \Bigg[\int_0^t \frac{(t-s)^{+-1}}{\Gamma(+)} g(s) ds - t^{+-1} \int_0^1 \frac{(1-s)^{+-1}}{\Gamma(+)} g(s) ds\Bigg], \tag{3}$$

and

$$z(t) = \psi_2(t, w(t), z(t)) \Bigg[\int_0^t \frac{(t-s)^{+-1}}{\Gamma(+)} g(s) ds - t^{+-1} \int_0^1 \frac{(1-s)^{+-1}}{\Gamma(+)} g(s) ds\Bigg]. \tag{4}$$

Proof. Using Lemma 3, we can write

$$\frac{w(t)}{\psi_1(t, w(t), z(t))} = \mathtt{I}g(t) + a_1 t^{-1},$$
$$\frac{z(t)}{\psi_2(t, w(t), z(t))} = \mathtt{I}h(t) + b_1 t^{-1},$$

where $a_1, b_1 \in \mathbb{R}$. Now by Lemma 4, we have

$$w(t) = \psi_1(t, w(t), z(t)) \Big[\mathtt{I}^{+} g(t) + \frac{a_1 \Gamma()}{\Gamma(+)} t^{+-1} + a_2 \Big], \tag{5}$$

$$z(t) = \psi_2(t, w(t), z(t)) \Big[\mathtt{I}^{+} h(t) + \frac{b_1 \Gamma()}{\Gamma(+)} t^{+-1} + b_2 \Big], \tag{6}$$

where $a_2, b_2 \in \mathbb{R}$. Using boundary conditions $w(0) = w(1) = z(0) = z(1) = 0$, we obtain $a_2 = b_2 = 0$,

$$a_1 = -\frac{\Gamma(+)}{\Gamma()} \int_0^1 (1-s)^{+-1} g(s)\, ds,$$

and

$$b_1 = -\frac{\Gamma(+)}{\Gamma()} \int_0^1 (1-s)^{+-1} h(s)\, ds.$$

Substituting the values of $a_j, b_j, j = 1, 2$ in (5) and (6), we get (3) and (4). □

3. Existence and uniqueness of solutions to the sequential coupled hybrid system

We will use the standard fixed point theorems, to study the fractional hybrid system (1). In this regard, we define the space

$$\mathtt{W} \times \mathtt{Z} = \{(w, z) : w, z \in C([0,1], \mathbb{R})\},$$

endowed with the norm $\|(w,z)\|_{\mathtt{W}\times\mathtt{Z}} = \|w\| + \|z\|$, where $\|w\| = \sup\{|w(t)| : t \in [0,1]\}$. It is clear that $(\mathtt{W}\times\mathtt{Z}, \|.\|_{\mathtt{W}\times\mathtt{Z}})$ is a Banach space. Define an operator $\mathtt{O} : \mathtt{W}\times\mathtt{Z} \to \mathtt{W}\times\mathtt{Z}$ by

$$\mathtt{O}(w,z)(t) = (\mathtt{O}_1(w,z)(t),\ \mathtt{O}_2(w,z)(t)),\ t \in [0,1],$$

where

$$\mathtt{O}_1(w,z)(t) = \psi_1(t, w(t), z(t))\, \mathfrak{N}_1(t), \tag{7}$$

in which

$$\mathfrak{N}_1(t) = \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{+-1}}{\Gamma(+)} \varphi_i(s, w(s), z(s))\, ds - \sum_{i=1}^{k} \int_0^1 \frac{[t(1-s)]^{+-1}}{\Gamma(+)} \varphi_i(s, w(s), z(s))\, ds,$$

and

$$\mathtt{O}_2(w,z)(t) = \psi_2(t, w(t), z(t))\, \mathfrak{N}_2(t), \tag{8}$$

in which

$$\mathfrak{N}_2(t) = \sum_{i=1}^{k} \int_0^t \frac{(t-s)^{+-1}}{\Gamma(+)} \phi_i(s, w(s), z(s))\, ds - \sum_{i=1}^{k} \int_0^1 \frac{[t(1-s)]^{+-1}}{\Gamma(+)} \phi_i(s, w(s), z(s))\, ds \Big].$$

Now, we prove the existence of solutions of the fractional hybrid system (1) by applying Leray-Schauder nonlinear alternative [39].

Lemma 6. *(Leray-Schauder alternative). Let $F : E \to E$ be a completely continuous operator (i.e.,a map that restricted to any bounded set in E is compact). Let*

$$(F) = \{u \in E : u = \lambda F(u) \text{ for some } 0 < \lambda < 1\}.$$

Then either the set (F) is unbounded, or F has at least one fixed point.

