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RESEARCH ARTICLE

Stability tests and solution estimates for non-linear differential equations

Osman Tunç

Department of Computer Programing, Baskale Vocational School, Van Yuzuncu Yil University, Van, Turkey
osmantunc89@gmail.com

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ABSTRACT

This article deals with certain systems of delay differential equations (DDEs) and a system of ordinary differential equations (ODEs). Here, five new theorems are proved on the fundamental properties of solutions of these systems. The results on the properties of solutions consist of sufficient conditions and they dealt with uniformly asymptotically stability (UAS), instability and integrability of solutions of unperturbed systems of DDEs, boundedness of solutions of a perturbed system of DDEs at infinity and exponentially stability (ES) of solutions of a system of nonlinear ODEs. Here, the techniques of proofs depend upon the Lyapunov- Krasovskiĭ functional (LKF) method and Lyapunov function (LF) method. For illustrations, in particular cases, four examples are constructed as applications. Some results of this paper are given at first time in the literature, and the other results generalize and improve some related ones in the literature.

1. Introduction

Functional differential equations (FDEs) which include delay differential equations and differential integral equations have been studied for at least 200 years. However, especially, it can be seen from the relevant literature that during the last seven decades numerous qualitative behaviors of various FDEs, in particular, delay deferential equations have been studied extensively and they are still being investigated by researchers. It is known that UAS, exponential stability, instability, integrability and boundedness of solutions are the most important fundamental properties of FDEs and ODEs. There are many publications on fundamental properties of solutions of FDEs, ODEs and so on. We cite here the papers [1–6], [7–9], [10], [11–31] and the books of ([32], [33–39]) fully or partially devoted to fundamental motions of trajectories of solutions of these classes of equations. In particular, UAS and boundedness of solutions at the infinity describe long time behaviors of solutions. Additionally, during the applications of FDEs and ODEs in control theory, engineering, medicine, etc., usually it is necessary to know qualitative estimates of solutions such as instability, integrability, exponentially stability and so on.

We would like to summarize two recent works of AS, UAS and some other fundamental motions of solutions of DDEs. Recently, Tian and Ren [13] took into consideration the below system of linear DDEs:

$$\frac{dx}{dt} = Ax(t) + Bx(t - h(t)). \tag{1}$$

In [13, Theorem 1], an LKF was defined for the system (1) and based on that LKF, a theorem was proved on the AS of the zero solution of (1). In [13], the method of proof is depending upon the definition of a very interesting suitable LKF. Later, Tunç et al. [23] dealt with the nonlinear system of DDEs:

$$\begin{aligned}\frac{dx}{dt} =& A(t)x(t) + BF(x(t - h(t))) \\ &+ E(t, x(t), x(t - h(t))).\end{aligned} \tag{2}$$

In [23], three theorems, which have sufficient conditions, were proved on the UAS and integrability of solutions, when $E(.) \equiv 0$ in (2), and the boundedness of the solutions of (2), when $E(.) \neq 0$. In [35], the method used in the proofs is based on the definitions of two suitable LKFs. For some interesting recent and applicable results on the fractional mathematical models, see [40–43].

In this article, by the virtue of the systems of DDEs (1) and (2), the related ones in the references of this paper and literature, we deal with the following nonlinear system of DDEs:

$$\begin{aligned}\frac{dx}{dt} =& A(t)x(t) + G(x(t)) + H(t, x(t)) \\ &+ BF(x(t-h(t))) + Q(t, x(t), x(t-h(t))),\end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}^+$, $\mathbb{R}^+ = [0, \infty)$, $h(t) \in C^1(\mathbb{R}^+, (0, \infty))$, $A(t) \in C(\mathbb{R}^+, \mathbb{R}^{n\times n})$, $G \in C(\mathbb{R}^n, \mathbb{R}^n)$,$G(0) = 0$, $H \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $H(t, 0) = 0$,$B \in \mathbb{R}^{n\times n}$, $F \in C(\mathbb{R}^n, \mathbb{R}^n)$, $F(0) = 0$, $Q \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and the variable delay $h(t)$ of (3) fulfills the inequalities:

$$\begin{aligned}&0 \le h_1 \le h(t) \le h_2, \\ &h_{12} = h_2 - h_1, \\ &0 \le h'(t) \le h_0 < 1.\end{aligned} \tag{4}$$

We would now like to explain the objectives of this paper.

1) In this paper, Theorem 1, Theorem 4 and Theorem 2 dealt with UAS, instability and integrability of solutions nonlinear system of DDEs (5):
$$\begin{aligned}\frac{dx}{dt} =& A(t)x(t) + G(x(t)) + H(t, x(t)) \\ &+ BF(x(t-h(t))).\end{aligned} \tag{5}$$
2) The ES of the following system of ODEs was discussed by Theorem 3, when $BF(x(t-h(t))) \equiv 0$ in (5):
$$\frac{dx}{dt} = A(t)x(t) + G(x(t)) + H(t, x(t)). \tag{6}$$
3) Theorem 5 dealt with the bounded solutions of the perturbed system (3).
4) In particular cases of the considered systems, four new examples are designed to show applications of Theorems 1-5.

2. Basic information

Assume that $C_0 = C_0([-\tau, 0], \mathbb{R}^n)$,$\tau > 0$, is the space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$. For any $a \in \mathbb{R}$, $a \ge 0$, $\forall t_0 \ge 0$ and $x \in C_0([t_0 - \tau, t_0 + a], \mathbb{R}^n)$, let $x_t = x(t + \theta)$ when $-\tau \le \theta \le 0$ and $t \ge t_0$.

Let $x \in \mathbb{R}^n$. The norm $\|.\|$ is defined as $\|x\| = \sum_{i=1}^{n} |x_i|$. Additionally, the matrix norm $\|A\|$ is defined as $\|A\| = \max_{1\le j\le n} \left(\sum_{i=1}^{n} |a_{ij}|\right)$, where $A \in \mathbb{R}^{n\times n}$.

