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RESEARCH ARTICLE

A simple method for studying asymptotic stability of discrete dynamical systems and its applications

Manh Tuan Hoang,[a*] Thi Kim Quy Ngo,[b] Ha Hai Truong,[c]

[a] *Department of Mathematics, FPT University, Hoa Lac Hi-Tech Park, Km29 Thang Long Blvd, Hanoi, Vietnam*
[b] *Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology (PTIT),Hanoi, Vietnam*
[c] *Department of Basic Sciences, Thai Nguyen University of Information and Communication Technology, Thai Nguyen, Vietnam*
tuanhm14@fe.edu.vn, quyntk@ptit.edu.vn, thhai@ictu.edu.vn

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ABSTRACT

In this work, we introduce a simple method to investigate the asymptotic stability of discrete dynamical systems, which can be considered as an extension of the classical Lyapunov's indirect method. This method is constructed based on the classical Lyapunov's indirect method and the idea proposed by Ghaffari and Lasemi in a recent work. The new method can be applicable even when equilibia of dynamical systems are non-hyperbolic. Hence, in many cases, the classical Lyapunov's indirect method fails but the new one can be used simply. In addition, by combining the new stability method with the Mickens' methodology, we formulate some nonstandard finite difference (NSFD) methods which are able to preserve the asymptotic stability of some classes of differential equation models even when they have non-hyperbolic equilibrium points. As an important consequence, some well-known results on stability-preserving NSFD schemes for autonomous dynamical systems are improved and extended. Finally, a set of numerical examples are performed to illustrate and support the theoretical findings.

1. Introduction

Many important processes and phenomena in real-world situations can be mathematically modeled by autonomous dynamical systems described by differential equations associated with the classical and fractional derivative operators [1–8]. While differential equation models with the classical derivatives have been formed and studied for a long time [1, 3, 5, 6, 8], mathematical models based on fractional differential equations have been strongly developed in recent years (see, for example, [9–28]). The stability analysis of differential equation models has been a central and prominent problem with many useful applications.

In this work, we consider general time-continuous dynamical systems of the form

$$\frac{dy}{dt} = f(y), \quad y(0) = y_0 \in \mathbb{R}^n,$$

where $y : [0, T] \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ are real-valued functions satisfying appropriate conditions to guarantee the existence and uniqueness of solutions of the system (see, for instance, [1,3,5,6,8]).

The stability analysis of the system (1) has played a prominent role in both theory and practice, especially in control theory and mathematical epidemiology [5, 29]. The continuous version of the classical Lyapunov's indirect method can be considered as the most successful approach to this problem (see [29] or also [1,3,5,6,8]). This method studies the asymptotic stability of an equilibrium

*Corresponding Author

point by analyzing the associated Jacobian matrix with respect to the left half-plane. More precisely, an equilibrium point y^* is locally asymptotically stable if all the eigenvalues of the Jacobian matrix $J(y^*)$ lie strictly in the left half-plane; and y^* is unstable if any of the eigenvalues lie in the strict right half-plane. Clearly, the Lyapunov's indirect theorem is only applicable for hyperbolic equilibrium points. Here, an equilibrium point y^* is said to be hyperbolic if none of the eigenvalues of $J(y^*)$ lies on the imaginary axis; otherwise, y^* is said to be non-hyperbolic. Hence, the method fails to determine the asymptotic stability of non-hyperbolic equilibrium points. This leads to a big restriction of the application of the method. For this reason, in a recent work [30], Ghaffari and Lasemi constructed a new method to examine the stability of continuous dynamical systems, which is based on the classical Lyapanov's indirect method. However, it studies the stability of an equilibrium point by analyzing the associated Jacobian matrix at a deleted neighborhood of the equilibrium point instead of at the equilibrium point. As a consequence, the new method can be applicable for non-hyperbolic equilibrium points in many cases. Therefore, the weakness of the classical theorem can be improved.

Similarly to the continuous version, the discrete version of the Lyapunov's indirect method can be considered as a powerful and effective approach to the stability problem of discrete dynamical systems (see, for instance, [5, 31]). This method investigates the stability of an equilibrium point by considering the position of eigenvalues of the associated Jacobian matrix with respect to the unit circle. More specifically, an equilibrium point is asymptotically stable if all the eigenvalues of the Jacobian matrix lie strictly inside the unit circle and is unstable if any of the eigenvalues lie outside the unit circle. Consequently, the method is only applicable when none of the eigenvalues of the Jacobian matrix lies on the unit circle. In this case, equilibrium points are said to be hyperbolic.

Motivated and inspired by the above reason, in this work we introduce a new and simple method to analyze the asymptotic stability of discrete dynamical systems, which can be considered as an extension of the classical Lyapunov's indirect method. This method is constructed based on the classical Lyapunov's indirect method and the idea proposed by Ghaffari and Lasemi in [30]. It is worth noting that the new method can be applicable even when equilibia of dynamical systems are non-hyperbolic. Consequently, in many cases, the classical Lyapunov's indirect method fails but the new theorem can be used simply. In addition, a relation between the new method and the classical Lyapunov's indirect one is also provided.

To illustrate the applicability of the new theorem, we combine it with the Mickens' methodology [32–36] to construct nonstandard finite difference (NSFD) methods, which have ability to preserve the asymptotic stability of some differential equation models even when they possess non-hyperbolic equilibrium points. We recall that the concept of NSFD schemes was first introduced by Mickens in 1980 to overcome drawbacks of standard finite difference ones [32–36]. Nowadays, NSFD schemes have been widely used as a powerful and efficient class of numerical methods for solving differential equations arising in real-world situations. We refer the readers to [32–39] and [40–54] for good reviews and some recent notable works related to NSFD schemes, respectively. Recently, we have successfully developed the Mickens' methodology to construct NSFD schemes for differential equation models arising in real-world applications [55–60]. In the construction of NSFD schemes, one of the most important problem is to formulate NSFD schemes preserving the asymptotic stability of equilibrium points of differential equation models (see, for instance, [43,51,55,61–64]). A common approach to this problem is the use of the continuous and discrete versions of the classical Lyapunov's indirect method. Following this approach, the continuous version is first used to determine the stability of equilibria, and then, the discrete version is applied to analyze the stability of NSFD schemes. However, as mentioned before, the classical Lyapunov' indirect method fails to conclude the asymptotic stability of non-hyperbolic equilibrium points. So, the construction of NSFD schemes for differential equation models having non-hyperbolic equilibrium points is still a challenge. This challenge was mentioned in some well-known works [43,61,62]. An indispensable condition in the previous results on stability-preserving NSFD methods [43,51,55,61–64] is that all equilibrium points of differential equation models must be hyperbolic. This problem leads to a big restriction in the application of these NSFD methods.

For the above reason, by combining the new stability theorem with the Mickens' methodology, we formulate some NSFD methods which can preserve the asymptotic stability of some classes of differential equation models even when they have non-hyperbolic equilibrium points. Consequently, the applicability of the new method is shown and the stability-preserving NSFD schemes formulated in [43, 55, 61, 62] are improved and extended. Therefore, the new method is reliable and it has advantages over the classical one. In numerical examples, we will see that in many cases the classical method is not working but the new method proves helpful.