Theorem 1. *Assume that hypotheses $(H_j)_{j=2,3}$ are satisfied. Furthermore, assume that*

$$\sum_{i=1}^{k} \left(\frac{\Pi_1 \gamma_i}{\Gamma(++1)} + \frac{\Pi_2 \gamma_i'}{\Gamma(++1)} \right) < \frac{1}{2},$$

and

$$\sum_{i=1}^{k} \left(\frac{\Pi_1 \omega_i}{\Gamma(++1)} + \frac{\Pi_2 \omega_i'}{\Gamma(++1)} \right) < \frac{1}{2}.$$

Then the system (1) has at least one solution on $[0,1]$.

Proof. In the first step, we show that the operator $\mathtt{O} : \mathtt{W}\times\mathtt{Z} \to \mathtt{W}\times\mathtt{Z}$ is completely continuous. By continuity of the functions $\psi_j, \varphi_i, \phi_i, j = 1, 2, i = 1, 2, ..., k$, it follows that the operator $\mathtt{O}$ is continuous.

Let $\Sigma \subset \mathtt{W}\times\mathtt{Z}$ be bounded. Then we can find positive constants $A_i, B_i, i = 1, 2, ..., k$ such that

$$|\varphi_i(t, w(t), z(t))| \le A_i, |\varphi_i(t, w(t), z(t))| \le B_i$$

for all $(w,z) \in \Sigma$. Then for any $(w,z) \in \Sigma$ we have

$$\begin{aligned}
&\|\mathtt{O}_1(w,z)\| \\
&\le \Pi_1 \Big[\frac{1}{\Gamma(+)} \sum_{i=1}^{k} \int_0^t (t-s)^{+-1} |\varphi_i(s, w(s), z(s))|\, ds \\
&\quad + \frac{t^{+-1}}{\Gamma(+)} \sum_{i=1}^{k} \int_0^1 (1-s)^{+-1} |\varphi_i(s, w(s), z(s))|\, ds \Big] \\
&\le \sum_{i=1}^{k} \frac{2\Pi_1 A_i}{\Gamma(++1)},
\end{aligned}$$

which yields

$$\|\mathtt{O}_1(w,z)\| \leq \sum_{i=1}^{k} \frac{2\Pi_1 A_i}{\Gamma(+ + 1)} < +\infty. \quad (9)$$

Also,

$$\|\mathtt{O}_2(w,z)\| \leq \sum_{i=1}^{k} \frac{2\Pi_2 B_i}{\Gamma(+ + 1)} < +\infty. \quad (10)$$

Hence, by (9) and (10), we deduce that the operator $\mathtt{O}$ is uniformly bounded.

Next, we show that $\mathtt{O}$ is equicontinuous. For all $0 \leq t_2 < t_1 \leq 1$, we have

$$\begin{aligned} &\left|\mathtt{O}_1(w,z)(t_1) - \mathtt{O}_1(w,z)(t_2)\right| \\ &\leq \sum_{i=1}^{k} \frac{\Pi_1 A_i}{\Gamma(+ + 1)}\Big(\left[(t_1 - t_2)^{+} + \left|t_1^{+} - t_2^{+}\right|\right] \\ &\quad + \left|t_1^{+-1} - t_2^{+-1}\right| \Big), \end{aligned} \quad (11)$$

and

$$\begin{aligned} &|\mathtt{O}_2(w,z)(t_1) - \mathtt{O}_2(w,z)(t_2)| \\ &\leq \sum_{i=1}^{k} \frac{\Pi_2 B_i}{\Gamma(+ + 1)}\Big(\Big[(t_1 - t_2)^{+} + \left|t_1^{+} - t_2^{+}\right|\Big] \\ &\quad + |t_1^{+-1} - t_2^{+-1}|\Big). \end{aligned} \quad (12)$$

From (11) and (12), $\|\mathtt{O}(w,z)(t_1) - \mathtt{O}(w,z)(t_2)\|_{\mathtt{W}\times\mathtt{Z}} \to 0$ as $t_2 \to t_1$. Thus, by using the Arzela-Ascoli theorem one can conclude that the operator $\mathtt{O} : \mathtt{W}\times\mathtt{Z} \to \mathtt{W}\times\mathtt{Z}$ is completely continuous.

Finally, it will be verified that the set

$$\Psi = \left\{ (w,z) \in \mathtt{W}\times\mathtt{Z}, (w,z) = \mathtt{O}(w,z), 0 \leq\leq 1 \right\}$$

is bounded. Let $(w,z) \in \Psi$. Then, for each $t \in [0,1]$, we can write

$$w(t) = \mathtt{O}_1(w,z)(t) \quad \text{and} \quad z(t) = \mathtt{O}_2(w,z)(t).$$

Then, we have

$$|w(t)| \leq \Pi_1 |\mathfrak{N}_1(t)|,$$

and

$$|z(t)| \leq \Pi_2 |\mathfrak{N}_2(t)|.$$

From (H_3), we obtain

$$|w(t)| \leq \frac{2\Pi_1}{\Gamma(+ + 1)} \left(\lambda_0 + \lambda_1 |w(t)| + \lambda_2 |z(t)|\right),$$

and

$$|z(t)| \leq \frac{2\Pi_2}{\Gamma(+ + 1)} \left(\gamma_0 + \gamma_1 |w(t)| + \gamma_2 |z(t)|\right).$$