For any $\phi \in C_0$, let

$$\|\phi\|_{C_0} = \sup_{\theta\in[-\tau,0]} \|\phi(\theta)\| = \|\phi(\theta)\|_{[-\tau,0]}$$

and

$$C_H = \{\phi : \phi \in C_0 \text{ and } \|\phi\|_{C_0} \le H < \infty\}.$$

In this article, without mention, let $x(t)$ represent x.

3. Stability and integrability

Let $Q(.) = 0$. Hence, we now have the nonlinear system of DDEs (5).

A. Assumptions

(H1) Let $a_A \in \mathbb{R}$, $a_A > 0$ with
$$a_{ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \le -a_A \text{ for all } t \in \mathbb{R}^+;$$

(H2) There exist positive constants h_0 and a_A from (3) and (H1), respectively, and f_F, g_G, h_H, $K_2 > 0$ such that
$$\|F(u) - F(v)\| \le f_F \|u - v\|,$$
$$\forall u,\ v \in \mathbb{R}^n, F(0) = 0,$$
$$sgn x_i G_i(x) < 0$$
as
$$x_i \neq 0, \forall x \in \mathbb{R}^n, G(0) = 0,$$
$$\|G(x)\| \ge g_G \|x\| \text{ for all } x \in \mathbb{R}^n,$$
$$H(t, 0) = 0, sgn x_i H_i(t, x) < 0$$
as
$$x_i \neq 0, \forall t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$\|H(t, x)\| \ge h_H \|x\|, \forall t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$(a_A + g_G + h_H)(1 - h_0) - f_F \|B\| \ge K_2.$$

Theorem 1. *We suppose that conditions (H1) and (H2) are held. Then, trivial solution of (5) is UA stable.*

Proof. We define an LKF $\Delta_1 := \Delta_1(t, x_t)$ by

$$\Delta_1(t, x_t) := \|x(t)\| + \gamma \int_{t-h(t)}^{t} \|F(x(s))\| ds, \tag{7}$$

where $\gamma \in \mathbb{R}$, $\gamma > 0$, it will be chosen after some calculations.

From the LKF (7), we have

$$\Delta_1(t,x_t) := |x_1(t)| + ... + |x_n(t)| + \gamma \int\limits_{t-h(t)}^{t} |f_1(x(s))| ds$$

$$+ ... + \gamma \int\limits_{t-h(t)}^{t} |f_n(x(s))| ds.$$

According to the LKF of (7), it satisfies that

$$\Delta_1(t,0) = 0, \gamma_1 \|x\| \le \Delta_1(t,x_t),$$

where

$$\gamma_1 \in (0,1), \gamma_1 \in \mathbb{R},$$

Let $\gamma_2 \ge 1$, $\gamma_2 \in \mathbb{R}$, and define

$$Z_1(t,x_t) := \int\limits_{t-h(t)}^{t} \|F(x(s))\| ds.$$

Next, we have

$$\gamma_1 \|x\| + \gamma Z_1(t,x_t) \le \Delta_1(t,x_t) \le \gamma_2 \|x\| + \gamma Z_1(t,x_t).$$

Using condition (H2) and some simple evaluations, we find that

$$\begin{aligned} &\|\Delta_1(t,x_t) - \Delta_1(t,y_t)\| \\ &\le \|x(t) - y(t)\| \\ &+ \gamma F_f h_2 \sup_{t-h(t)\le s\le t} \|x(s) - y(s)\| \\ &\le M_0 \sup_{t-h(t)\le s\le t} \|x(s) - y(s)\|, \end{aligned}$$

where

$$M_0 := 1 + \gamma F_f h_2.$$

According to the above inequality, it is followed that

$$|\Delta_1(t,x_t) - \Delta_1(t,y_t)| \le M_0 \|x(s) - y(s)\|_{[t-h(t),t]}.$$

Thus, the locally Lipschitz condition in x_t is satisfied by the LKF $\Delta_1(t,x_t)$. Thus, condition (A1) of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) is held.

For the next step, by virtue of the definition of $Z_1(t,x_t)$ and condition (H2), we have

$$\begin{aligned} Z_1(t,x_t) &= \int\limits_{t-h(t)}^{t} \|F(x(s))\| ds \\ &\le f_F h(t) \sup_{t-h(t)\le s\le t} \|x(s)\| \\ &\le f_F h_2 \sup_{t-h(t)\le s\le t} \|x(s)\|. \end{aligned}$$

Using some simple calculations and condition (H2), we have

$$\begin{aligned} Z_1(t_2,x_t) - Z_1(t_1,x_t) &= \int\limits_{t_2-h(t_2)}^{t_2} \|F(x(s))\| ds - \int\limits_{t_1-h(t_1)}^{t_1} \|F(x(s))\| ds \\ &= \int\limits_{t_1}^{t_2} \|F(x(s))\| ds - \int\limits_{t_1-h(t_1)}^{t_2-h(t_2)} \|F(x(s))\| ds \\ &\le \int\limits_{t_1}^{t_2} \|F(x(s))\| ds \\ &\le f_F \sup_{t_1\le s\le t_2} \|x(s)\| (t_2 - t_1) = M(t_2 - t_1), \end{aligned}$$

$$M_1 = f_F \sup_{t_1\le s\le t_2} \|x(s)\|, 0 < t_1 < t_2 < \infty.$$

The obtained inequality demonstrates that the second condition, i.e., (A2), of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) is satisfied.