The plan of this work is as follows:

In Section 2, some concepts and preliminaries are provided. The new stability method is introduced in Section 3. In Section 4, we construct stability-preserving NSFD schemes for some classes of differential equation models having non-hyperbolic equilibrium points. Numerical examples are performed in Section 5. Some conclusions and remarks are presented in the last section.

2. Preliminaries

In this section, we provide some concepts and preliminaries related to stability theory of dynamical systems and NSFD methods, which will be used in the next sections.

2.1. Stability of dynamical systems

The following theorem is known as the Lyapunov's indirect method for continuous dynamical systems.

Theorem 1. *([3, Theorem 4.7]) Let $y^* = 0$ be an equilibrium point for the nonlinear system*

$$\frac{dy}{dt} = f(y), \tag{1}$$

where $f : D \to \mathbb{R}^n$ is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial y}(y)\bigg|_{y=0}.$$

Then,

(1) *The origin is asymptotically stable if $Re\lambda_i < 0$ for all eigenvalues of A.*
(2) *The origin is unstable if $Re\lambda_i > 0$ for one or more of the eigenvalues of A.*

Definition 1. *([8, Definition 2.3.6]) An equilibrium point y^* of the system (1) is said to be hyperbolic if none of the eigenvalues of $df(y^*)$ lies on the imaginary axis.*

The following extension of Theorem 1 was proposed by Ghaffari and Lasemi in [30].

Theorem 2. *Let N be a deleted neighborhood of origin that contains no equilibrium points of the system (1). Let y_0 be the initial condition inside N:$\{i.e, y_0 \in N\}$, and $A = \frac{\partial f}{\partial y}(y)\Big|_{y=y_0}$, then;*

(1) *The origin is asymptotically stable if for any y_0 in N all eigenvalues of A are in the open left-half complex plane.*
(2) *The origin is unstable if for any y_0 in N one or more of the eigenvalues of A are in the open right-half complex plane.*

We now consider a general dynamical system governed by difference equations of the form

$$y_{n+1} = g(y_n), \quad y_0 = c \in \mathbb{R}^n, \tag{2}$$

where $G : D \to \mathbb{R}^n$ and $D \in \mathbb{R}^n$ is the domain of definition of g.

Definition 2. *([8, Definition 1.3.6]) An equilibrium point y^* of the system (2) is said to be hyperbolic if none of the eigenvalues of $dg(y^*)$ lie on the unit circle.*

Theorem 3. *([8, Theorem 1.3.7]) Let $g \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}^n)$. Then an equilibrium point y^* of the system (2) is asymptotically stable of the eigenvalues of $dg(y^*)$ lie strictly inside the unit circle. If any of the eigenvalues lie outside the unit circle the equilibrium point is unstable.*

2.2. Nonstandard finite difference methods

Consider a one-step numerical scheme with a step size h, that approximates the solution $y(t_n)$ of the system (1) in the form:

$$D_h(y_n) = F_h(f; y_n), \tag{3}$$

where $D_h(y_n) \approx dy/dt, F(f; y_n) \approx f(y)$, and $t_n = t_0 + nh$. The following definition is derived from the Mickens' methodology.

Definition 3. *(See [37, Definition 1], [45, Definition 3.3], [64, Definition 3]) The one-step finite-difference scheme (3) for solving System (1) is a NSFD method if at least one of the following conditions is satisfied:*

- *$D_h(y_n) = \dfrac{y_{n+1} - y_n}{\phi(h)}$, where $\phi(h) = h + \mathcal{O}(h^2)$ is a non-negative function*
- *$F(f, y_n) = g(y_n, y_{n+1}, h)$, where $g(y_n, y_{n+1}, h)$ is a non-local approximation of the right-hand side of System (1).*

Definition 4. *([61, Definition 4]) The finite-difference method is called "weakly" nonstandard if the traditional denominator h in the first-order discrete derivative $D_h(y_n)$ is replaced by a non-negative function $\phi(h)$ such that $\phi(h) = h + \mathcal{O}(h^2)$.*

The advantage and power of NSFD schemes over the standard ones are expressed in the following definitions.

Definition 5. *(See [37, Definition 2]) Assume that the solutions of Eq.* (1) *satisfy some property* $\mathcal{P}$. *The numerical scheme* (3) *is called (qualitatively) stable with respect to property* $\mathcal{P}$ *(or* $\mathcal{P}$*-stable), if for every value of* $h > 0$ *the set of solutions of* (3) *satisfies property* $\mathcal{P}$.

Definition 6. *(See [34]) Consider the differential equation* $y' = f(y)$. *Let a finite difference scheme for the equation be* $y_{n+1} = F(y_n, h)$. *Let the differential equation and/or its solutions have property* $\mathcal{P}$. *The discrete model equation is dynamically consistent with the differential equation if it and/or its solutions also have property* $\mathcal{P}$.

3. New stability method for discrete dynamical systems

In this section, we introduce a new method to study the asymptotic stability of discrete dynamical systems and give a relation between it and the Lyapunov's indirect method.

Theorem 4. *Assume that* $y^* \in \mathbb{R}^n$ *is an equilibrium point of the dynamical system* (2), *that is,* $g(y^*) = y^*$. *Let* N^* *be a deleted neighborhood of the equilibrium* y^* *that contains no equilibrium points of the system. Let* y_0 *be any point belonging to* N *and denote* $A^* = \dfrac{\partial g}{\partial y}(y)\Big|_{y=y_0}$. *Then,*

(1) *The equilibrium point* y^* *is asymptotically stable if for any* y_0 *in* N^* *all eigenvalues of* A^* *lie strictly inside the unit circle.*
(2) *The equilibrium* y^* *is unstable if for any* y_0 *in* N^* *one or more of the eigenvalues of* A^* *lie outside the unit circle.*

Remark 1. *The proof of this Theorem is based on the proof of the classical Lyapunov's indirect method (see, for instance, [3, 8, 31]).*

Proof. **Proof of Part (i).** First, it follows from the mean value theorem that

$$\begin{aligned} g_i(g(y)) &= g_i(y) + \frac{\partial g_i}{\partial y}(\xi_i)(g(y) - y) \\ &= g_i(y) + \frac{\partial g_i}{\partial y}(y_0)(g(y) - y) \\ &\quad + \left(\frac{\partial g_i}{\partial y}(\xi_i) - \frac{\partial g_i}{\partial y}(y_0)\right)(g(y) - y), \end{aligned}$$

where ξ_i is a point in the line segment connecting $g(y)$ to the y. Hence, we can write

$$g(g(y)) = g(y) + A^*(g(y) - y) + h(y), \tag{4}$$

where

$$\begin{aligned} A^* &= \frac{\partial g}{\partial y}(y)\Big|_{y=y_0}, \\ h_i(y) &= \left(\frac{\partial g_i}{\partial y}(\xi_i) - \frac{\partial g_i}{\partial y}(y_0)\right)(g(y) - y), \end{aligned}$$

for $i = 1, 2, \ldots, n$ and $h_i(y)$ is the ith row of $h(y)$. The function $h_i(y)$ satisfies

$$|h_i(y)| \le \left\| \frac{\partial g_i}{\partial y}(\xi_i) - \frac{\partial g_i}{\partial y}(y_0) \right\| \big\|(g(y) - y)\big\|.$$