Hence, we have

$$\|w\| \leq \sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)} \left(\lambda_i + \gamma_i \|w\| + \omega_i \|z\|\right),$$

and

$$\|z\| \leq \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)} \left(\lambda_i' + \gamma_i' \|w\| + \omega_i' \|z\|\right),$$

which imply that

$$\begin{aligned} &\|w\| + \|z\| \\ &\leq \sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)}\lambda_i + \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)}\lambda_i' \\ &+ \left(\sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)}\gamma_i + \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)}\gamma_i'\right)\|w\| \\ &+ \left(\sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)}\omega_i + \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)}\omega_i'\right)\|z\|. \end{aligned}$$

Consequently,

$$\begin{aligned} \|(w,z)\|_{\mathtt{W}\times\mathtt{Z}} \leq \frac{1}{G}\Big[&\sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)}\lambda_i \\ &+ \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)}\lambda_i'\Big], \end{aligned}$$

for all $t \in [0,1]$, where $G = \min\{\mathfrak{d}_1, \mathfrak{d}_2\}$, in which

$$\mathfrak{d}_1 = 1 - \left(\sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)}\gamma_i + \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)}\gamma_i'\right),$$

and

$$\mathfrak{d}_2 = 1 - \left(\sum_{i=1}^{k} \frac{2\Pi_1}{\Gamma(+ + 1)}\omega_i + \sum_{i=1}^{k} \frac{2\Pi_2}{\Gamma(+ + 1)}\omega_i'\right).$$

This shows that the set Ψ is bounded. Hence all the conditions of Lemma 6 are satisfied and consequently the operator $\mathtt{O}$ has at least one fixed point, which corresponds to a solution of the system (1). This completes the proof. □

In the next result, we establish the existence of uniqueness solutions to the fractional hybrid system (1) by using Banach's fixed point theorem.

Theorem 2. *Assume that $(H_j)_{j=1,2}$ hold and that*

$$\begin{aligned} &\sum_{i=1}^{k} \frac{\pi_i}{\Gamma(+ + 1)} < \frac{1}{4\Pi_1}, i = 1, 2, \cdots, k, \\ &\sum_{i=1}^{k} \frac{\vartheta_i}{\Gamma(+ + 1)} < \frac{1}{4\Pi_2}, i = 1, 2, \cdots, k. \end{aligned} \quad (13)$$

Then the problem (1) *has a unique solution on* $[0,1]$.

Proof. Define $\sup_{t\in[0,1]} |\varphi_i(t,0,0)| = \Lambda_i < \infty$ and $\sup_{t\in[0,1]} |\phi_i(t,0,0)| = \nabla_i < \infty$, $i = 1, 2, ..., k$ such that $\max\{\wp_1, \wp_2\} \leq, i = 1, 2, ..., k$, where

$$\wp_1 = \sum_{i=1}^{k} \frac{\Pi_1 \Lambda_i}{\Gamma(+ + 1)} \left[\frac{1}{4} - \sum_{i=1}^{k} \frac{\Pi_1 \pi_i}{\Gamma(+ + 1)} \right]^{-1},$$

and

$$\wp_2 = \sum_{i=1}^{k} \frac{\Pi_2 \nabla_i}{\Gamma(+ + 1)} \left[\frac{1}{4} - \sum_{i=1}^{k} \frac{\Pi_2 \vartheta_i}{\Gamma(+ + 1)} \right]^{-1}.$$

Firstly, we show that $\mathtt{OB} \subset \mathtt{B}$, where $\mathtt{B} = \left\{ (w, z) \in \mathtt{W} \times \mathtt{Z} : \|(w, z)\|_{\mathtt{W}\times\mathtt{Z}} \leq \right\}$. For all $(w, z) \in \mathtt{B}$ and $t \in [0, 1]$, we have

$$\begin{aligned} &|\varphi_i (t, w(t), z(t))| \\ &\leq |\varphi_i (t, w(t), z(t)) - \varphi_i (t, 0, 0)| + |\varphi_i (t, 0, 0)| \\ &\leq \pi_i (|w(t)| + |z(t)|) + \Lambda_i \leq \pi_i (\|w\| + \|z\|) + \Lambda_i \\ &\leq \pi_i \|(w, z)\| + \Lambda_i \leq \pi_i + \Lambda_i, \ i = 1, 2, ..., k. \end{aligned}$$

Similarly, we have

$$\begin{aligned} &|\phi_i (t, w(t), z(t))| \\ &\leq |\phi_i (t, w(t), z(t)) - \phi_i (t, 0, 0)| + |\phi_i (t, 0, 0)| \\ &\leq \vartheta_i (|w(t)| + |z(t)|) + \nabla_i \leq \vartheta_i (\|w\| + \|z\|) + \nabla_i \\ &\leq \vartheta_i \|(w, z)\| + \nabla_i \leq \vartheta_i + \nabla_i, \ i = 1, 2, ..., k, \end{aligned}$$