The differentiating the LKF $\Delta_1(t,x_t)$ of (7) and taking into account (5), we arrive that

$$\begin{aligned} \frac{d}{dt}\Delta_1(t,x_t) = &\sum_{i=1}^{n} x'_i(t) sgn x_i(t+0) + \gamma \|F(x(t))\| \\ &- \gamma(1 - h'(t)) \|F(x(t-h(t)))\|. \end{aligned} \tag{8}$$

By virtue of conditions (H1) and (H2), we obtain

$$\begin{aligned} &\sum_{i=1}^{n} sgn x_i(t+0) x'_i(t) \\ &\le \sum_{i=1}^{n} \left(a_{ii}(t) + \sum_{j=1, j\ne i}^{n} |a_{ji}(t)| \right) |x_i(t)| \\ &\quad - \|G(x(t))\| - \|H(t,x(t))\| \\ &\quad + \|B\| \ \|F(x(t-h(t)))\| \\ &\le -(a_A + g_G + h_H) \|x(t)\| \\ &\quad + \|B\| \ \|F(x(t-h(t)))\|. \end{aligned} \tag{9}$$

Thereby, putting the inequality (9) into (8) and using the condition $0 \le h'(t) \le h_0 < 1$, we have

$$\begin{aligned} \frac{d}{dt}\Delta_1(t,x_t) \le &- a_A \|x(t)\| - g_G \|x(t)\| - h_H \|x(t)\| \\ &+ \|B\| \ \|F(x(t-h(t)))\| \\ &+ \gamma \|F(x(t))\| \\ &- \gamma(1 - h'(t)) \|F(x(t-h(t)))\| \\ \le &-(a_A + g_G + h_H) \|x(t)\| \\ &+ \|F(x(t-h(t)))\| \ \|B\| \\ &+ \gamma f_F \|x(t)\| \\ &- \gamma(1 - h_0) \|F(x(t-h(t)))\|. \end{aligned}$$

Let $\gamma = \|B\|(1-h_0)^{-1}$. Then, it follows that

$$\begin{aligned}&\frac{d}{dt}\Delta_1(t,x_t)\\ &\leq -\Big[(a_A+g_G+h_H)-(1-h_0)^{-1}f_F\|B\|\Big]\|x(t)\|\\ &= -\frac{1}{1-h_0}\left[(a_A+g_G+h_H)(1-h_0)-f_F\|B\|\right]\|x(t)\|.\end{aligned}$$

Using the condition (H2), clearly, we have

$$\frac{d}{dt}\Delta_1(t,x_t) \leq -K_2\|x(t)\| < 0, \quad \|x(t)\| \neq 0. \tag{10}$$

Thus, it is obvious that $\frac{d}{dt}\Delta_1(t,x_t)$ is negative definite. From the inequality (10), it follows that assumption (A3) of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) is satisfied. Thus, all the assumptions of ([32, Theorem 4.2.9], Tunç et al. [23, Theorem 1]) are held. Hence, the zero solution of (5) is UA stable. □

Theorem 2. *If the conditions (H1) and (H2) are held, then the solutions of (5) satisfies that* $\int\limits_{t_0}^{\infty}\|x(s)\|ds < \infty$.

Proof. As in the proof of the above first theorem, we utilize the LKF $\Delta_1(t,x_t)$. According to conditions (H1) and (H2) we have

$$\frac{d}{dt}\Delta_1(t,x_t) \leq -K_2\|x(t)\|.$$

This result confirms that the LKF $\Delta_1(t,x_t)$ is decreasing, i.e.,

$$\Delta_1(t,x_t) \leq \Delta(t_0,\phi(t_0)) \text{ for all } \ t \geq t_0.$$

Integrating this inequality, it follows that

$$K_2\int\limits_{t_0}^{t}\|x(s)\|ds \leq \Delta(t_0,\phi(t_0)) - \Delta_1(t,x_t) \leq K_3,$$

$$t \geq t_0,$$

where $K_3 = \Delta(t_0,\phi(t_0))$. Then,

$$\int\limits_{t_0}^{t}\|x(s)\|ds \leq K_2^{-1}\Delta(t_0,\phi(t_0)) \equiv K_2^{-1}K_3.$$

Let $t \to +\infty$. Hence,

$$\int\limits_{t_0}^{\infty}\|x(s)\|ds \leq K_2^{-1}K_3 < \infty.$$

Thus, the proof of Theorem 2 is finished. □

Example 1. *Let us take into account the below system of non-linear DDEs:*

$$\begin{aligned}\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} &= \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\ &+\begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix}\\ &+\begin{pmatrix} -2x_1-\frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2-\frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix}\\ &+\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \sin x_1(t-\frac{1}{2}|\arctan(t)|) \\ \sin x_2(t-\frac{1}{2}|\arctan(t)|) \end{pmatrix},\end{aligned} \tag{11}$$

where $h(t) = \frac{1}{2}|\arctan t|$ *is the delay function,* $t \geq 2^{-1}\pi$.

A comparison between the systems of DDEs (11) and DDEs (5) gives that

$$A(t) = \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4} \end{pmatrix}.$$

By the virtue of the matrix $A(t)$*, we derive that*

$$a_{ii}(t) + \sum_{j=1,j\neq i}^{n}|a_{ji}(t)| = -25 < -24 = -a_A$$

because of

$$\begin{aligned}a_{11}(t)+|a_{21}(t)| &= -25-\frac{1}{1+t^4}+\frac{1}{1+t^4}\\ &= -25 < -24 = -a_A\end{aligned}$$

and

$$\begin{aligned}a_{22}(t)+|a_{12}(t)| &= -\frac{1}{1+t^4}-25+\frac{1}{1+t^4}\\ &= -25 < -24 = -\ a_A.\end{aligned}$$

Hence,

$$a_{ii}(t) + \sum_{j=1,j\neq i}^{2}|a_{ji}(t)| < -a_A = -24, \forall t \in \mathbb{R}^+.$$

