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RESEARCH ARTICLE

A new approach on approximate controllability of Sobolev-type Hilfer fractional differential equations

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ABSTRACT

The approximate controllability of Sobolev-type Hilfer fractional control differential systems is the main emphasis of this paper. We use fractional calculus, Gronwall's inequality, semigroup theory, and the Cauchy sequence to examine the main results for the proposed system. The application of well-known fixed point theorem methodologies is avoided in this paper. Finally, a fractional heat equation is discussed as an example.

1. Introduction

Differential systems of fractional order are found to be useful models for a variety of physical, biological, and engineering challenges. As a result, they have gotten greater attention from researchers in the last two decades. Fractional derivatives are a stronger tool for illustrating memory and hereditary features. As a result, they've found widespread use in physics, electrodynamics, economics, aerodynamics, control theory, viscoelasticity, and heat conduction. In recent years, significant advances in the theory and applications of fractional systems have been made, one can review the books [1–4]. The notation of exact and approximate controllability is useful in analysis and design control frameworks. In [5] authors studied the existence and controllability of fractional integro-differential system of order $1 < r < 2$ via measure of noncompactness using fixed point theory approach. In [6–13] Anurag et al. studied the controllability of semilinear deterministic and stochastic systems of integral and fractional order with several important extensions using different approaches. The numerical model of numerous physical phenomena, such as the movement of liquid through split rocks, thermodynamics, and so on, is usually Sobolev-type. (see [14–17]).

Another type of fractional order derivative introduced by Hilfer [18] is the Caputo fractional and Riemann-Liouville derivative. Several authors have focused on the Hilfer fractional derivative including [19–27] for the existence and controllability of deterministic and stochastic fractional order systems. Many academics have recently considered the exact and approximate controllability of

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systems characterized by impulsive functional inclusions, integro-differential equations, semilinear functional equations, neutral functional differential equations, and evolution inclusions, to name a few examples, see [23, 24, 27] and references in that. In [28–34] Ravi et. al. studied the existence, uniqueness, controllability, and optimal control of fractional differential control systems and their real-life mathematical applications using various types of approaches.

Consider the following Sobolev-type Hilfer fractional control system as below.

$$yD^{\wp,\varpi}_{0+}[Lz(\sigma)] = Az(\sigma) + Bv(\sigma) + F(\sigma, z(\sigma), v(\sigma)),\ \sigma \in J = (0, c], \quad (1)$$

$$J^{(1-\wp)(1-\varpi)}_{0+} z(0) = z_0, \quad (2)$$

$D^{\wp,\varpi}_{0+}$is the Hilfer fractional derivative, $0 \leq \wp \leq 1$; $\frac{1}{2} < \varpi < 1$;is the Banach Space X with $\|\cdot\|$, and $z(\cdot)$ is the Banach Space X with $\|\cdot\|$. The non densely defined closed linear operator $A : D(A) \subseteq X \to X$ yields an integrated semigroup $\{T(t)\}_{t\geq 0}$ in Banach Space X with $\|\cdot\|$. The function $F : J \times X \times U \to X$ is a purely nonlinear function and $B : X \to U$ is a bounded linear operator.

This article makes the following major contributions:

- Using two separate situations, investigate the sufficient conditions for the approximate controllability of the suggested systems (1)-(2).
- In the first case, we assume that $B = I$ (where I is an identity operator) and in the second case, we assume that $B \neq I$.
- controllability results are achieved using Gronwall's inequality and the Cauchy sequence.
- Results are obtained with weaker conditions (Lipschitz) on nonlinearity and can be extended for the delay differential equations.
- The suggested method is simple in terms of hefty estimations as compared to standard ways such as the fixed point theory approach.
- The results are advanced and weighted enough as contribution in control differential equations.

We have divided this paper into the following sections: Section (1) provides a review of some essential concepts and preparatory outcomes Section (2). The main discussion of our manuscript is presented in Section (3). Finally, in Section (4), an application for drawing the theory of the primary outcomes is discussed.