By continuity of $(\partial g / \partial y)$, we obtain that

$$\frac{\|h(y)\|}{\|g(y) - y\|} \to 0 \quad \text{as} \quad \|y - y_0\| \to 0.$$

Therefore, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|h(y)\| \le \epsilon \|g(y) - y\| \quad \text{if} \quad \|y - y_0\| < \delta. \tag{5}$$

We now use the function

$$V(y) = \big(g(y) - y\big)^T R \big(g(y) - y\big),$$

as a Lyapunov function candidate for the nonlinear system (2), where R is a symmetric positive definite matrix. The variation of V relative to (2) is given by

$$\begin{aligned} \Delta V(y) &:= V(g(y)) - V(y) \\ &= \big[g(g(y)) - g(y)\big]^T R \big[g(g(y)) - g(y)\big] \\ &\quad - \big[g(y) - y\big]^T R \big[g(y) - y\big]. \end{aligned}$$

From (4), we have that

$$\begin{aligned} &\Delta V(y) \\ &= \Big[A^*\big(g(y) - y\big) + h(y)\Big]^T R \Big[A^*\big(g(y) - y\big) + h(y)\Big] \\ &- \big(g(y) - y\big)^T R \big(g(y) - y\big) \\ &= \big(g(y) - y\big)^T \big(A^{*T} R A^* - R\big)\big(g(y) - y\big) \\ &+ 2(h(y))^T R A^* \big(g(y) - y\big) + (h(y))^T R h(y). \end{aligned}$$

Since all eigenvalues of the matrix A^* lie strictly inside the unit circle, for every positive definite symmetric matrix T, there is a unique symmetric and positive definite matrix R such that (see Theorem 4.30 in [31] or Lemma B.12 in [8])

$$A^{*T}RA^* - R = -T, \tag{6}$$

which implies that

$$\begin{aligned}&\big(g(y)-y\big)^T\big(A^{*T}RA^*-R\big)\big(g(y)-y\big)\\ &=\big(g(y)-y\big)^T(-T)(g(y)-y)\\ &\le -\lambda_{\min}(T)\|g(y)-y\|^2,\end{aligned}$$

where $\lambda_{\min}(T)$ denotes the minimum eigenvalue of matrix T. Note that $\lambda_{\min}(T)$ is real and positive because T is symmetric and positive definite. Therefore,

$$\begin{aligned}\Delta V(y) &\le -\lambda_{\min}(T)\|g(y)-y\|^2\\ &\quad + 2h^T(y)RA^*\big(g(y)-y\big)+h^T(y)Rh(y).\end{aligned}$$

It follows from the estimate (5) that

$$\begin{aligned}&2(h(y))^TRA^*\big[g(y)-y\big]\\ &\le 2\|h(y)\|\|R\|\|A^*\|\|g(y)-y\|\\ &\le 2\epsilon\|A^*\|\|R\|\|g(y)-y\|^2,\\ &(h(y))^TRh(y)\le\|R\|\|h(y)\|^2\le\|R\|\epsilon^2\|g(y)-y\|^2.\end{aligned}$$

for all $\|y-y_0\|<\delta$. Thus,

$$\begin{aligned}&\Delta V(y)\\ &<\Big(-\lambda_{\min}(T)+2\epsilon\|A^*\|\|R\|+\epsilon^2\|R\|\Big)\|g(y)-y\|^2\end{aligned}$$

for all $\|y-y_0\|<\delta$. We now choose ϵ small enough such that $\lambda_{\min}(T)>2\epsilon\|A^*\|\|R\|+\epsilon^2\|R\|$. Then, $\Delta V(y)<0$. Therefore, for any $y_0\in N$ there always exists $\epsilon>0$ such that $\Delta V(y)<0$. Thus, by the classical Lyapunov's direct method, we conclude that the equilibrium y^* is asymptotically stable. The proof of this part is complete.

Proof of part (ii). Assume that at y_0 the matrix A^* has an eigenvalue which lies outside the unit circle. By [31, Corollary 4.31], then there exists a real symmetric matrix R that is not positive semidefinite for which $A^{*T}RA^*-R=-T$ is negative define. Thus, the Lyapunov function $V(y)=\big(g(y)-y\big)^TR\big(g(y)-y\big)$ is negative at points arbitrarily close to the origin. Furthermore, we also obtain

$$\begin{aligned}\Delta V(y) &= -\big(g(y)-y\big)^TT\big(g(y)-y\big)\\ &\quad+2\big(g(y)-y\big)^T(A^*)^TRh(y)+V(h(y)).\end{aligned}$$

Similarly to the proof of Part (i), if we choose ϵ small enough then $\Delta V(y)\le-\gamma\|g(y)-y\|^2$ for some $\gamma>0$. Therefore, by [31, Theorem 4.27], the equilibrium y^* is unstable. The proof of this part is complete. □

Remark 2. *From the continuity of polynomial roots (see [65, Theorem 3.9.1]), it is easy to verify that if the classical Lyapunov's indirect method is applicable, so is Theorem 4. In other words, the classical Lyapunov's indirect theorem is a consequence of Theorem 4.*

Example 1. *Consider the difference equation*

$$y_{n+1}=y_n+ay_n^3,\quad a\in\mathbb{R}. \tag{7}$$

The equation (7) has a unique equilibrium point $y^*=0$. The Jacobian matrix at y^* is given by $J(y^*)=1$. So, y^* is non-hyperbolic and the classical Lyapunov's indirect method fails to conclude the stability of y^*. However, Theorem 4 is applicable. Indeed, let $y_0\neq0$. The Jacobian matrix at y_0 is given by

$$J(y_0)=1+3ay_0^2.$$

Hence, by Theorem 4, we conclude that:

(1) If $a>0$, then y^* is unstable.
(2) If $a<0$, then y^* is asymptotic stable.

4. Stability-preserving NSFD methods

In this section, we construct NSFD methods which can preserve the stability of not only hyperbolic equilibrium points but also non-hyperbolic equilibrium ones of the system (1). For this purpose, we introduce the following hypotheses for the system (1):

(H1) The set of equilibrium points of the system (1) is finite.

(H2) For each equilibrium point, there is a deleted neighborhood in which none of the eigenvalues of the Jacobian matrix lies on the imaginary axis.

The hypothesis (H2) means that Theorem 4 is applicable for the system (1). Obviously, this condition is satisfied automatically for hyperbolic equilibrium points.

Theorem 5. *Assume that the hypotheses (H1) and (H2) are satisfied for the system* (1). *Then, the following NSFD scheme*

$$\frac{y_{n+1}-y_n}{\phi(h)}=\Big[I-\frac{\phi(h)}{2}\frac{\partial f}{\partial y}(y_n)\Big]^{-1}f(y_n) \tag{8}$$

is dynamically consistent with respect to the asymptotic stability of the system (1).