Using (3), we can write

$$\begin{aligned} |\mathtt{O}_1 (w, z)(t)| &\leq \Pi_1 \sup_{t\in[0,1]} \left\{ \mathfrak{N}_1(t) \right\} \\ &\leq \frac{\sum_{i=1}^{k} 2\Pi_1 \pi_i}{\Gamma(+ + 1)} + \frac{\sum_{i=1}^{k} 2\Pi_1 \nabla_i}{\Gamma(+ + 1)}, \end{aligned}$$

which implies that

$$\begin{aligned} &\|\mathtt{O}_1(w, z)\| \\ &\quad\leq \sum_{i=1}^{k} \frac{\Pi_1 \pi_i}{\Gamma(+ + 1)} + \sum_{i=1}^{k} \frac{\Pi_1 \Lambda_i}{\Gamma(+ + 1)} \\ &\quad\leq \frac{}{4}. \end{aligned}$$

Also, by (3), we have

$$\begin{aligned} &\|\mathtt{O}_2 (w, z)\| \\ &\quad\leq \sum_{i=1}^{k} \frac{\Pi_2 \vartheta_i}{\Gamma(+ + 1)} + \sum_{i=1}^{k} \frac{\Pi_2 \nabla_i}{\Gamma(+ + 1)} \\ &\quad\leq \frac{}{4}. \end{aligned}$$

From the definition of $\| \cdot \|_{\mathtt{W}\times\mathtt{Z}}$, we have

$$\begin{aligned} &\|O(w, z)\|_{\mathtt{W}\times\mathtt{Z}} \\ &\leq \sum_{i=1}^{k} \left(\frac{\Pi_1 \pi_i}{\Gamma(+ + 1)} + \frac{\Pi_1 \vartheta_i}{\Gamma(+ + 1)} \right) \\ &\qquad + \sum_{i=1}^{k} \left(\frac{\Pi_2 \Lambda_i}{\Gamma(+ + 1)} + \frac{\Pi_2 \nabla_i}{\Gamma(+ + 1)} \right) \\ &\leq \frac{}{2}, \end{aligned}$$

which implies that $OB \subset B$. Next, for $(w_1, z_1), (w_2, z_2) \in B$ and for each $t \in [0, 1]$, we have

$$\begin{aligned} &\left| \mathtt{O}_1(w_1, z_1)(t) - \mathtt{O}_1(w_2, z_2)(t) \right| \\ &\quad\leq \Pi_1 \sup_{t\in[0,1]} \left\{ \sum_{i=1}^{k} \int_0^t \frac{(t - s)^{+-1}}{\Gamma(+)} \Upsilon(s) ds \right. \\ &\qquad \left. + t^{+-1} \sum_{i=1}^{k} \int_0^1 \frac{(1 - s)^{+-1}}{\Gamma(+)} \Upsilon_2(s) ds \right\}. \end{aligned}$$

where

$$\Upsilon_2(s) = |\varphi_i (s, w_1 (s), z_1 (s)) - \varphi_i (s, w_2 (s), z_2 (s))|,$$
$$\Upsilon_2(s) = |\varphi_i (s, w_1 (s), z_1 (s)) - \varphi_i (s, w_2 (s), z_2 (s))|.$$

From (H_1), we can write

$$\begin{aligned} &\|\mathtt{O}_1 (w_1, z_1) - \mathtt{O}_1 (w_2, z_2)\| \\ &\quad\leq \sum_{i=1}^{k} \frac{2\Pi_1 \pi_i}{\Gamma(+ + 1)} \|(w_1 - w_2, z_1 - z_2)\|_{\mathtt{W}\times\mathtt{Z}}. \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} &\|\mathtt{O}_2 (w_1, z_1) - \mathtt{O}_2 (w_2, z_2)\| \\ &\quad\leq \sum_{i=1}^{k} \frac{2\Pi_2 \vartheta_i}{\Gamma(+ + 1)} \|(w_1 - w_2, z_1 - z_2)\|_{\mathtt{W}\times\mathtt{Z}}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\|\mathtt{O} (w_1, z_1) - \mathtt{O} (w_2, z_2)\|_{\mathtt{W}\times\mathtt{Z}} \\ &= \|\mathtt{O}_1 (w_1, z_1) - \mathtt{O}_1 (w_2, z_2)\| \\ &\qquad + \|\mathtt{O}_2 (w_1, z_1) - \mathtt{O}_2 (w_2, z_2)\| \\ &\leq \left[\sum_{i=1}^{k} \left(\frac{2\Pi_1 \pi_i}{\Gamma(+ + 1)} + \frac{2\Pi_2 \vartheta_i}{\Gamma(+ + 1)} \right) \right] \\ &\qquad \times \left\|(w_1 - w_2, z_1 - z_2)\right\|_{\mathtt{W}\times\mathtt{Z}}. \end{aligned}$$