2. Preliminaries

Let the spaces of all continuous functions is denoted by $C(J, X)$. Take $\eta = \wp + \varpi - \wp\varpi$, then $(1 - \eta) = (1 - \wp)(1 - \varpi)$. We now define $C_{1-\eta}(J, X) = \{z : \sigma^{1-\eta} z(\sigma) \in C(J, X)\}$ along $\|\cdot\|_\eta$ by $\|z\|_C = \sup\{\sigma^{1-\eta}\|z(\sigma)\|, \sigma \in J, \eta = (\wp + \varpi - \wp\varpi)\}$. It is clear that $C_{1-\eta}(J, X)$ is a Banach space.

The linear operators $A : D(A) \subset X \to X$ and $L : D(A) \subset X \to X$ satisfies the properties discussed in $A : D(A) \subset X \to X$. [17]:

- $(\mathbf{P_1})$ A and L are closed linear operators.
- $(\mathbf{P_2})$ $D(L) \subset D(A)$ and L is bijective.
- $(\mathbf{P_3})$ $L^{-1} : X \to D(L)$ is continuous.

Additionally, because $(\mathbf{P_1})$ and $(\mathbf{P_2})$, L^{-1} is closed, by $(\mathbf{P_3})$ and from closed graph theorem, we have the boundedness of $AL^{-1} : X \to X$. Define $||L^{-1}|| = \widetilde{L}_1$ and $\|L\| = \widetilde{L}_2$.

Introducing acquaint essential facts relevant to fractional theory. (The readers can check [18,35]).

Definition 1. *[3] "The left sided Riemann-Liouville fractional integral of order ϖ having lower limit c for $F : [c, +\infty) \to \mathbb{R}$ is presented as*

$$J^{\varpi}_{c^+} F(\varrho) = \frac{1}{\Gamma(\varpi)} \int_c^{\varrho} \frac{F(\tau)}{(\varrho - \tau)^{1-\varpi}} d\tau,\ \varrho > c;\ \varpi > 0,$$

if the right side is pointwise determined on $[c, +\infty)$, where $\Gamma(\cdot)$ denotes gamma function."

Definition 2. *[3] "The left-sided Riemann-Liouville fractional derivative of order $\varpi \in [k-1, k)$, $k \in X$ for $F : [c, +\infty) \to \mathbb{R}$ is given by*

$${}^{L}D^{\varpi}{}_{c^+}\ F(\varrho) = \frac{1}{\Gamma(k - \varpi)} \frac{d^k}{d\varrho^k} \int_c^{\varrho} \frac{F(\tau)}{(\varrho - \tau)^{\varpi+1-k}} d\tau, \quad \varrho > c,\ k - 1 < \varpi < k."$$

Definition 3. *[3] "The left-sided Hilfer fractional derivative of order $0 \leq \wp \leq 1$ and $0 < \varpi < 1$ function of $F(\varrho)$ is given by*

$$D^{\wp,\varpi}_{c^+} F(\varrho) = (J^{\wp(1-\varpi)}_{c^+} D(J^{(1-\wp)(1-\varpi)}_{d^+} F))(\varrho)."$$

Remark 1. *[3] "*

(i) *Given $\varpi = 0$, $0 < \wp < 1$ also $c = 0$, the Hilfer fractional derivative identical with standard Riemann-Liouville fractional derivative:*

$$D^{0,\varpi}_{0+} F(\varrho) = \frac{d}{d\varrho} J^{1-\varpi}_{0+}\ F(\varrho) = {}^{L}D^{\varpi}{}_{0+}\ F(\varrho).$$

(ii) *Given $\varpi = 1$, $0 < \wp < 1$ also $c = 0$, the Hilfer fractional derivative identical with standard Caputo derivative:*

$$D_{0^+}^{1,\varpi} F(\varrho) = J_{0^+}^{1-\varpi} \frac{d}{d\varrho} F(\varrho) = {}^{c}D^{\varpi}{}_{0^+}\ F(\varrho).\text{”}$$