Proof. Suppose that y^* is an equilibrium point of the system (1) and N is a deleted neighborhood of y^*. For each $y_0\in N$, let us denote by $\lambda_i(y_0)$ and $\mu_i(y_0)$ $(1\le i\le n)$ are eigenvalues of $\dfrac{\partial f}{\partial y}(y_0)$ and $\dfrac{\partial g}{\partial y}(y_0)$, respectively, where g is given by

$$g(y_n)=y_n+\phi\Big[I-\frac{\phi(h)}{2}\frac{\partial f}{\partial y}(y_n)\Big]^{-1}f(y_n).$$

Then, we have

$$\mu_i(y_0) = \Big(1 + \frac{\phi}{2}\lambda_i(y_0)\Big)\Big(1 - \frac{\phi}{2}\lambda_i(y_0)\Big)^{-1}.$$

Hence, $|\mu_i(y_0)| < 1$ if and only if

$$\Big|1 + \frac{\phi}{2}\lambda_i(y_0)\Big| < \Big|1 - \frac{\phi}{2}\lambda_i(y_0)\Big|,$$

or equivalently,

$$2\phi Re(\lambda(y_0)) < 0. \tag{9}$$

We consider two cases of the stability of y^*.

Case 1. y^* is an asymptotically stable equilibrium point of (1). Then, by Theorem 2, there is a deleted neighborhood N of y^* in which $Re(\lambda_i(y_0)) < 0$ for all $i = 1, 2, \dots, n$. Therefore, the inequality (9) is satisfied for all $y_0 \in N$. By Theorem 4, we conclude that y^* is an asymptotically stable equilibrium point of (8).

Case 2. y^* is an unstable equilibrium point of (1). Then, there is a deleted neighborhood N of y^* such that for all $y_0 \in N$, there exists some j $(1 \le j \le n)$ for which $Re(\lambda_j(y_0)) > 0$. Consequently, the inequality (9) does not hold. Therefore, by Theorem 4, y^* is an unstable stable equilibrium point of (8).

Combining Case 1 and Case 2, we conclude that the scheme (8) preserves the stability of the system (1) for all finite step sizes. The proof is complete. □

Remark 3.

- *If $\phi(h)$ is small enough, then $I - \frac{\phi}{2}\frac{\partial f}{\partial y}(y_n) \approx I$. Hence, the existence of the solution of the scheme (8) is ensured. To make sure the scheme (8) is defined for all finite step sizes, we can use the following family of nonstandard denominator functions*

$$\phi(h) = \frac{1 - e^{-\tau h}}{\tau}, \quad \tau > 0$$

  since they are bounded from above by τ^{-1}. Note that the standard denominator function $\phi(h) = h$ is not bounded from above for $h > 0$.
- *In the case it is hard to determine $\Big[I - \frac{\phi}{2}\frac{\partial f}{\partial y}(y_n)\Big]^{-1}$, we can compute the numerical solutions of the scheme (8) as follows.*
  (1) *Set $\delta_n = y_{n+1} - y_n$.*
  (2) *Solve the following linear system*

$$\Big[I - \frac{\phi}{2}\frac{\partial f}{\partial y}(y_k)\Big]\delta_n = \phi f(y_n).$$

  (3) *Compute $y_{n+1} = y_n + \delta_n$.*

The following theorem is proved similarly to Theorem 5.

Theorem 6. *Assume that the hypotheses (H1) and (H2) are satisfied for the system* (1). *Then, the nonstandard implicit trapezoidal scheme*

$$\frac{y_{n+1} - y_n}{\phi(h)} = \frac{1}{2}f(y_n) + \frac{1}{2}f(y_{n+1}) \tag{10}$$

and the nonstandard implicit midpoint scheme

$$\frac{y_{n+1} - y_n}{\phi(h)} = f\left(\frac{y_n + y_{n+1}}{2}\right) \tag{11}$$

are dynamically consistent with respect to the asymptotic stability of the system (1).

Remark 4. *The numerical schemes* (8), (10) *and* (11) *can preserve the asymptotic stability of the system* (1) *for all denominator functions $\phi(h) = h + \mathcal{O}(h^2)$. When $\phi(h) = h$, these schemes becomes standard ones. However, in real-world applications, differential equation models possess not only the stability but also other essential mathematical features, for examples, the positivity. Therefore, nonstandard denominator functions are needed for dynamics consistency. Moreover, they can ensure the existence of the solutions of the schemes* (10) *and* (11).

5. Some applications and numerical experiments

In this section, we conduct numerical simulations to illustrate and support the theoretical findings.

Example 2. *Consider the following scalar differential equation*

$$\dot{y} = ay^3, \quad a \in \mathbb{R}. \tag{12}$$

In this case, the equation has a unique equilibrium point $y^* = 0$, which is non-hyperbolic. It was shown in [30] that

(1) if $a > 0$, y^* is unstable;
(2) if $a < 0$, y^* is asymptotically stable.

Note that the set $\mathbb{R}_+ := \{y \in \mathbb{R} | y \ge 0\}$ is a positively invariant set of the equation (12). Therefore, our objective is to construct an NSFD scheme, which is dynamically consistent with respect to the positivity and stability of (12). For convenience, we only consider the case $a < 0$. The case $a \ge 0$ can be considered in a same way.

Applying the Mickens' methodology, we obtain the following NSFD scheme for (12)

$$\frac{y_{n+1} - y_n}{\phi(h)} = ay_{n+1}y_n^2,$$

or equivalently

$$y_{n+1} = \frac{y_n}{1 - \phi a y_n^2}. \tag{13}$$

The equation (13) implies that $y_n \ge 0$ for all $n \ge 1$ whenever $y_0 \ge 0$. So, the positivity of

(12) is preserved. We now analyze the stability of (13). The Jacobian matrix associated with (13) is given by

$$J(y) = \frac{1+\phi a y^2}{1-\phi a y^2}.$$

Hence, $J(0) = 1$. In this case, $y^* = 0$ is a non-hyperbolic equilibrium point. So, the classical Lyapunov's indirect method fails to conclude the stability of y^*. However, by Theorem 4 we have that y^* is asymptotically stable since

$$J(y) = 1 + \frac{2\phi a y^2}{1-\phi a y^2} \in (-1, 1) \quad \text{for all} \quad y \neq 0.$$

Consequently, we obtain a positvity and stability preserving NSFD scheme for the equation (12).

Example 3. *Consider the following nonlinear system*

$$\begin{aligned} \dot{x} &= -x^3 - x + y, \\ \dot{y} &= x - 2y^3 - y. \end{aligned} \tag{14}$$

The system (14) has a unique equilibrium point, that is, $E^* = (0, 0)$. Moreover,

$$J(0,0) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Hence, E^* is a non-hyperbolic equilibrium point. So, the classical Lyapunov's indirect method cannot conclude the stability of E^*. However, by using a Lyapunov function given by

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2,$$

we have

$$\dot{V} = x\dot{x} + y\dot{y} = -x^4 - 2y^4 - (x-y)^2.$$

Hence, E^* is asymptotically stable. Also, since

$$\begin{aligned} \dot{x}\big|_{x=0} &= y \geq 0, \\ \dot{y}\big|_{y=0} &= x \geq 0, \end{aligned}$$

we conclude that the set $\mathbb{R}_2^+$ is a positively invariant set of (14) (see Theorem B.7 in [66]).