As for the next step, we get

$$\begin{aligned}G(x) = G(x_1,x_2) &= \begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix}\\ &= \begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix},\end{aligned}$$

$$\begin{aligned}sgnx_1G_1(x) &= sgnx_1G_1(x_1,x_2)\\ &= -2x_1^2-\frac{x_1^2}{1+x_1^2} < 0, \ x_1 \neq 0,\\ sgnx_2G_2(x) &= sgnx_2G_2(x_1,x_2)\\ &= -2x_2^2-\frac{x_2^2}{1+x_2^2} < 0, \ x_2 \neq 0,\end{aligned}$$

$$\begin{aligned}\|G(x)\| = \|G(x_1,x_2)\| &= \left\|\begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix}\right\|\\ &= \left\|\begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix}\right\|\\ &= \left|-2x_1-\frac{x_1}{1+x_1^2}\right| + \left|-2x_2-\frac{x_2}{1+x_2^2}\right|\end{aligned}$$

$$\geq 2|x_1| - \frac{|x_1|}{1+x_1^2} + 2|x_2| - \frac{|x_2|}{1+x_2^2}$$
$$\geq |x_1| + |x_2| = \|x\|, \ g_G = 1 > 0.$$

Additionally, we have

$$H(t,x) = H(t,x_1,x_2) = \begin{pmatrix} -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix}$$

$$sgnx_1 H_1(t,x) = sgnx_1 H_1(t,x_1,x_2) = -2x_1^2 - \frac{x_1^2}{1+\exp(t)+x_1^2} < 0, \ x_1 \neq 0,$$

$$sgnx_2 H_1(t,x) = sgnx_2 H_1(t,x_1,x_2) = -2x_2^2 - \frac{x_2^2}{1+\exp(t)+x_2^2} < 0, \ x_2 \neq 0.$$

$$\|H(t,x)\| = \|H(t,x_1,x_2)\| = \left\| \begin{pmatrix} -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} \right\|$$

$$= \left| -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \right| + \left| -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \right|$$

$$\geq 2|x_1| - \frac{|x_1|}{1+\exp(t)+x_1^2} + 2|x_2| - \frac{|x_2|}{1+\exp(t)+x_2^2}$$

$$\geq |x_1| + |x_2| = \|x\|, \ h_H = 1 > 0.$$

$$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \|B\| = 5.$$

$$F(x(t - \frac{1}{2}|arctg(t)|) = F(x_1(t - \frac{1}{2}|arctg(t)|, x_2(t - \frac{1}{2}|arctg(t)|) = \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix}$$

$$F(0) = 0, h(t) = \frac{1}{2}|\arctan(t)|.$$

Let

$$u = x(t - \frac{1}{2}|\arctan(t)|, v = y(t - \frac{1}{2}|\arctan(t)|,$$

$$u_1 = x_1(t - \frac{1}{2}|\arctan(t)|), v_1 = y_1(t - \frac{1}{2}|\arctan(t)|),$$

and

$$u_2 = x_2(t - \frac{1}{2}|\arctan(t)|)$$

$$v_2 = y_2(t - \frac{1}{2}|\arctan(t)|), t \geq \frac{\pi}{2}.$$

Then,

$$\|F(u) - F(v)\| = \|F(u_1,u_2) - F(v_1,v_2)\| = \left\| \begin{pmatrix} \sin u_1 - \sin v_1 \\ \sin u_2 - \sin v_2 \end{pmatrix} \right\|$$
$$= |\sin u_1 - \sin v_1| + |\sin u_2 - \sin v_2|$$
$$\leq 2\left|\frac{u_1 - v_1}{2}\right| + 2\left|\frac{u_2 - u_2}{2}\right|$$
$$= \|u - v\|, f_F = 1.$$

As for the variable delay $h = h(t)$,

$$h(t) = \frac{1}{2}|\arctan(t)|,$$

$$0 < 0.001 = h_1 = \frac{1}{2}|\arctan(t)| \leq \frac{\pi}{4} = h_2,$$

$$h_{12} = h_2 - h_1 = \tfrac{\pi}{4} - 0.001,$$

$$h'(t) = \frac{1}{2+2t^2},$$

$$0 \leq h'(t) \leq \frac{1}{2} = h_0 < 1.$$

Next, we derive that

$$(a_A + g_G + h_H)(1 - h_0) - f_F\|B\| = (24 + 1 + 1)(1 - 2^{-1}) - 5 = 13 - 5 = 8 \geq K_2.$$

By the virtue of the above estimates, it follows that the conditions (H1) and (H2) of Theorem 1 are held. For this reason, the solution $(x_1(t), x_2(t)) = (0,0)$ of the system of DDEs (11) is UA stable. Furthermore, $\|x(t)\|$, the norm of solutions of (11) are integrable.

B. Assumption

For the exponentially stability of the system of ODEs (6), we need the below conditions.

(H3) There exist constants h_0 from (4), a_A from (H1), and $f_F > 0$, $g_G > 0$, $H_0 > 0$, $K_2 > 0$, $e_E > 0$ such that

$$G(0) = 0, sgnx_i G_i(x) < 0 as x_i \neq 0, \ \text{ for all } x \in \mathbb{R}^n,$$
$$\|G(x)\| \geq g_G\|x\| \ \text{ for all } x \in \mathbb{R}^n,$$
$$H(t,0) = 0, sgnx_i H_i(t,x) < 0$$
as
$$x_i \neq 0, \ \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$\|H(t,x)\| \geq h_H\|x\| \ \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$
$$(a_A + g_G + h_H) \geq e_E.$$

Theorem 3. *We suppose that conditions (H1) and (H3) are held. Then the trivial solution of the system (6) is exponentially stable.*

Proof. Define a Lyapunov function (LF) $\Delta_2 := \Delta_2(t,x)$ by

$$\Delta_2(t,x) := \|x(t)\|. \tag{12}$$

This function is equivalent to

$$\Delta_2(t,x) := |x_1(t)| + ... + |x_n(t)|.$$

From this point of view, we see that the LF $\Delta_2(t,x)$ is positive definite. The derivative of the LF $\Delta_2(t,x)$ of (12) along the system of ODEs (6) gives that

$$\frac{d}{dt}\Delta_2(t,x) = \sum_{i=1}^{n} x'_i(t) sgn x_i(t+0).$$

Using conditions (H1), (H3) and doing some simple calculations, we obtain

$$\begin{aligned}&\sum_{i=1}^{n} sgn x_i(t+0) x'_i(t)\\ &\le \sum_{i=1}^{n}\left(a_{ii}(t) + \sum_{j=1, j\neq i}^{n} |a_{ji}(t)|\right)|x_i(t)|\\ &\quad - \|G(x(t))\| - \|H(t,x(t))\|\\ &\le -\left(a_A + g_G + h_H\right)\|x(t)\|\\ &= -\left(a_A + g_G + h_H\right)\Delta_2(t,x).\end{aligned}$$

