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Minimax fractional programming problem involving nonsmooth generalized α-univex functions

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Abstract. In this paper, we introduce a new class of generalized α-univex functions where the involved functions are locally Lipschitz. We extend the concept of α-type I invex [S. K. Mishra, J. S. Rautela, On nondifferentiable minimax fractional programming under generalized α-type I invexity, J. Appl. Math. Comput. 31 (2009) 317-334] to α-univexity and an example is provided to show that there exist functions that are α-univex but not α-type I invex. Furthermore, Karush-Kuhn-Tucker-type sufficient optimality conditions and duality results for three different types of dual models are obtained for nondifferentiable minimax fractional programming problem involving generalized α-univex functions. The results in this paper extend some known results in the literature.

Keywords: Nondifferentiable minimax fractional programming; α-univexity; sufficient optimality conditions; duality.
AMS Classification: 29A51, 49J35, 90C32

1. Introduction

Fractional programming models have become a subject of wide interest since they provide a universal apparatus for a wide class of models. For example, it can be used in engineering, corporate planning, agricultural planning, public policy decision making, financial analysis of a firm, health care, and educational planning. In these sorts of problems the objective function is usually given as a ratio of functions in fractional programming form (see Stancu Minasion [20]).

The problems, in which both a minimization and a maximization process of fractional objectives are performed, are usually called in decision science as generalized minimax fractional programming problems. These problems have arisen in game theory [3], goal programming [4], minimum risk problems [21], economics [22] and multiobjective programming [23].

Nonlinear programming problems containing square roots of positive semidefinite quadratic forms have arisen in stochastic programming, in multifacility location problems, and in portfolio selection problems, among others. A fairly extensive list of references pertaining to various aspects of these problems is given in Zalmai [26]. Generalizations of convexity related to optimality conditions and duality for minimax fractional programming problems have been of much interest in the recent past and many contributions have been made to this development. For example, see [1, 5, 8-20, 24] and the references cited therein. Yadav and Mukherjee [24] formulated two dual models for minimax fractional programming problem and established some duality results. In view of some omissions and inconsistencies in Yadav and Mukherjee [24], Chandra and Kumar [5] constructed two dual models, and proved various duality theorems under convexity assumptions.

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The necessary and sufficient conditions for generalized minimax programming were first developed by Schmitendorf [19]. Bector and Bhatia [1] relaxed the convexity assumptions in the sufficient optimality condition in [19] and also employed the optimality conditions to construct several dual models which involve pseudo-convex and quasi-convex functions, and derived weak and strong duality theorems.

Liu [13, 14] obtained the necessary and sufficient optimality conditions and derived duality theorems for a class of nonsmooth multiobjective fractional programming problems involving (F,ρ)-convex and pseudoinvex functions. Lai and Lee [12] focus his study on nondifferentiable minimax fractional programming problems and its two parameter-free dual models. They also established weak, strong and strict converse duality theorems under the assumptions of pseudo/quasi-convex functions. In the formulation of the dual models in [12] optimality conditions given in [11] are used. Zheng and Cheng [25] introduce a new class of nonsmooth generalized (F,ρ,θ)-*d*-univex function and derived sufficient optimality conditions and duality theorems for nondifferentiable minimax fractional programming problem and its three different types of dual models.

To relax the definition of invex function recently Noor [18] introduced the concept of α-invex functions. Mishra and Rautela [17] study a nondifferentiable minimax fractional programming problem under the assumption of generalized α-type I invex which has been defined in the setting of Clarke's derivative and established sufficient optimality conditions and duality theorems for the three different type of dual problems.

Bector *et al*. [2] established optimality and duality results for a nonlinear multiobjective programming problem involving univex functions which have been defined by relaxing the definition of an invex function by Bector *et al*. [2] itself.

In this paper, firstly we introduce the concept of nonsmooth α-univex functions and a counter example is given to show that there exists a function which is nonsmooth α-univex but not α-type I invex given in [17]. Then we establish sufficient optimality conditions for nondifferentiable minimax fractional programming problems involving the aforesaid functions. Finally, weak, strong and strict converse duality theorems are discussed in order to relate the efficient solutions of primal problem and its three different types of dual models.

This paper is organized as follows. Section 2 is devoted to some definitions and notations. In Section 3, we derive the sufficient optimality conditions for nondifferentiable minimax fractional programming problems under the assumption of generalized α-univex functions. Duality results are presented in Sections 4-6. This work extends the works of Mishra and Rautela [17] and partially the results of Jayswal [10] to the nonsmooth case.

2. Preliminaries

Throughout this paper, let R^n be the *n*-dimensional Euclidean space and R^n_+ be its non-negative orthant. Let *X* be a nonempty subset of R^n. First, we recall the following definitions.

Definition 2.1 [6] A function $f: X \to R$ is said to Lipschitz near $x \in X$ if for some $K > 0$,

$$|f(y) - f(z)| \le K\|y - z\|,$$

$\forall y, z$ within a neighbourhood of x.

We say that $f: X \to R$ is locally Lipschitz on X if it is Lipschitz near any point of X.