Our object is to construct an NSFD scheme preserving the positivity and stability of the system (14). For this purpose, applying the Mickens' methodology, we propose the following NSFD scheme for (14)

$$\begin{aligned} \frac{x_{n+1} - x_n}{\phi(h)} &= -x_{n+1}x_n^2 - x_{n+1} + y_n, \\ \frac{y_{n+1} - y_n}{\phi(h)} &= x_n - 2y_{n+1}^2 y_n - y_{n+1}. \end{aligned} \tag{15}$$

The system of difference equations (15) can be rewritten in the explicit form

$$\begin{aligned} x_{n+1} &= \frac{x_n + \phi y_n}{1 + \phi + \phi x_n^2}, \\ y_{n+1} &= \frac{y_n + \phi x_n}{1 + \phi + 2\phi y_n^2}, \end{aligned}$$

which implies that the set $\mathbb{R}_+^2$ is a positively invariant set of (15).

We now investigate the stability of (15). The system (15) has a unique equilibrium point, that is $E^* = (0, 0)$. The Jacobian matrix associated with (15) is

$$J(x,y) = \begin{pmatrix} \dfrac{1+\phi-\phi x^2 - 2\phi^2 xy}{(1+\phi+\phi x^2)^2} & \dfrac{\phi}{1+\phi+\phi x^2}, \\ \dfrac{\phi}{1+\phi+2\phi y^2} & \dfrac{1+\phi-2\phi y^2-4\phi^2 xy}{(1+\phi+2\phi y^2)^2} \end{pmatrix}. \tag{16}$$

Hence,

$$J(0,0) = \begin{pmatrix} \dfrac{1}{1+\phi} & \dfrac{\phi}{1+\phi} \\ \dfrac{\phi}{1+\phi} & \dfrac{1}{1+\phi}. \end{pmatrix}.$$

This implies that $E^* = (0,0)$ is a non-hyperbolic equilibrium point. So, it is not suitable to use the classical Lyapunov's indirect method for investigating the stability of E^*. For this reason, we will apply Theorem 4. By some simple algebraic manipulations, we have

$$\begin{aligned} &Trace(J(x,y)) < 1, \\ &1 + Trace(J(x,y)) + \det(J(x,y)) > 0, \\ &1 - Trace(J(x,y)) + \det(J(x,y)) > 0, \end{aligned}$$

for all (x, y) in some appropriate deleted neighborhood of the origin. By the Jury condition [1], all eigenvalues of $J(x, y)$ lie strictly inside the unit circle. Consequently, the stability of E^* is proved.

We now compare the NSFD scheme (15) with the standard Euler and second-order Runge-Kutta (RK2) schemes. Figures 1 and 2 depict numerical solutions generated by the Euler and RK2 schemes. It is clear that the obtained numerical solutions are negative. So, the positivity of the system is violated.

Conversely, from Figures 3-5, we observe that the numerical solutions obtained by the NSFD scheme (15) preserves the positivity and stability of the system for all the chosen step sizes. Also, the dynamics of the numerical solutions does not dependent on the chosen step sizes.

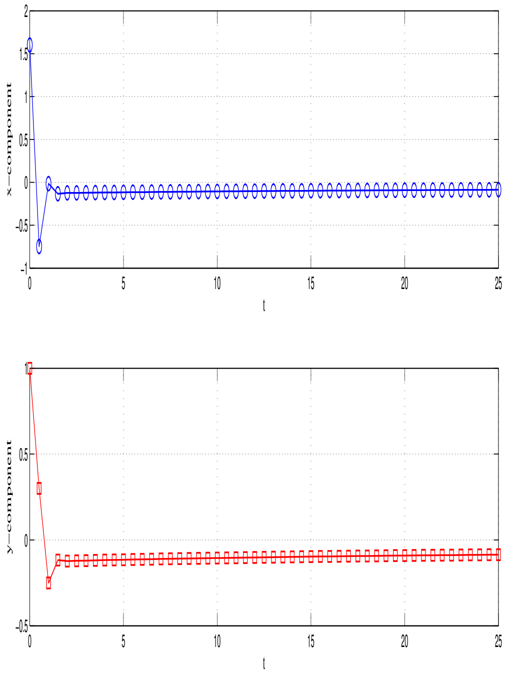

Figure 1. The numerical solutions obtained by the Euler scheme with $h = 0.5$ after 50 iterations in Example 3.

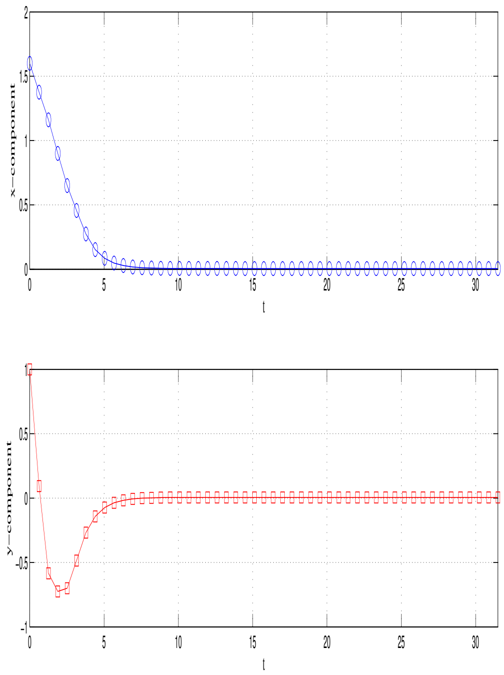

Figure 2. The numerical solutions obtained by the RK2 scheme with $h = 0.63$ after 50 iterations in Example 3.

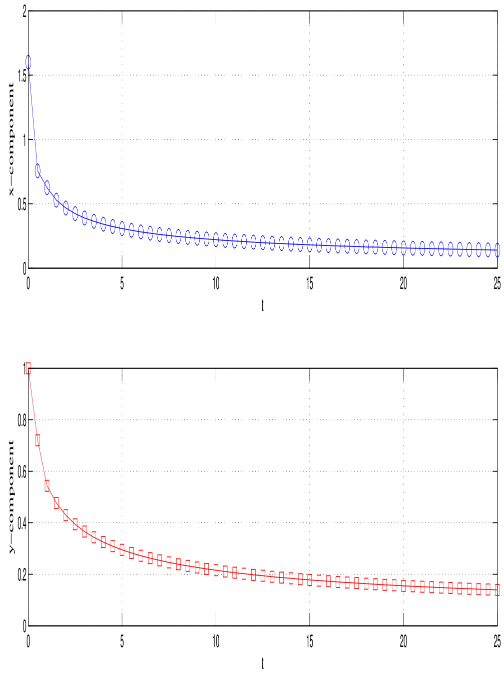

Figure 3. The numerical solutions obtained by the NSFD scheme with $h = 0.5$ after 50 iterations in Example 3.