Thanks to (13), we conclude that $\mathtt{O}$ is a contraction mapping. Hence, by the Banach fixed point theorem, there exists a unique fixed point which is a solution of system (1). This completes the proof. □

4. Stability in Ulam-Hyers-Rassias sense

In the following section, we consider the Ulam's type stability of the fractional hybrid system (1). For $t \in [0,1]$, we give the following inequalities:

$$\begin{cases} \left| {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{w_1(t)}{\psi_1(t,w_1(t),z_1(t))}\right]\right] - \sum_{i=1}^{k}\varphi_i(t,w_1(t),z_1(t))\right| \le d_1, \\ \left| {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{z_1(t)}{\psi_2(t,w_1(t),z_1(t))}\right]\right] - \sum_{i=1}^{k}\phi_i(t,w_1(t),z_1(t))\right| \le d_2, \end{cases} \tag{14}$$

and

$$\begin{cases} \left| {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{w_1(t)}{\psi_1(t,w_1(t),z_1(t))}\right]\right] - \sum_{i=1}^{k}\varphi_i(t,w_1(t),z_1(t))\right| \le d_1 u(t), \\ \left| {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{z_1(t)}{\psi_2(t,w_1(t),z_1(t))}\right]\right] - \sum_{i=1}^{k}\phi_i(t,w_1(t),z_1(t))\right| \le d_2 u(t), \end{cases} \tag{15}$$

where $d_j, j=1,2$ are positive reals numbers and $u:[0,1]\to\mathbb{R}^+$, is continuous function.

Definition 4. *[40] System (1) is Ulam-Hyers stable if there exists a real number $\rho_{\varphi_i,\phi_i} = (\rho_{\varphi_i},\rho_{\phi_i}) > 0, i=1,2,...,k$ such that for each $d=\max(d_1,d_2)>0$ and for each solution $(w_1,z_1)\in \mathtt{W}\times\mathtt{Z}$ of the inequality (14) there exists a solution $(w,z)\in\mathtt{W}\times\mathtt{Z}$ of the system (1) with*

$$|(w_1(t)-w(t), z_1(t)-z(t))| \le \rho_{\varphi_i,\phi_i} d,$$

for $t\in[0,1]$, $i=1,2,\cdot,k$.

Definition 5. *[40] System (1) is Ulam-Hyers-Rassias stable with respect to $u\in C([0,1],\mathbb{R})$ if there exists a real number $\varsigma_{\varphi_i,\phi_i,u} = (\varsigma_{\varphi_i,u},\varsigma_{\phi_i,u}) > 0$ such that for each $d=\max(d_1,d_2)>0$ and for each solution $(w_1,z_1)\in\mathtt{W}\times\mathtt{Z}$ of the inequality (15) there exists a solution $(w,z)\in\mathtt{W}\times\mathtt{Z}$ of the system (1) with*

$$|(w_1(t)-w(t), z_1(t)-z(t))| \le \varsigma_{\varphi_i,\phi_i,u} d u(t),$$

for $t\in[0,1]$, $i=1,2,\cdot,k$.

Theorem 3. *Assume that $(H_j)_{j=1,2}$ hold. If*

$$\begin{cases} \sum_{i=1}^{k}\dfrac{\pi_i}{\Gamma(+ + 1)} < \dfrac{1}{\Pi_1}, \\ \sum_{i=1}^{k}\dfrac{\vartheta_i}{\Gamma(+ + 1)} < \dfrac{1}{\Pi_2}, \end{cases} \tag{16}$$

then the problem (1) is Ulam-Hyers stable.

Proof. Let $(w_1,z_1)\in\mathtt{W}\times\mathtt{Z}$ be a solution of the inequality (14) and let $(w,z)\in\mathtt{W}\times\mathtt{Z}$ be the unique solution of the system

$$\begin{cases} {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{w(t)}{\psi_1(t,w(t),z(t))}\right]\right] = \sum_{i=1}^{k}\varphi_i(t,w(t),z(t)), \\ {}^{\mathtt{RL}}\mathtt{D}\left[{}^{\mathtt{C}}\mathtt{D}\left[\dfrac{z(t)}{\psi_2(t,w(t),z(t))}\right]\right] = \sum_{i=1}^{k}\phi_i(t,w(t),z(t)), \\ w(0)=w_1(0), w(1)=w_1(1), \\ z(0)=z_1(0), z(1)=z_1(1). \end{cases}$$

By Lemma 5, we have

$$w(t) = \psi_1(t,w(t),z(t))\left[\sum_{i=1}^{k}\mathtt{I}^{+}g_i^{w}(t) + \frac{a_1\Gamma()}{\Gamma(+)}t^{+-1} + a_2\right],$$

and

$$z(t) = \psi_2(t,w(t),z(t))\left[\sum_{i=1}^{k}\mathtt{I}^{+}h_i^{z}(t) + \frac{b_1\Gamma()}{\Gamma(+)}t^{+-1} + b_2\right],$$

such that

$$g_i^{w}(t) = \varphi_i(t,w(t),z(t)),\ i=1,2,\cdots,k,$$
$$h_i^{z}(t) = \phi_i(t,w(t),z(t)),\ i=1,2,\cdots,k.$$