Hence,

$$\frac{d}{dt}\Delta_2(t,x) \le -\left(a_A + g_G + h_H\right)\Delta_2(t,x)$$

Integrating the last inequality, we derive that

$$\begin{aligned}&\|x(t)\| = \Delta_2(t,x(t))\\ &\le \Delta_2(t_0,x(t_0))\exp\left[-\ (a_A + g_G + h_H)\right](t-t_0).\end{aligned}$$

According to this inequality,

$$\begin{aligned}\|x(t)\| \le &\Delta_2(t_0,x(t_0))\\ &\times \exp\left[-\ (a_A + g_G + h_H)\right](t-t_0),\ t \ge t_0.\end{aligned}$$

This inequality verifies that the zero solution of (6) is exponentially stable. □

Example 2. *Consider the following two dimensional system of non-linear ODEs, which is a special case of (6):*

$$\begin{aligned}\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} &= \begin{pmatrix} -25 - \frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25 - \frac{1}{1+t^4} \end{pmatrix}\\ &\times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\\ &+ \begin{pmatrix} -2x_1 - \frac{x_1}{1+x_1^2} \\ -2x_2 - \frac{x_2}{1+x_2^2} \end{pmatrix}\\ &+ \begin{pmatrix} -2x_1 - \frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2 - \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} \quad (13)\end{aligned}$$

A comparison between the systems of ODEs (13) and ODEs (6) gives that $BF(x(t-h(t))) \equiv 0$.Next, $A(t)$, $G(x(t))$and $H(t,x(t))$ are the same as in Example 1. The estimates for the functions $A(t)$, $G(x(t))$ and $H(t,x(t))$ remain the same and correct. As for the final step for this example, it follows that

$$(a_A + g_G + h_H) = (24 + 1 + 1) = 26 > 25 = e_E.$$

According to the above discussions, it follows that conditions (H1) and (H3) of Theorem 3 are satisfied. Thus, the solution $(x_1(t), x_2(t)) = (0,0)$ of the system of ODEs (13) is exponentially stable.

4. Instability

C. Assumption

As for the instability of (5), we need the below conditions.

(H4) There exists a constant positive constant $\bar{a}_A$ such that

$$a_{ii}(t) - \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \ge \bar{a}_A \text{ for all } t \in \mathbb{R}^+.$$

(H5) There exist constants h_0 from (4), $\bar{a}_A$ from (H4) and $f_F > 0$, $g_G > 0$, $H_0 > 0$, $K_2 > 0$ such that

$$\begin{gathered} F(0) = 0, \|F(v)\| \le f_F\|v\| \text{ for all} v \in \mathbb{R}^n,\\ G(0) = 0, sgn x_i G_i(x) > 0 \text{ as } x_i \neq 0, \text{ for all } x \in \mathbb{R}^n,\\ \|G(x)\| \ge g_G\|x\| \text{ for all } x \in \mathbb{R}^n,\\ H(t,0) = 0, sgn x_i H_i(t,x) > 0\\ \text{as}\\ x_i \neq 0, \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,\\ \|H(t,x)\| \ge h_H\|x\| \text{ for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,\\ \bar{a}_A + g_G\ + h_H - (1-h_0)^{-1} f_F\|B\| > 0. \end{gathered}$$

Theorem 4. *We suppose that conditions (H4) and (H5) are held. Then, the trivial solution of the system of DDEs (5) is unstable.*

Proof. Define a new LKF $\Delta_3 := \Delta_3(t,x_t)$ by

$$\Delta_3(t,x_t) := \|x(t)\| - \gamma_1 \int_{t-h(t)}^{t} \|F(x(s))\|\,ds, \quad (14)$$

where $\gamma_1 \in \mathbb{R}$, $\gamma_1 > 0$. It will be determined at the below.

Next, the LKF (14) is equivalent to

$$\begin{aligned}&\Delta_3(t,x_t) := |x_1(t)| + ... + |x_n(t)|\\ &- \gamma_1 \int_{t-h(t)}^{t} |f_1(x(s))|ds - ... - \gamma_1 \int_{t-h(t)}^{t} |f_n(x(s))|ds.\end{aligned}$$

From this point of view, the LKF $\Delta_3(t,x_t)$ satisfies the following relation:

$$\Delta_3(t,x_t) \geq \|x(t)\| - \gamma_1 f_F \int_{t-h(t)}^{t} \|(x(s))\| \, ds$$
$$\geq \|x(t)\| - \gamma_1 f_F h(t) \sup_{t-h(t)\leq s\leq t} \|x(s)\|$$
$$\geq \|x(t)\| - \gamma_1 f_F h_1 \sup_{t-h(t)\leq s\leq t} \|x(s)\|$$
$$= [1-\gamma_1 f_F h_1] \sup_{t-h(t)\leq s\leq t} \|x(s)\| > 0$$

provided that $\|x(t)\| = \sup_{t-h(t)\leq s\leq t} \|x(s)\|$, $h_1 < (\gamma_1 f_F)^{-1}$ and $\|x(t)\| \neq 0$.