Remark 2. *We show the mild solution of* (1)-(2)*in the following way using the Wright function $M_\varpi(s)$.*

$$M_\varpi(s) = \sum_{k=1}^{\infty} \frac{(-s)^{k-1}}{(k-1)!\Gamma(1-k\varpi)},\ 0 < \varpi < 1,\ s \in C,$$

and satisfies

$$\int_0^\infty s^\zeta M_\varpi(s) ds = \frac{\Gamma(1+\zeta)}{\Gamma(1+\varpi\zeta)},\ \textit{for } s \geq 0.$$

Lemma 1. *There exists $F : J \times X \times U \to X$ such that the system* (1)-(2)*is satisfied.*

$$\begin{aligned} z(\sigma) &= L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 \\ &+ \int_0^\sigma L^{-1}\mathscr{R}_\varpi(\sigma)F(\zeta, z(\zeta), v(\zeta))d\zeta \\ &+ \int_0^\sigma L^{-1}\mathscr{R}_\varpi(\sigma)Bv(\zeta)d\zeta, \quad \sigma \in J, \end{aligned}$$

where

$$\mathscr{P}_{\wp,\varpi}(\sigma) = J_{0^+}^{v(1-\varpi)}(\sigma)^{\varpi-1}\mathscr{S}_\varpi(\sigma);$$

$$\mathscr{R}_\varpi(\sigma) = \sigma^{\varpi-1}\mathscr{S}_\varpi(\sigma);$$

$$\mathscr{S}_\varpi(\sigma) = \int_0^\infty \varpi\omega M_\varpi(\omega)S(\sigma^\varpi\omega)d\omega.$$

Definition 4. *([36]) A function $z : [0, c] \to X$ is called the mild solution of* (1)-(2) *provided that $z(0) = z_0 \in X$ and fulfills*

$$\begin{aligned} z(\sigma) &= L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1}L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)Bv(\zeta)d\zeta \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1}L^{-1}\mathscr{S}_\varpi(\sigma-\zeta) \\ &\quad F(\zeta, z(\zeta), v(\zeta))d\zeta,\ \sigma \in J, \end{aligned} \tag{3}$$

where

$$\mathscr{P}_{\wp,\varpi}(\sigma) = \int_0^\infty \xi_\varpi(\omega)M(\sigma^\varpi\omega)d\omega,$$

$$\mathscr{S}_\varpi = \varpi\int_0^\infty \omega\xi_\varpi(\omega)M(\sigma^\varpi\omega)d\omega,$$

and for $\omega \in (0, \infty)$

$$\xi_\varpi(\omega) = \frac{1}{\varpi}\omega^{-1-\frac{1}{\varpi}}\overline{z}_\varpi(\omega^{-\frac{1}{\varpi}}) \geq 0,$$

$$\overline{z}_\varpi(\omega) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\sigma^{-n\varpi-1}\frac{\Gamma(n\varpi+1)}{n!}sin(n\pi\varpi),$$

where ξ_ϖ is a probability density function defined on $(0, \infty)$, i.e,

$$\xi_\varpi(\omega) \geq 0, \quad \omega \in (0, \infty) \quad \textit{also} \quad \int_0^\infty \xi_\varpi(\omega)d\omega = 1.$$

Lemma 2. *([36]) “The operators $\mathscr{P}_{\wp,\varpi}$ and $\mathscr{S}_\varpi$ fulfills:*

(i) *For $\sigma \geq 0$, $\mathscr{P}_{\wp,\varpi}$ and $\mathscr{S}_\varpi$ are linear and bounded, that is, for every $z \in X$, $\|\mathscr{P}_{\wp,\varpi}(\sigma)z\| \leq \frac{M\sigma^{\eta-1}}{\Gamma(\wp(1-\varpi)+\varpi)}\|z\|$ and $\|\mathscr{S}_\varpi(\sigma)z\| \leq \frac{M}{\Gamma(\varpi)}\|z\|$, where $\mathscr{P}_{\wp,\varpi}(\sigma) = J_{0^+}^{\wp(1-\varpi)}\mathscr{R}_\varpi(\sigma)$, $\mathscr{R}_\varpi(\sigma) = \sigma^{\varpi-1}\mathscr{S}_\varpi(\sigma)$.*