Integrating (14), we obtain

$$\left| w_1(t) - \psi_1(t,w(t),z(t))\left[\sum_{i=1}^{k}\mathtt{I}^{+}g_i^{w}(t) + \frac{a_3\Gamma()}{\Gamma(+)}t^{+-1} + a_4\right]\right| \le \frac{d_1 t^{+}}{\Gamma(+ + 1)} \le \frac{d_1}{\Gamma(+ + 1)},$$

and

$$\left|z_1(t) - \psi_2(t, w(t), z(t))\left[\sum_{i=1}^{k} \mathtt{I}^{+} h_i^z(t) + \frac{b_3\Gamma()}{\Gamma(+)} t^{+-1} + b_4\right]\right| \leq \frac{d_2 t^{+}}{\Gamma(++1)} \leq \frac{d_2}{\Gamma(++1)}.$$

From $(H_j)_{j=1,2}$, we have

$$\begin{aligned}
&|w_1(t) - w(t)| \\
&\leq \left|w_1(t) - \psi_1(t, w(t), z(t))\left[\sum_{i=1}^{k} \mathtt{I}^{+} g_i^w(t) + \frac{a_3\Gamma()}{\Gamma(+)} t^{+-1} + a_4\right]\right| \\
&\quad + |\psi_1(t, w(t), z(t))| \sum_{i=1}^{k} \mathtt{I}^{+} |g_i^{w_1}(t) - g_i^w(t)| \\
&\leq \frac{d_1}{\Gamma(++1)} + \sum_{i=1}^{k} \Pi_1 \mathtt{I}^{+} |g_i^{w_1}(t) - g_i^w(t)|,
\end{aligned}$$

this implies that

$$|w_1(t) - w(t)| \leq \frac{d_1}{\Gamma(++1)} + \sum_{i=1}^{k} \frac{\Pi_1 \pi_i}{\Gamma(++1)}\Big[|w_1(t) - w(t)| + |z_1(t) - z(t)|\Big].$$

Similarly, we get

$$|z_1(t) - z(t)| \leq \frac{d_2}{\Gamma(++1)} + \sum_{i=1}^{k} \frac{\Pi_2 \vartheta_i}{\Gamma(++1)}\Big[|w_1(t) - w(t)| + |z_1(t) - z(t)|\Big].$$

Thus,

$$|(w_1(t), z_1(t)) - (w(t), z(t))| \leq \frac{\frac{1}{\Gamma(++1)} + \frac{1}{\Gamma(++1)}}{\min\{\mathfrak{r}_1, \mathfrak{r}_2\}} d := \rho_{\varphi_i, \phi_i} d,$$

where

$$\mathfrak{r}_1 = \frac{1}{\Pi_1} - \sum_{i=1}^{k} \frac{\pi_i}{\Gamma(++1)},$$

$$\mathfrak{r}_2 = \frac{1}{\Pi_2} - \sum_{i=1}^{k} \frac{\vartheta_i}{\Gamma(++1)}.$$

Hence the system (1) is Ulam-Hyers stable. □

Theorem 4. *Assume that $(H_j)_{j=1,2}$ and (16) hold. Suppose there exist $\upsilon_{1u} > 0, \upsilon_{2u} > 0$ such that*

$$\mathtt{I}^{+} u(t) \leq \upsilon_{1u} u(t), \mathtt{I}^{+} u(t) \leq \upsilon_{2u} u(t), t \in [0,1], \tag{17}$$

where $u \in C([0,1], \mathbb{R}_+)$ is nondecreasing. Then the system (1) is Ulam-Hyers-Rassias stable.

Proof. Let $(w_1, z_1) \in \mathtt{W} \times \mathtt{Z}$ is a solution of the inequality (15) and let us assume that $(w, z) \in \mathtt{W} \times \mathtt{Z}$ is a solution of system (1). Thus, we have

$$\begin{aligned}
w(t) &= \psi_1(t, w(t), z(t))\left[\sum_{i=1}^{k} \mathtt{I}^{+} g_i^w(t) + \frac{a_1\Gamma()}{\Gamma(+)} t^{+-1} + a_2\right], \\
z(t) &= \psi_2(t, w(t), z(t))\left[\sum_{i=1}^{k} \mathtt{I}^{+} h_i^z(t) + \frac{b_1\Gamma()}{\Gamma(+)} t^{+-1} + b_2\right],
\end{aligned}$$