Next, the differentiating the LKF $\Delta_3(t,x_t)$ of (14) along (5) leads that

$$\frac{d}{dt}\Delta_3(t,x_t) = \sum_{i=1}^{n} x'_i(t) sgn x_i(t+0) - \gamma \|F(x(t))\| + \gamma \|F(x(t-h(t)))\| \times (1-h'(t)). \quad (15)$$

For the first term of (15), using conditions (H4), (H5) and doing some elementary calculations, we obtain

$$\sum_{i=1}^{n} sgn x_i(t+0) x'_i(t)$$
$$\geq \sum_{i=1}^{n} a_{ii} |x_i(t)| - \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} |a_{ji}| \ |x_i(t)| + \sum_{i=1}^{n} G_i(x(t)) sgn x_i(t+0) + \sum_{i=1}^{n} H_i(t,x(t)) sgn x_i(t+0) - \sum_{i=1}^{n}\sum_{j=1}^{n} |b_{ij}| \ |F_j(x(t-h(t)))|$$
$$= \sum_{i=1}^{n} \left(a_{ii}(t) - \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \right) |x_i(t)| + \|G(x(t))\| + \|H(t,x(t))\| - \|B\| \ \|F(x(t-h(t)))\|$$
$$\geq \bar{a}_A \|x(t)\| + g_G \ \|x(t)\| + h_H \ \|x(t)\| - \|B\| \ \|F(x(t-h(t)))\|. \quad (16)$$

Combining the inequalities (15), (16) and using the condition $0 \leq h'(t) \leq h_0 < 1$, we derive that

$$\frac{d}{dt}\Delta_3(t,x_t) \geq \bar{a}_A \|x(t)\| + g_G \ \|x(t)\| + h_H \ \|x(t)\| - \|B\| \ \|F(x(t-h(t)))\| - \gamma_1 \|F(x(t))\| + \gamma_1 \|F(x(t-h(t)))\| \times (1-h'(t))$$
$$\geq (\bar{a}_A + g_G + h_H) \|x(t)\| - \|B\| \ \|F(x(t-h(t)))\| - \gamma_1 f_F \|x(t)\| + \gamma_1 (1-h_0) \|F(x(t-h(t)))\|.$$

Let $\gamma_1 = (1-h_0)^{-1} \|B\|$. Then,

$$\frac{d}{dt}\Delta_3(t,x_t) \geq \left(\bar{a}_A + g_G + h_H - (1-h_0)^{-1} f_F \|B\| \right) \times \|x(t)\| > 0.$$

Thus, the zero solution of the nonlinear system of DDEs (5) is unstable. $\square$

Example 3. *Let us consider the system:*

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} 25+\frac{1}{1+t^4} & \frac{1}{1+t^4} \\ \frac{1}{1+t^4} & 25+\frac{1}{1+t^4} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} + \begin{pmatrix} 2x_1 + \frac{x_1}{1+x_1^2} \\ +2x_2 + \frac{x_2}{1+x_2^2} \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \sin x_1(t-\frac{1}{2}|\arctan(t)|) \\ \sin x_2(t-\frac{1}{2}|\arctan(t)|) \end{pmatrix}, \quad (17)$$

where $h(t) = \frac{1}{2}|\arctan t|$ *is the delay function,* $t \geq 2^{-1}\pi$.

A comparison between the systems of DDEs (17) and DDEs (5) gives that

$$A(t) = \begin{pmatrix} 25+\frac{1}{1+t^4} & \frac{1}{1+t^4} \\ \frac{1}{1+t^4} & 25+\frac{1}{1+t^4} \end{pmatrix}.$$

By the virtue of the matrix $A(t)$*, we derive that*

$$a_{ii}(t) - \sum_{j=1, j\neq i}^{n} |a_{ji}(t)| \geq 25 = \bar{a}_A$$

since

$$a_{11}(t) - |a_{21}(t)| = 25 + \frac{1}{1+t^4} - \frac{1}{1+t^4} \geq 25 = \bar{a}_A$$

and

$$a_{22}(t) - |a_{12}(t)| = \frac{1}{1+t^4} + 25 - \frac{1}{1+t^4} \geq 25 = \bar{a}_A.$$

Hence,

$$a_{ii}(t) - \sum_{j=1, j\neq i}^{2} |a_{ji}(t)| \geq \bar{a}_A = 25, \forall t \in \mathbb{R}^+.$$

As for the next step, we get

$$G(x) = G(x_1,x_2) = \begin{pmatrix} G_1(x_1,x_2) \\ G_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} 2x_1 + \frac{x_1}{1+x_1^2} \\ 2x_2 + \frac{x_2}{1+x_2^2} \end{pmatrix}$$

$$sgn x_1 G_1(x) = sgn x_1 G_1(x_1,x_2) = 2x_1^2 + \frac{x_1^2}{1+x_1^2} > 0, \ x_1 \neq 0,$$

$$sgnx_2G_2(x) = sgnx_2G_2(x_1, x_2)$$
$$= 2x_2^2 + \frac{x_2^2}{1+x_2^2} > 0,\ x_2 \neq 0,$$
$$\|G(x)\| = \|G(x_1, x_2)\|$$
$$= \left\| \begin{pmatrix} G_1(x_1, x_2) \\ G_2(x_1, x_2) \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2x_1 + \frac{x_1}{1+x_1^2} \\ 2x_2 + \frac{x_2}{1+x_2^2} \end{pmatrix} \right\|$$
$$= \left|2x_1 + \frac{x_1}{1+x_1^2}\right| + \left|2x_2 + \frac{x_2}{1+x_2^2}\right|$$
$$\geq 2|x_1| - \frac{|x_1|}{1+x_1^2} + 2|x_2| - \frac{|x_2|}{1+x_2^2}$$
$$\geq |x_1| + |x_2| = \|x\|,\ g_G = 1 > 0.$$