(ii) *The operators $\{\mathscr{P}_{\wp,\varpi}(\sigma)\}_{\sigma\geq 0}$ and $\{\mathscr{S}_\varpi(\sigma)\}_{\geq 0}$ are strongly continuous.*

(iii) *For every $z \in X$, $\mu, \varpi \in (0, 1]$, one can get*

$$A\mathscr{S}_\varpi(\sigma)z = A^{1-\mu}\mathscr{S}_\varpi(\sigma)A^\mu z,\ 0 \leq \sigma \leq c;$$

$$\|A^\mu\mathscr{S}_\varpi(\sigma)\| \leq \frac{\varpi C_\mu\Gamma(2-\mu)}{\sigma^{\varpi\mu}\Gamma(1+\varpi(1-\mu))},\ 0 < \sigma \leq c.\text{”}$$

Definition 5. *[6] “The reachable set of (1)-(2) is given by*
$K_c(F) = \{z(c) \in X : z(\sigma)$ represents mild solution of (1)-(2)$\}$.
In case $F \equiv 0$, then the system (1)-(2) reduces to the corresponding linear system. The reachable set in this case is denoted by $K_c(0)$.”

Definition 6. *[6] “If $\overline{K_c(F)} = X$, then the semilinear control system is approximately controllable on $[0, c]$. Here $\overline{K_c(F)}$ represents the closure of $K_c(F)$. It is clear that, if $\overline{K_c(0)} = X$, then linear system is approximately controllable.”*

3. Controllability results

3.1. Controllability of semilinear system: when $B = I$

The linear system has an approximate controllability is proven to reach from the semilinear system under specified nonlinear term constraints in this study. Clearly, $X = U$.

Let us consider the subsequent linear system

$$D_{0^+}^{\wp,\varpi}[Lw(\sigma)] = Aw(\sigma) + u(\sigma),\ \sigma \in J = (0, c], \tag{4}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)}w(0) = z_0, \tag{5}$$

and the semilinear system

$$\begin{aligned} D_{0^+}^{\wp,\varpi}[Lz(\sigma)] &= Az(\sigma) + v(\sigma) \\ &+ F(\sigma, z(\sigma), v(\sigma)),\ \sigma \in J, \end{aligned} \tag{6}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)}z(0) = z_0, \tag{7}$$

We need to present the following assumptions to prove the primary aim of this section, which is the approximate controllability of (6)-(7):

Assumption 1. *The linear system (4)-(5) is approximately controllable.*

Assumption 2. *$F(\sigma, z(\sigma), v(\sigma))$ is a nonlinear function that, in z and v, satisfies the Lipschitz condition.*

$\|F(\sigma, z, v) - F(\sigma, w, u)\| \leq l(\|z-w\| + \|v-u\|)$,

where $l > 0$, $\forall\, z, w \in X$, $\sigma \in [0, c]$.

Theorem 1. *Under the assumptions (1)-(2), the system (6)-(7) is approximately controllable provided that $l < 1$.*

Proof. Assume that $w(\sigma)$, along with the control u, is the mild solution of (4)-(5). Assume that the semilinear system of the following kind:

$$D_{0+}^{\wp,\varpi} z(\sigma) = Az(\sigma) + F(\sigma, z(\sigma), v(\sigma)) + u(\sigma) - F(\sigma, w(\sigma), v(\sigma)), \quad (8)$$

$$J_{0+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \quad (9)$$

Compare (6)-(7) and (8)-(9), the control function $v(\sigma)$ is chosen in such a way that

$$v(\sigma) = u(\sigma) - F(\sigma, w(\sigma), v(\sigma)). \quad (10)$$

We consider for the given $u(\sigma)$ and $w(\sigma)$, there exists $v(\varrho)$ fulfilling (10) (We need to verify the existence and uniqueness of v).