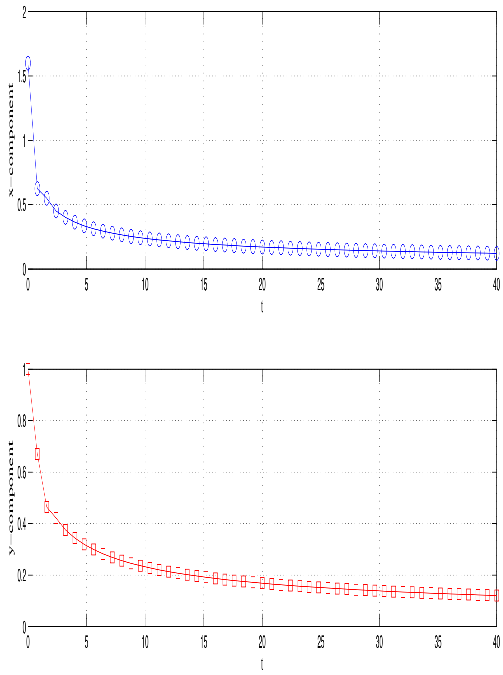

Figure 4. The numerical solutions obtained by the NSFD scheme with $h = 0.8$ after 50 iterations in Example 3.

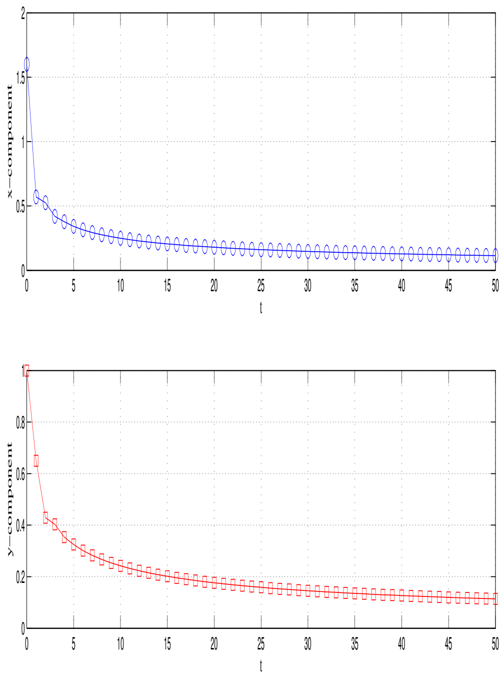

Figure 5. The numerical solutions obtained by the NSFD scheme with $h = 1$ after 50 iterations in Example 3.

If applying the scheme (8) for the system (14) we obtain

$$\begin{pmatrix} \dfrac{x_{n+1} - x_n}{\phi} \\ \\ \dfrac{y_{n+1} - y_n}{\phi} \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{\phi}{2}(3x_n^2 + 1) & -\dfrac{\phi}{2} \\ \\ -\dfrac{\phi}{2} & 1 + \dfrac{\phi}{2}(6y_n^2 + 1) \end{pmatrix} \times \begin{pmatrix} -x_n^3 - x_n + y_n \\ \\ x_n - 2y_n^3 - y_n \end{pmatrix}. \tag{17}$$

The scheme (17) is defined for all denominator functions ϕ since

$$\det \begin{pmatrix} 1 + \dfrac{\phi}{2}(3x_n^2 + 1) & -\dfrac{\phi}{2} \\ \\ -\dfrac{\phi}{2} & 1 + \dfrac{\phi}{2}(6y_n^2 + 1) \end{pmatrix} > 0.$$

Example 4. *Consider the nonlinear system*

$$\begin{aligned} \dot{x} &= -x^5 - x + y, \\ \dot{y} &= -x - y^3 - y. \end{aligned} \tag{18}$$

It is easy to verify that the system (18) has a unique equilibrium point $E^* = (0, 0)$, which is non-hyperbolic. However, by a Lyapunov function given by $V(x, y) = x^2 + y^2$, we have that E^* is asymptotically stable. Our objective is to construct an NSFD scheme which is dynamically consistent with respect to the stability of the system (18). For this purpose, we propose the following NSFD scheme

$$\begin{aligned} \frac{x_{n+1} - x_n}{\phi(h)} &= -x_{n+1}x_n^4 - x_{n+1} + y_n, \\ \frac{y_{n+1} - y_n}{\phi(h)} &= -x_n - y_{n+1}y_n^2 - y_{n+1}. \end{aligned} \tag{19}$$

The explicit form of the scheme (19) is given by

$$\begin{aligned} x_{n+1} &= \frac{x_n + \phi y_n}{1 + \phi + \phi x_n^4}, \\ y_{n+1} &= \frac{y_n - \phi x_n}{1 + \phi + \phi y_n^2}. \end{aligned}$$

The trivial equilibrium point $E^* = (0, 0)$ is also a non-hyperbolic equilibrium point of the scheme (19). So, the classical Lyapunov's indirect method fails to conclude the stability of E^*. However, by the new theorem 4, we can show that E^* is a asymptotically stable equilibrium point of the NSFD scheme (19). Figures 6-8 sketch numerical solutions generated by the NSFD scheme (19) with three different step sizes. In these figures, each blue curve represents a phase plane corresponding to a specific initial data, the red circle represents the position of the stable equilibrium point and the yellow arrows show the evolution of the model. Clearly, the stability of the system (18) is confirmed.

We can also obtain a stability-preserving numerical scheme for the system (18) by using the scheme (8). In this case, the scheme (8) is given by

$$\begin{pmatrix} \dfrac{x_{n+1} - x_n}{\phi(h)} \\ \\ \dfrac{y_{n+1} - y_n}{\phi(h)} \end{pmatrix} = \begin{pmatrix} 1 + \dfrac{\phi}{2}(5x_n^4 + 1) & -\dfrac{\phi}{2} \\ \\ \dfrac{\phi}{2} & 1 + \dfrac{\phi}{2}(3y_n^2 + 1) \end{pmatrix} \times \begin{pmatrix} -x_n^5 - x_n + y_n \\ \\ -x_n - y_n^3 - y_n \end{pmatrix}. \tag{20}$$

Note that

$$\det \begin{pmatrix} 1 + \dfrac{\phi}{2}(5x_n^4 + 1) & -\dfrac{\phi}{2} \\ \\ \dfrac{\phi}{2} & 1 + \dfrac{\phi}{2}(3y_n^2 + 1) \end{pmatrix} > 0,$$

which implies that the scheme (20) is defined for all denominator function $\phi(h)$.

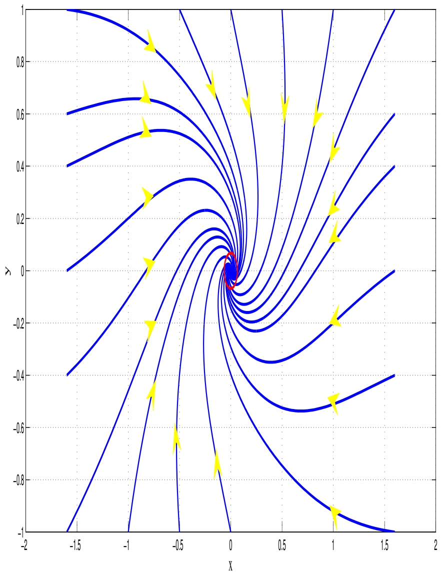

Figure 6. The numerical solutions generated by the NSFD scheme with $h = 0.01$ and $t \in [0, 100]$ in Example 4.