Additionally, we have

$$H(t,x) = H(t, x_1, x_2)$$
$$= \begin{pmatrix} 2x_1 + \frac{x_1}{1+\exp(t)+x_1^2} \\ 2x_2 + \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix},$$
$$sgnx_1H_1(t,x) = sgnx_1H_1(t, x_1, x_2)$$
$$= 2x_1^2 + \frac{x_1^2}{1+\exp(t)+x_1^2} > 0,\ x_1 \neq 0,$$
$$sgnx_2H_1(t,x) = sgnx_2H_1(t, x_1, x_2)$$
$$= 2x_2^2 + \frac{x_2^2}{1+\exp(t)+x_2^2} > 0,\ x_2 \neq 0.$$
$$\|H(t,x)\| = \|H(t, x_1, x_2)\|$$
$$= \left\| \begin{pmatrix} 2x_1 + \frac{x_1}{1+\exp(t)+x_1^2} \\ 2x_2 + \frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} \right\|,$$
$$= \left|2x_1 + \frac{x_1}{1+\exp(t)+x_1^2}\right|$$
$$+ \left|2x_2 + \frac{x_2}{1+\exp(t)+x_2^2}\right|$$
$$\geq 2|x_1| - \frac{|x_1|}{1+\exp(t)+x_1^2}$$
$$+ 2|x_2| - \frac{|x_2|}{1+\exp(t)+x_2^2}$$
$$\geq |x_1| + |x_2|$$
$$= \|x\|,\ h_H = 1 > 0.$$
$$B = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \|B\| = 5.$$
$$F(x(t - \frac{1}{2}|arctg(t)|)$$
$$= F(x_1(t - \frac{1}{2}|arctg(t)|, x_2(t - \frac{1}{2}|arctg(t)|)$$
$$= \begin{pmatrix} \sin x_1(t - \frac{1}{2}|\arctan(t)|) \\ \sin x_2(t - \frac{1}{2}|\arctan(t)|) \end{pmatrix}$$
$$F(0) = 0, h(t) = \frac{1}{2}|\arctan(t)|.$$

Let

$$u = x(t - \frac{1}{2}|\arctan(t)|, u_1 = x_1(t - \frac{1}{2}|\arctan(t)|)$$

and

$$u_2 = x_2(t - \frac{1}{2}|\arctan(t)|), t \geq \frac{\pi}{2}.$$
$$\|F(u)\| = \|F(u_1, u_2)\| = \left\| \begin{pmatrix} \sin u_1 \\ \sin u_2 \end{pmatrix} \right\|$$
$$= |\sin u_1| + |\sin u_2|$$
$$\leq |u_1| + |u_2|$$
$$= \|u\|,$$
$$f_F = 1.$$

As for the variable delay

$$h = h(t) = \frac{1}{2}|\arctan(t)|,$$

the verifications in Example 1 for this function are the same there, too.

Finally, we have that

$$(\bar{a}_A + g_G + h_H)(1 - h_0) - f_F\|B\|$$
$$= (25 + 1 + 1)(1 - 2^{-1}) - 5$$
$$= 13.5 - 5 = 8.5 > 0.$$

By the virtue of the above estimates, it follows that the conditions (H4) and (H5) of Theorem 4 are satisfied. For this reason, the solution $(x_1(t), x_2(t)) = (0, 0)$ *of the system of DDEs (17) is unstable.*

5. Boundedness

For the bounded solutions of (3), we need to modify condition (H2) as the below:

(H6) There exist positive constants h_0 and a_A from (4) and $(H1)$, respectively, f_F, g_G, h_H and a continuous function $q_Q \in C(\mathbb{R}, \mathbb{R})$ such that

$$F(0) = 0,$$
$$\|F(u) - F(v)\| \leq f_F\|u - v\| \text{ for all } u,\ v \in \mathbb{R}^n,$$
$$G(0) = 0, sgnx_iG_i(x) < 0$$

as

$$x_i \neq 0, \text{ for all } x \in \mathbb{R}^n,$$
$$\|G(x)\| \geq g_G\|x\| \text{ for all } x \in \mathbb{R}^n,$$
$$H(t, 0) = 0, sgnx_iH_i(t,x) < 0$$

as

$$x_i \neq 0, \quad \text{for all } t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^n,$$

$\|H(t,x)\| \geq h_H\|x\|$ for all $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$,

$$\|Q(t, x(t), x(t - h(t)))\| \leq |q_Q(t)| \quad \|x(t)\|,$$
$$(a_A + g_G + h_H - |q_Q(t)|)$$
$$\times(1 - h_0) - f_F\|B\| \geq 0.$$

Theorem 5. *If conditions (H1) and (H6) are held, then the solutions of the system of DDEs (3) are bounded as $t \to +\infty$.*

Proof. By virtue of conditions (H1), (H6) and the LKF $\Delta_1(t, x_t)$, we derive that

$$\frac{d}{dt}\Delta_1(t,x_t) \le -\frac{1}{1-h_0}\Big[(a_A+g_G+h_H)(1-h_0) - f_F\,\|B\|\Big]\,\|x(t)\| + \|Q(t,x(t),x(t-h(t)))\| \le -\frac{1}{1-h_0}\Big[(a_A+g_G+h_H-|q_Q(t)|) \times (1-h_0) - f_F\,\|B\|\Big]\,\|x(t)\|.$$

Hence, from condition (H6), it is clear that

$$\frac{d}{dt}\Delta_1(t,x_t) \le 0.$$

Integrating this inequality, we obtain

$$\Delta_1(t,x_t) \le \Delta_1(t_0,\phi(t_0)) \equiv K_4 > 0,\ \ \phi(t_0) \ne 0. \tag{18}$$

By virtue of the LKF $\Delta_1(t, x_t)$ and (18), we derive that

$$\|x(t)\| \le K_4.$$

Next, it follows that

$$\lim_{t\to+\infty} \||x(t)|\| \le \lim_{t\to+\infty} K_4 = K_4.$$

Thus, the solutions of the system of nonlinear DDEs (3) are bounded as $t \to +\infty$. This is the end of proof of Theorem 5. □

Example 4. *Consider the following perturbed system of DDEs:*

$$\begin{pmatrix} x'_1 \\ x'_2 \end{pmatrix} = \begin{pmatrix} -25-\frac{1}{1+t^4} & -\frac{1}{1+t^4} \\ -\frac{1}{1+t^4} & -25-\frac{1}{1+t^4} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} -2x_1-\frac{x_1}{1+x_1^2} \\ -2x_2-\frac{x_2}{1+x_2^2} \end{pmatrix} + \begin{pmatrix} -2x_1-\frac{x_1}{1+\exp(t)+x_1^2} \\ -2x_2-\frac{x_2}{1+\exp(t)+x_2^2} \end{pmatrix} + \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} \sin x_1(t-\frac{1}{2}|\arctan(t)|) \\ \sin x_2(t-\frac{1}{2}|\arctan(t)|) \end{pmatrix} + \begin{pmatrix} \frac{4\sin x_1}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} \\ \frac{4\sin x_2}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \end{pmatrix}, \tag{19}$$

where $h(t) = \frac{1}{2}|\arctan t|$ is time-varying delay, $t \ge 2^{-1}\pi$.