From inequality (15), we have

$$\left|w_1(t) - \psi_1(t, w(t), z(t))\left[\sum_{i=1}^{k} \mathtt{I}^{+} g_i^w(t) + \frac{a_3\Gamma()}{\Gamma(+)} t^{+-1} + a_4\right]\right| \leq d_1 \mathtt{I}^{+} u(t),$$

and

$$\left|z_1(t) - \psi_2(t, w(t), z(t))\left[\sum_{i=1}^{k} \mathtt{I}^{+} h_i^z(t) + \frac{b_3\Gamma()}{\Gamma(+)} t^{+-1} + b_4\right]\right| \leq d_2 \mathtt{I}^{+} u(t).$$

Now, using $(H_j)_{j=1,2}$ and (17), we get

$$|w_1(t) - w(t)| \leq d_1 \upsilon_{1u} u(t) + \sum_{i=1}^{k} \frac{\Pi_1 \pi_i}{\Gamma(++1)}(|w_1(t) - w(t)| + |z_1(t) - z(t)|),$$

and

$$|z_1(t) - z(t)| \leq d_2 \upsilon_{2u} u(t) + \sum_{i=1}^{k} \frac{\Pi_2 \vartheta_i}{\Gamma(++1)}(|w_1(t) - w(t)| + |z_1(t) - z(t)|).$$

Consequently,

$$|(w_1(t), z_1(t)) - (w(t), z(t))| \leq \frac{\upsilon_{1u} + \upsilon_{2u}}{\min\{\Bbbk_1, \Bbbk_2\}} du(t) := \varsigma_{\varphi_i, \phi_i, u} du(t),$$

where

$$\Bbbk_1 = 1 - \sum_{i=1}^{k} \frac{\Pi_1 \pi_i}{\Gamma(++1)},$$

and

$$\Bbbk_2 = 1 - \sum_{i=1}^{k} \frac{\Pi_2 \vartheta_i}{\Gamma(++1)}.$$

Hence the system (1) is stable in Ulam-Hyers-Rassias sense. □

5. Application

To illustrate our main results, we treat the following example.

Example 1. *Consider the following fractional hybrid system:*

$$\begin{cases} {}^{RL}D^{\frac{4}{5}}\left[{}^{C}D^{\frac{2}{3}}\left[\dfrac{w(t)}{\frac{\sin w(t)+1}{15} + 1 + \frac{1}{13}e^{-t^2}\cos z(t)}\right]\right] \\ = \dfrac{\cos(2\pi w(t))}{60\pi} + \dfrac{|z(t)|}{30(1+|z(t)|)} + \arctan\left(t^2+2t+1\right) \\ + \dfrac{|w(t)|}{32(e^t+3\sqrt{\pi})(1+|w(t)|)} + \dfrac{\sin^2 z(t)}{16(5t^2+2(1+3\sqrt{\pi}))} \\ + \dfrac{\ln(1+t)}{3}, \\ {}^{RL}D^{\frac{5}{6}}\left[{}^{C}D^{\frac{3}{4}}\left[\dfrac{z(t)}{\frac{3}{7}t\cos w(t) + \dfrac{1}{7+z(t)}}\right]\right] \\ = \dfrac{\cos(w(t)+z(t))}{19(\ln(1+t)+2\sqrt{\pi})} + \dfrac{\left(1+2e^{1+t}\right)}{2} \\ + \dfrac{|w(t)|}{3(\pi t+3)^2(1+|w(t)|)} + \dfrac{\tan^{-1} z(t)}{27} + \sinh\left(1+13e^t\right), \\ w(0)=w(1)=0,\ z(0)=z(1)=0, \end{cases} \tag{18}$$

and the following inequalities

$$\begin{cases} \left| {}^{RL}D^{\frac{4}{5}}\left[{}^{C}D^{\frac{2}{3}}\left[\dfrac{w(t)}{\psi_1(t,w(t),z(t))}\right]\right] - \displaystyle\sum_{i=1}^{2}\varphi_i(t,w(t),z(t)) \right| \leq d_1, \\ \left| {}^{RL}D^{\frac{5}{6}}\left[{}^{C}D^{\frac{3}{4}}\left[\dfrac{z(t)}{\psi_2(t,w(t),z(t))}\right]\right] - \displaystyle\sum_{i=1}^{2}\phi_i(t,w(t),z(t)) \right| \leq d_2, \end{cases}$$

and

$$\begin{cases} \left| {}^{RL}D^{\frac{4}{5}}\left[{}^{C}D^{\frac{2}{3}}\left[\dfrac{w(t)}{\psi_1(t,w(t),z(t))}\right]\right] - \displaystyle\sum_{i=1}^{2}\varphi_i(t,w(t),z(t)) \right| \leq d_1 u(t), \\ \left| {}^{RL}D^{\frac{5}{6}}\left[{}^{C}D^{\frac{3}{4}}\left[\dfrac{z(t)}{\psi_2(t,w(t),z(t))}\right]\right] - \displaystyle\sum_{i=1}^{2}\phi_i(t,w(t),z(t)) \right| \leq d_2 u(t), \end{cases}$$

where

$$\varphi_1(t,w,z) = \frac{\cos(2\pi w)}{60\pi} + \frac{|z|}{30(1+|z|)} + \arctan\left(t^2+2t+1\right),$$