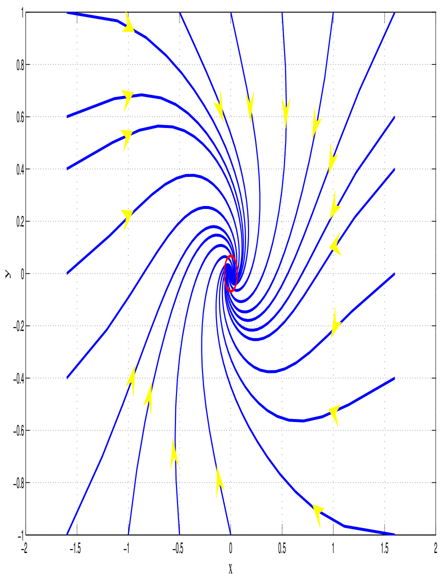

Figure 7. The numerical solutions generated by the NSFD scheme with $h = 0.1$ and $t \in [0, 100]$ in Example 4.

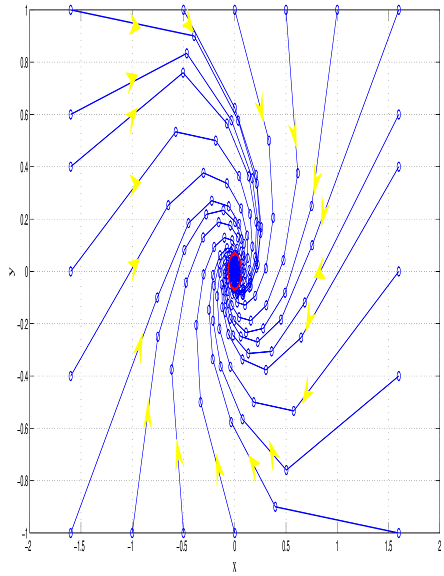

Figure 8. The numerical solutions generated by the NSFD scheme with $h = 0.5$ and $t \in [0, 100]$ in Example 4.

Example 5. *Consider the following system ([30])*

$$\begin{aligned} \dot{x} &= -x^3 + y, \\ \dot{y} &= -4x - y^3. \end{aligned} \tag{21}$$

It was proved in [30] that this system has a unique equilibrium point $E^* = (0, 0)$, which is non-hyperbolic and also asymptotically stable. Numerical solutions generated by the standard Euler and RK2 schemes are sketched in Figures 9-11. Clearly, these schemes cannot preserve the dynamics of the system (21). We now utilize the NSFD scheme (8) to solve the system (21). In this case, we have

$$I - \frac{\phi}{2}\frac{\partial f}{\partial y} = \begin{pmatrix} 1 + \dfrac{3\phi}{2}x^2 & -\dfrac{\phi}{2} \\ \\ 2\phi & 1 + \dfrac{3\phi}{2}y^2 \end{pmatrix},$$

which implies that

$$\det\Big(I - \frac{\phi}{2}\frac{\partial f}{\partial y}\Big) = 1 + \frac{3\phi}{2}x^2 + \frac{3\phi}{2}y^2 + \frac{9\phi^2}{4}x^2y^2 + \phi^2 > 0.$$

Hence, the scheme (8) is defined for all denominator functions and step sizes. Numerical solutions obtained by the NSFD scheme (8) with $\phi(h) = 1 - e^{-h}$ are depicted in Figures 12-14. It is clear that the dynamics of the system (21) is preserved.

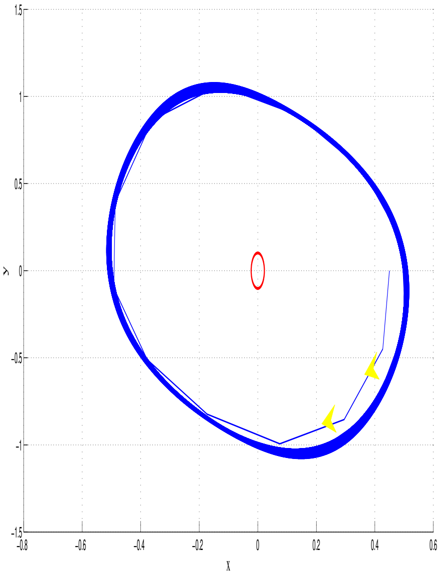

Figure 9. The numerical solution generated by the Euler scheme with $h = 0.2$ and $t \in [0, 1000]$ in Example 5.

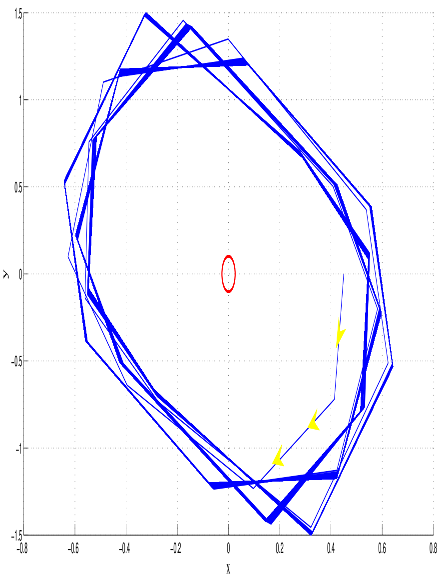

Figure 10. The numerical solution generated by the Euler scheme with $h = 0.4$ and $t \in [0, 1000]$ in Example 5.

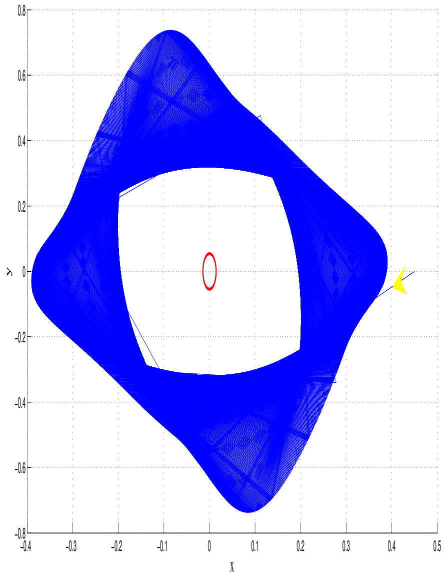

Figure 11. The numerical solutions generated by the RK2 scheme with $h = 0.7$ and $t \in [0, 980]$ in Example 5.

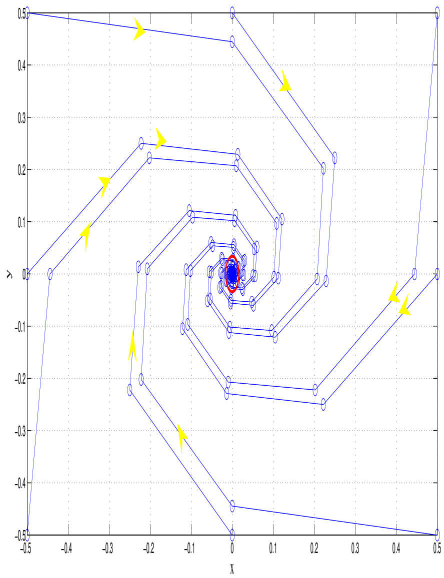

Figure 12. The numerical solutions generated by the NSFD scheme with $h = 1.0$ and $t \in [0, 1000]$ in Example 5.