The mild solution of (4)-(5) is given by

$$w(\sigma) = L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 + \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)u(\zeta)d\zeta \quad (11)$$

and for the system (8)-(9) is given by

$$\begin{aligned} z(\sigma) &= L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)F(\zeta, z(\zeta), v(\zeta))d\zeta \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)u(\zeta)d\zeta \\ &- \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)F(\zeta, w(\zeta), v(\zeta))d\zeta \end{aligned} \quad (12)$$

From (11) and (12), we get

$$w(\sigma) - z(\sigma) = \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta) \{F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\}d\zeta \quad (13)$$

Applying norm on both sides, one can get

$$\begin{aligned} &\|w(\sigma) - z(\sigma)\|_X \\ &\leq \int_0^\sigma (\sigma-\zeta)^{\varpi-1} \|L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)\| \\ &\|F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\|d\zeta \\ &\leq \frac{M\widetilde{L}_1}{\Gamma(\varpi)} \int_0^\sigma (\sigma-\zeta)^{\varpi-1} \\ &\|F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\|d\zeta \end{aligned} \quad (14)$$

Using assumption (2), we get

$$\|w(\sigma) - z(\sigma)\|_X \leq \frac{M\widetilde{L}_1 l}{\Gamma(\varpi)} \int_0^\sigma (\sigma-\zeta)^{\varpi-1} \|w(\zeta) - z(\zeta)\|d\zeta$$

By referring the Gronwall's inequality, $w(\sigma) = z(\sigma)$, $\forall\ \sigma \in [0, c]$. As a result, the linear system's solution w along the control u is a semilinear system's solution z along the control v, i.e., $K_c(F) \supset K_c(0)$. Because $K_c(0)$ is dense in X (according to assumption 1), $K_c(F)$ is dense in X as well, implying that system (6)-(7) is approximate controllable. The proof is finished.

We need to verify that there exists a $v(\sigma) \in X$ such that $v(\sigma) = u(\sigma) - F(\sigma, w(\sigma), v(\sigma))$, $\forall\ \sigma \in [0, c]$.

Assume that $v_0 \in X$ and $v_{n+1} = u - F(\sigma, w(\sigma), v_n)$: $n = 0, 1, 2, ...$ Thus, one can get

$$v_{n+1} - v_n = F(\sigma, w(\sigma), v_{n-1}) - F(\sigma, w(\sigma), v_n).$$

Hence, by referring assumption (2),

$$\|v_{n+1} - v_n\|_X = l\|v_n - v_{n-1}\|_X \leq l^n \|v_1 - v_0\|_X. \quad (15)$$

When $n \to \infty$ (since $l < 1$), the R.H.S of (15)goes to zero. As a result, $\{v_n\}$ is a Cauchy sequence in X that converges to $v \in X$.

Next,

$$\begin{aligned} &\|u - v_{n+1} - F(\sigma, w(\sigma), v)\|_X = \\ &\quad \|F(\sigma, w(\sigma), v_n) - F(\sigma, w(\sigma), v)\|_X \\ &\quad \leq l\|v_n - v\|. \end{aligned} \quad (16)$$

Because, R.H.S of (16) approaches to zero when $n \to \infty$, one can obtain

$$\begin{aligned} F(\sigma, w(\sigma), v) &= \lim_{n\to\infty}(u - v_{n+1}) = u - v \\ &\Rightarrow v = u - F(\sigma, w(\sigma), v). \end{aligned}$$

Now, we will show that v is unique. For proving it let $v_1 = u - F(\sigma, w(\sigma), v_1)$ and $v_2 = u - F(\sigma, w(\sigma), v_2)$. Then using assumption (2), we get

$$\begin{aligned} \|v_2 - v_1\| &= \|F(\sigma, w(\sigma), v_1) - F(\sigma, w(\sigma), v_2)\| \\ &\leq l\|v_2 - v_1\| \Rightarrow (1-l)\|v_2 - v_1\| \leq 0. \end{aligned}$$

But $0 < l < 1$ therefore $\|v_2 - v_1\| = 0 \Rightarrow v_2 = v_1$. Hence v is unique. $\square$