A comparison between the systems of DDEs (19) and DDEs (3) shows that the functions $A(t)$, $G(x(t))$, $H(t,x(t))$, $F(x(t-h(t)))$ and the constant matrix B are the same as in Example 1. From this point of view, the relations for the functions $A(t)$, $G(x(t))$, $H(t,x(t))$, $F(x(t-h(t)))$ and the matrix B remain the same and correct as in Example 1.

For the remain calculations, we consider the function

$$Q(t,x,x(t-\frac{1}{2}|\arctan(t)|)) = \begin{pmatrix} \frac{4\sin x_1}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} \\ \frac{4\sin x_2}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \end{pmatrix}$$

$$\|Q(t,x,x(t-\frac{1}{2}|\arctan(t)|))\| = \left\|\begin{pmatrix} \frac{4\sin x_1}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} \\ \frac{4\sin x_2}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \end{pmatrix}\right\| = \frac{4|\sin x_1|}{4+|\arctan(t)|+x_1^2(t-\frac{1}{2}|\arctan(t)|)} + \frac{4|\sin x_2|}{4+|\arctan(t)|+x_2^2(t-\frac{1}{2}|\arctan(t)|)} \le [|x_1|+|x_2|] = |q_Q(t)|\,\|x\|,$$

where

$$|q_Q(t)| = 1,$$
$$\|x\| = |x_1| + |x_2|.$$

Next,

$$(\bar{a}_A + g_G + h_H - |q_Q(t)|)(1-h_0) - f_F\,\|B\| = (24+1+1-1)(1-2^{-1})-5 = 12.5-5 = 7.5 > 0.$$

Thus, conditions (H1) and (H6) of Theorem 6 are held. By virtue of the given discussions, we conclude that all the solutions of (19) are bounded as $t \to \infty$.

6. Contributions

In this section, we make comments to the contributions of Theorems 1-5.

1) It follows that the systems of (1) and (2) are particular cases of the systems of DDEs (3) and DDEs (5). This is an improvement and a new contribution (see, [17, 23]).
2) In [13, Theorem 1], the authors proved a theorem on the AS of the linear system of DDEs (1) using a suitable LKF as basic tool. Next, in [23], the authors proved three results on the UAS, the integrability and the boundedness of the solutions

of the nonlinear system of DDEs (2) using a suitable LKF.

In this paper, we proved five new theorems related to the UAS, the instability and the integrability of solutions of the nonlinear system of DDEs (5) by Theorem 1, Theorem 4 and Theorem 2, the exponentially stability of zero solution of the system of nonlinear ODEs (6) by Theorem 3 and the boundedness of solutions of the system of nonlinear DDEs (3) by Theorem 5, respectively.

To prove Theorems 1, 2 and 5, the LKF

$$\Delta_1(t, x_t) := \|x(t)\| + \gamma \int_{t-h(t)}^{t} \|F(x(s))\| \, ds,$$

to prove Theorem 3, the LF

$$\Delta_2(t, x) := \|x(t)\|$$

and to prove Theorem 4, the LKF

$$\Delta_3(t, x_t) := \|x(t)\| - \gamma_1 \int_{t-h(t)}^{t} \|F(x(s))\| \, ds$$

were used as basic tools.

Indeed, these LKFs and LF lead very suitable conditions for Theorem 1-Theorem 5. Next, the instability and the ES results are new, the other three results are nonlinear generalizations of the former results in the literature. These are some other contributions to the topic and literature.

3) In this paper, we provide four examples, which satisfy the conditions of Theorems 1-5, and, in particular cases, we also show the applications of the Theorem 1-Theorem 5.
4) The LKF $\Delta_1(t, x_t)$ implies to eliminate the need to use the Gronwall's inequality for the boundedness of solutions at infinity. Hence, the boundedness result, Theorem 5, has weaker conditions and it is also more general as well as has simple conditions, which are more convenient for applications.

7. Conclusion

In this article, the unperturbed nonlinear system of DDEs (5) with variable delay, the perturbed nonlinear system of DDEs (3) with variable delay and the system of ODEs (6) were taken into consideration. Here, five new results, i.e., Theorem 1 –Theorem 5, which are dealt with the qualitative behaviors of trajectories of solutions called UAS, instability and integrability of solutions of the unperturbed system of DDEs (5), the boundedness of solutions of the perturbed system of DDEs (3) and the exponentially stability of solutions of the system of ODEs (6), were proved using the LKF method for the delay systems (3), (5) and the second method of Lyapunov for the system of ODEs (6), respectively. In the proof of the boundedness result, i.e., Theorem 5, it was not needed to use the Gronwall's inequality. This case allows weaker conditions. Indeed, the novelty and the contributions of the results of this paper are that the results of this article are new and they have weaker conditions than those available in the relevant literature. This idea can be seen form the items 1)-4). Finally, four examples, Example 1-Example 4, were given to make clear the applications of our results.

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Osman Tunç *is an associate professor at Department of Computer Programing , Baskale Vocational School , Van Yuzuncu Yil University. He received his PhD degree from the Van Yuzuncu Yil University Van, Turkey. His research interests include ODEs and Integro-Differential Equations.*
https://orcid.org/0000-0001-7809-210X