$$\varphi_2(t,w,z) = \frac{|w|}{32(e^t+3\sqrt{\pi})(1+|w|)} + \frac{\ln(1+t)}{3} + \frac{\sin^2 z}{16(5t^2+2(1+3\sqrt{\pi}))},$$

$$\phi_1(t,w,z) = \frac{\cos(w+z)}{19(\ln(1+t)+2\sqrt{\pi})} + \frac{\left(1+2e^{1+t}\right)}{2},$$

$$\phi_2(t,w,z) = \frac{|w|}{3(\pi t+3)^2(1+|w|)} + \frac{\tan^{-1} z}{27} + \sinh\left(1+13e^t\right),$$

and

$$\psi_1(t,w,z) = \frac{1}{5}(\sin w + 1) + 1 + \frac{1}{13}e^{-t^2}\cos z,$$

$$\psi_2(t,w,z) = \frac{2}{7}t\cos w + \frac{1}{7+z}.$$

For $(w_i, z_i) \in \mathbb{R}^2, i = 1,2$ *and* $t \in [0,1]$, *we have*

$$|\varphi_1(t,w_1,z_1) - \varphi_1(t,w_2,z_2)| \leq \frac{1}{30}(|w_1-w_2| + |z_1-z_2|),$$

$$|\varphi_2(t,w_1,z_1) - \varphi_2(t,w_2,z_2)| \leq \frac{1}{32(1+3\sqrt{\pi})}(|w_1-w_2| + |z_1-z_2|),$$

$$|\phi_2(t,w_1,z_1) - \phi_2(t,w_2,z_2)| \leq \frac{1}{38\sqrt{\pi}}(|w_1-w_2| + |z_1-z_2|),$$

$$|\phi_1(t,w_1,z_1) - \phi_2(t,w_2,z_2)| \leq \frac{1}{27}(|w_1-w_2| + |z_1-z_2|),$$

and

$$|\psi_1(t,w,z)| \leq \frac{27}{65}, \quad |\psi_2(t,w,z)| \leq \frac{3}{7}.$$

So, we take $\pi_1 = \frac{1}{30}, \pi_2 = \frac{1}{32}$, $\vartheta_1 = \frac{1}{32(1+3\sqrt{\pi})}$, $\vartheta_2 = \frac{1}{38\sqrt{\pi}}$, $\Pi_1 = \frac{27}{65}$ *and* $\Pi_2 = \frac{3}{7}$.

Hence, we obtain

$$\sum_{i=1}^{2}\frac{\pi_i}{\Gamma(++1)}=4.972\,2\times10^{-2}<\frac{1}{4\Pi_1}=0.103\,85,$$

and

$$\sum_{i=1}^{2}\frac{\vartheta_i}{\Gamma(++1)}=1.401\,9\times10^{-2}<\frac{1}{4\Pi_2}=0.107\,14.$$

By Theorem 1, we conclude that the system (18) has a unique solution. And from Theorem 3 we deduce that (18) is Ulam-Hyers stable with

$$|(w_2(t),z_2(t))-(w_1(t),z_1(t))|\leq 0.365\,68d,$$

for $t\in[0,1]$, $d>0$. *Let* $u(t)=t^{\frac{\sqrt{5}}{2}}$, *then*

$$\mathrm{I}^{\frac{4}{5}+\frac{2}{3}}u_1(t)=\mathrm{I}^{\frac{4}{5}+\frac{2}{3}}t^{\frac{\sqrt{5}}{2}}\leq\frac{\Gamma\left(\frac{\sqrt{5}+2}{2}\right)}{\Gamma\left(\frac{\sqrt{5}}{2}+\frac{37}{15}\right)}t^{\frac{\sqrt{5}}{2}}=\upsilon_{1u}u(t),$$

and

$$\mathrm{I}^{\frac{5}{6}+\frac{3}{4}}u_2(t)=\mathrm{I}^{\frac{5}{6}+\frac{3}{4}}t^{\frac{\sqrt{5}}{2}}\leq\frac{\Gamma\left(\frac{\sqrt{5}+2}{2}\right)}{\Gamma\left(\frac{\sqrt{5}}{2}+\frac{31}{12}\right)}t^{\frac{\sqrt{5}}{2}}=\upsilon_{2u}u(t).$$

Thus, the condition (17) of Theorem 4 is satisfied with $u(t)=t^{\frac{\sqrt{5}}{2}}$ *and* $\upsilon_{1u}=0.289\,01$, $\upsilon_{2u}=0.252\,74$. *Hence from Theorem 4, the system (18) is Ulam-Hyers-Rassias stable with*

$$|(w_1(t),z_1(t))-(w(t),z(t))|\leq 0.948\,04dt^{\frac{\sqrt{5}}{2}},$$

for $t\in[0,1]$, $d>0$.

Remark 1. *One can easily figure out that problem* (18) *is not commented by any of the relevant existing results in the literature.*

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