3.2. Controllability of semilinear system: when $B \neq I$

The approximate controllability of the semilinear system under simple conditions B and F as indicated by assumptions (3)-(6). Let us consider the subsequent linear system

$$D_{0^+}^{\wp;\varpi}[Lw(\sigma)] = Aw(\sigma) + Bu(\sigma),\ \sigma \in J, \tag{17}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} w(0) = z_0, \tag{18}$$

and the semilinear system

$$D_{0^+}^{\wp,\varpi}[Lz(\sigma)] = Az(\sigma) + Bv(\sigma) + F(\sigma, z(\sigma), v(\sigma)),\ \sigma \in J = (0, c], \tag{19}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \tag{20}$$

We must make the following assumptions to prove the fundamental aim of this section, namely, the approximate controllability of (19)-(20):

Assumption 3. *The linear system (17)-(18) is approximately controllable.*

Assumption 4. *Assumption (2) is fulfilled.*

Assumption 5. $Range(F) \subseteq \overline{Range(B)}$.

Assumption 6. *There exists* $\xi > 0$ *such that* $\|Bv\| \geq \xi\|v\|$, $\forall\ v \in U$

Theorem 2. *Under the assumptions (3)-(6), the system (19)-(20) is approximately controllable, provided that* l *fulfills* $l < \xi$.

Proof. Assume that $w(\sigma)$ and the control u are the mild solution of (17)-(18). Assume that the semilinear system that follows is

$$D_{0^+}^{\wp,\varpi}[Lz(\sigma)] = Az(\sigma) + F(\sigma, z(\sigma), v(\sigma)) + Bu(\sigma) - F(\sigma, w(\sigma), v(\sigma)), \tag{21}$$

$$J_{0^+}^{(1-\wp)(1-\varpi)} z(0) = z_0, \tag{22}$$

In the above, the control function v in (21)-(22) fulfills $Bv(\sigma) = Bu(\sigma) - F(\sigma, w(\sigma), v(\sigma))$, and assumption (5), concludes that the considered equation is well defined. By employing assumption (6) and the way of approached followed in Theorem 2, we can easily prove that provided that $l < \xi$, $\exists\ v(\sigma) \in U$ such that $Bv(\sigma) = Bu(\sigma) - F(\sigma, w(\sigma), v(\sigma))$.

The mild solutions for (17)-(18) and (21)-(22) are given by

$$w(\sigma) = L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 + \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)Bu(\zeta)d\zeta \tag{23}$$

and

$$\begin{aligned} z(\sigma) &= L^{-1}\mathscr{P}_{\wp,\varpi}(\sigma)Lz_0 \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)F(\zeta, z(\zeta), v(\zeta))d\zeta \\ &+ \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)Bu(\zeta)d\zeta \\ &- \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta)F(\zeta, w(\zeta), v(\zeta))d\zeta \end{aligned} \tag{24}$$

From equations (23) and (24), one can get

$$w(\sigma) - z(\sigma) = \int_0^\sigma (\sigma-\zeta)^{\varpi-1} L^{-1}\mathscr{S}_\varpi(\sigma-\zeta) \times \{F(\zeta, w(\zeta), v(\zeta)) - F(\zeta, z(\zeta), v(\zeta))\}d\zeta. \tag{25}$$

Equation (25) is the same when compared with (13). From Theorem 2, one can easily verify $w(\sigma) = z(\sigma)$, $\forall\ \sigma \in [0, c]$, i.e., the reachable set of (17)-(18) is dense in the reachable set of (19)-(20), which is dense in X, by referring assumption (3) and which concludes the proof. $\square$