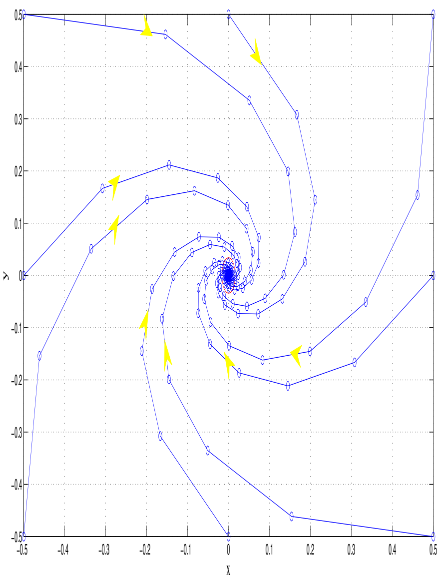

Figure 13. The numerical solutions generated by the NSFD scheme with $h = 0.5$ and $t \in [0, 1000]$ in Example 5.

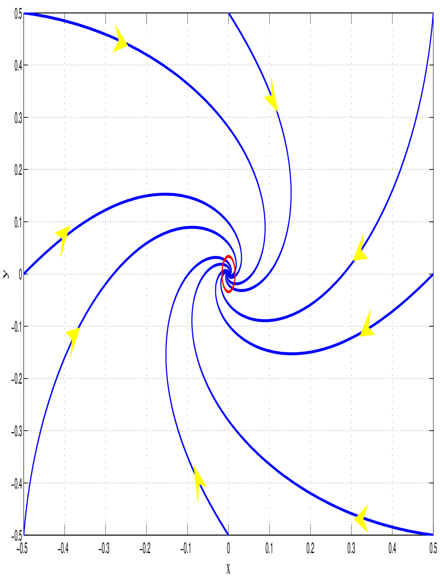

Figure 14. The numerical solutions generated by the NSFD scheme with $h = 0.01$ and $t \in [0, 1000]$ in Example 5.

Example 6 (Stabilization of nonlinear systems by feedback)**.** *Consider the following discrete dynamical systems described by the nonlinear difference equation*

$$y_{n+1} = y_n + ay_n^3, \quad a > 0. \tag{22}$$

It was proved in Example 1 that the equilibrium point $y^* = 0$ is unstable. Our objective is to find a control that stabilizes this system. More clearly, we need to determine a feedback control $u_n = h(y_n)$ in such a way that $y^* = 0$ of the corresponding closed-loop system is asymptotically stable. For this purpose, we consider

$$u_n = Cy_n^3, \quad C \in \mathbb{R}.$$

Then, the corresponding closed-loop system is given by

$$y_{n+1} = y_n + (a + C)y_n^3. \tag{23}$$

The Jacobian matrix of (23) evaluating at $y^* = 0$ is

$$J(0) = 1.$$

Consequently, the classical Lyapunov stability theorem fails to conclude the stability of (23). However, the new method (Theorem 4) can be used easily. Indeed, the Jacobian matrix (23) is given by

$$J(y) = 1 + 3(a + C)y^2,$$

which implies that $J(y) < 1$ if $C > -a$. On the other hand, $J(y) > -1$ whenever $y^2 < \dfrac{-2}{3(a + C)}$. Therefore, by using Theorem 4 we deduce that (23) is locally asymptotically stable if $C > -a$. This means that the desired feedback control u_n is determined.

Let us consider a more complicated system. Consider the following nonlinear system

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{3}x_n^3, \\ y_{n+1} &= y_n + \frac{1}{2}x_n^2 + \frac{1}{5}y_n^5. \end{aligned} \tag{24}$$

This system has a unique equilibrium point $E^* = (x^*, y^*) = (0, 0)$. The Jacobian of the system is given by

$$J(x, y) = \begin{pmatrix} 1 + x^2 & 0 \\ x & 1 + y^4. \end{pmatrix}$$

Therefore, the classical Lyapunov stability theorem cannot conclude the stability of E^*. However, E^* is unstable by applying Theorem 4.

To stabilize the system (24), we use a feedback control $u_n = (\alpha x_n^3, \beta y_n^5)$, where $\alpha, \beta \in \mathbb{R}$. Then, the closed-loop system is given by

$$\begin{aligned} x_{n+1} &= x_n + \frac{1}{3}x_n^3 + \alpha x_n^3, \\ y_{n+1} &= y_n + \frac{1}{2}x_n^2 + \frac{1}{5}y_n^5 + \beta y_n^5. \end{aligned} \tag{25}$$

The Jacobian matrix of (25) is

$$J(x, y) = \begin{pmatrix} 1 + (3\alpha + 1)x^2 & 0 \\ x & 1 + (5\beta + 1)y^4. \end{pmatrix}$$

Hence, the classical Lyapunov stability theorem is not applicable to determine the stability of (25), *but it follows from Theorem 4 that the closed-loop system is locally asymptotically stable if*

$$\alpha < -\frac{1}{3}, \qquad \beta < -\frac{1}{5}.$$

Hence, the system (24) *is stabilized.*

6. Conclusions and remarks

In this work, based on the classical Lyapunov's indirect method and the idea proposed by Ghaffari and Lasemi in [30], we have introduce a new and simple method for investigating the asymptotic stability of discrete dynamical systems (Theorem 4), which can be considered as an extension of the classical Lyapunov's indirect method. It is worth noting that the new method can be applicable even when equilibia of dynamical systems are non-hyperbolic. Hence, in many cases, the classical Lyapunov's indirect method fails but the new one can be used simply. Next, using the new theorem, we have constructed NSFD methods which are able to preserve the asymptotic stability of differential equation models having non-hyperbolic equilibrium points (Theorems 5 and 6). As an important consequence, some well-known results on positivity-preserving NSFD schemes for autonomous dynamical systems formulated in [43, 55, 61, 62] have been improved and extended. Finally, a set of numerical examples are performed to illustrate and support the theoretical findings.

In the near future, we will study practice applications of the new method to problems arising in control theory, economic and applied sciences. In addition, extensions of the new stability method for nonlinear systems associated with fractional-order operators will be also considered.

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Manh Tuan Hoang *received the Ph.D. degree in Applied Mathematics from Graduate University of Science and Technology, Vietnam Academy of Science and Technology (VAST) in 2021, the M.S in Applied Mathematics in 2015 and the B.S degree in Mathematics in 2012 from VNU University of Science. Currently, he is a lecturer-researcher at the Department of Mathematics, FPT University. His research interests are the qualitative theory and numerical analysis of differential equations and mathematical methods in information technology.*

https://orcid.org/0000-0001-6089-3451

Thi Kim Quy Ngo *received the Ph.D. degree in Applied Mathematics from Graduate University of Science and Technology, Vietnam Academy of Science and Technology (VAST) in 2017. Currently, she is a lecturer at the Department of Scientific Fundamentals, Posts and Telecommunications Institute of Technology (PTIT). Her research interests are the qualitative theory and numerical simulation of high-order differential equations with applications.*

https://orcid.org/0000-0002-8605-862X

Ha Hai Truong *received the Ph.D. degree in Applied Mathematics from Institute of Information Technology, Vietnam Academy of Science and Technology (VAST) in 2013. Currently, she is a lecturer at the Department of Basic Sciences, Thai Nguyen University of Information and Communication Technology. Her research interests are numerical methods for high-order differential equations and their applications.*

https://orcid.org/0000-0002-3673-8651