4. Example

Consider $U = L_2[0, \pi]$. Also, define the operator $B : D(B) \subset U \to U$ as

$$Bx = x'',\ \ x \in D(B),$$

$D(B) = \{x \in U :\ x,\ x'$ are absolutely continuous, $x" \in U,\ x(0) = x(\pi) = 0\}$. Assume that $A : D(A) \subset X \to X$, $L : D(L) \subset X \to X$, and $Lx = x - x''$ are the operators determined by $Ax = x''$ and $Lx = x - x''$, respectively, and that $D(A)$ and $D(L)$ are presented by

$$\{x \in X : x, x' \text{ are absolutely continuous, } x(0) = x(\pi) = 0\}.$$

Additionally, A and L are given by

$$Ax = \sum_{m=1}^{\infty} m^2 \langle x, u_m\rangle u_m,\ x \in D(A),$$

$$Lx = \sum_{m=1}^{\infty} (1+m^2)\langle x, u_m\rangle u_m, x \in D(L),$$

where $u_m(y) = \sqrt{\frac{2}{\pi}}\sin(my)$, $m = 1, 2, 3, \cdots$ is the orthonormal of vectors of A. Additionally, for $z \in X$, we have

$$L^{-1}z = \sum_{m=1}^{\infty} \frac{1}{(1+m^2)}\langle z, u_m\rangle u_m,$$

and

$$AL^{-1}z = \sum_{m=1}^{\infty} \frac{m^2}{(1+m^2)} \langle z, u_m \rangle u_m.$$

The operator B has eigen values $\lambda_m = -m^2 \;\; m \in \mathbb{N}$ and corresponding eigenfunction is given by u_n. Hence the spectral representation of B is presented as

$$Bz = \sum_{m=1}^{\infty} -m^2 \langle x, u_n \rangle u_n, \;\; x \in D(B).$$

Further, $\mathcal{S}(\varrho)$ which is a C_0-semigroup generated by B has e_n as the eigenfunctions corresponding to eigenvalues $\exp(\lambda_m t)$, that is

$$\mathcal{S}(\varrho)x = \sum_{m=1}^{\infty} \exp(-m^2\varrho) \langle x, u_n \rangle u_n, \;\; x \in U.$$

Define by

$$\widehat{U} = \left\{ v | v = \sum_{m=2}^{\infty} v_m u_m, \; with \; \sum_{m=2}^{\infty} v_m^2 < \infty \right\},$$

where $\widehat{U}$ is an infinite dimensional space with a norm of

$$\|v\|_{\widehat{U}} = \left(\sum_{m=2}^{\infty} v_m^2 \right)^{\frac{1}{2}}$$

Define $B : \widehat{U} \to U$ by

$$Bv = 2v_2 e_1 + \sum_{m=2}^{\infty} v_m u_m, \quad v = \sum_{m=2}^{\infty} v_m u_m \in \widehat{U}.$$

where B is a linear continuous map.

Assume that the Hilfer fractional semilinear control heat system is as follows:

$$D_{0^+}^{\wp,\varpi}\Big[z(\varrho,x) - \frac{\partial^2 z(\varrho,x)}{\partial z^2}\Big]$$
$$= \frac{\partial^2 z(\varrho,x)}{\partial z^2} + Bu(\varrho,x) + \gamma(\varrho, z(\varrho,x)); \quad 0 < \varrho \leq \iota, \tag{26}$$

$$z(\varrho,0) = z(\varrho,\pi) = 0, \quad \varrho > 0;$$
$$J_{0^+}^{(1-\gamma)}(z(0,x)) = z_0(x), \quad 0 \leq x \leq \pi, \tag{27}$$

The Hilfer fractional derivative of order $\wp \in (0,1)$ and type $\varpi \in [0,1]$ is denoted by $D_{0^+}^{\alpha,\eta}$. If the assumptions (1)-(6) hold, the above system (26)-(27) is approximate controllable.

5. Conclusion

The focus of this study is on the Sobolev-type approximate controllability of Hilfer fractional semilinear control systems. The results were obtained using Gronwall's inequality, the Cauchy sequence, and the fixed point technique was avoided. With appropriate changes, these conclusions may be extended to include many types of delay for both deterministic and stochastic systems.

Remark 3. *One can replace the Lipschitz condition on the nonlinearity by monotonic nonlinearity or integral contractor type nonlinearity and obtained a different set of sufficient conditions for the approximate controllability of the proposed system.*

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