Definition 2.2 [6] If $f: X \to R$ is locally Lipschitz at $x \in X$, the *generalized derivative* (in the sense of Clarke) of f at $x \in X$ in the direction $v \in R^n$, denote by $f^0(x;v)$, is given by

$$f^0(x;v) = \limsup_{\substack{y\to x \\ \lambda \downarrow 0}} \frac{f(y+\lambda v) - f(y)}{\lambda}.$$

Definition 2.3 [6] The *Clarke's generalized gradient* of f at $x \in X$, denoted by $\partial f(x)$, is defined as follows:

$$\partial f(x) = \{\xi \in R^n : f^0(x;v) \ge \xi^T v \quad \forall v \in R^n\}.$$

It follows that, for any $v \in R^n$

$$f^0(x;v) = \max\{\xi^T v : \xi \in \partial f(x)\}.$$

Definition 2.4 [18] A subset *X* is said to be α-invex set, if there exists $\eta: X \times X \to R^n$, $\alpha(x,u): X \times X \to R_+ \setminus \{0\}$ such that

$$u + \lambda\alpha(x,u)\eta(x,u) \in X, \ \forall x,u \in X, \ \lambda \in [0,1].$$

It is well known that the α-invex set need not be a convex set, see Noor [18].

Definition 2.5 [18] The function f on the α-invex set is said to be α-preinvex with respect to η, if

$$f(u+\lambda\alpha(x,u)\eta(x,u)) \le (1-\lambda)f(u)+\lambda f(x),\quad \forall x,u\in X,\ \lambda\in[0,1].$$

Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(u)=-|u|$ is not a convex function, but it is a preinvex function with respect to η and $\alpha(x,u)=1$, where

$$\eta(x,u)=\begin{cases} x-u, \text{ if } x\le 0, u\le 0 \text{ and } x\ge 0, u\ge 0,\\ u-x, \text{ otherwise.}\end{cases}$$

The following example shows that α-preinvex function exist.

Example 2.1 [7] Let $X=R$. For any $x,u\in X$, let $\alpha(x,u)=1$, $\eta(x,u)=e^x-e^u$ and $f(u)=c$, where $c\in R$ is a constant. Then X is an α-invex set with respect to α and η and

$$f(u+\lambda\alpha(x,u)\eta(x,u)) = (1-\lambda)f(u)+\lambda f(x),\quad \forall x,u\in X,\ \forall\lambda\in[0,1],$$

which indicates that F is α-preinvex with respect to α and η on X.

From now onwards, unless otherwise is specified, we assume that X is a nonempty α-invex set with respect to α and η.

Consider the following nondifferentiable minimax fractional programming problem:

$$\textbf{(P)}\qquad \inf_{x\in R^n}\sup_{y\in Y}\frac{f(x,y)+\langle x,\ Ax\rangle^{1/2}}{g(x,y)-\langle x,\ Bx\rangle^{1/2}},$$

subject to $h(x)\le 0$,

where $f,\ g: R^n\times R^m\to R$ and $h: R^n\to R^p$ are locally Lipschitz functions, A and B be $n\times n$ positive semi-definite matrices and Y, an α-invex set, is a compact subset of R^m.

Let $\Im_P$ be the set of all feasible solutions of (P). For each $(x,\ y)\in R^n\times R^m$, define

$$\phi(x,y)=\frac{f(x,y)+\langle x,\ Ax\rangle^{1/2}}{g(x,y)-\langle x,\ Bx\rangle^{1/2}}.$$

Suppose that for each $(x,\ y)\in R^n\times Y$,

$$f(x,y)+\langle x,\ Ax\rangle\ge 0$$

and $$g(x,y)-\langle x,\ Bx\rangle > 0.$$

Denote

$$\bar{Y}(x)=\left\{\bar{y}\in Y:\frac{f(x,\bar{y})+\langle x,\ Ax\rangle^{1/2}}{g(x,\bar{y})-\langle x,\ Bx\rangle^{1/2}} = \sup_{y\in Y}\frac{f(x,y)+\langle x,\ Ax\rangle^{1/2}}{g(x,y)-\langle x,\ Bx\rangle^{1/2}}\right\},$$

$$J=\{1,2,\dots,p\},\ J(x)=\{j\in J: h_j(x)=0\}.$$

Let K be a triplet such that

$$K(x)=\Big\{(s,t,\bar{y})\in N\times R_+^s\times R^{ms}: 1\le s\le n+1,\ t=(t_1,t_2,\dots,t_s)\in R_+^s \text{ with } \sum_{i=1}^{s} t_i=1 \text{ and } \bar{y}=(\bar{y}_1,\bar{y}_2,\dots,\bar{y}_s) \text{ and } \bar{y}_i\in\bar{Y}(x),\ \forall i=1,\dots,s\Big\}.$$

Since Y is a compact subset of R^m, it follows that for each $x_0\in\Im_P$, $\bar{Y}(x_0)\ne\phi$. Thus, for any $\bar{y}_i\in\bar{Y}(x_0)$, we have a positive constant $k_0=\phi(x_0,\ \bar{y}_i)$.

We shall make use of the following generalized Schwartz inequality:

$$\langle x, Av\rangle \le \langle x, Ax\rangle^{1/2}\langle v, Av\rangle^{1/2} \qquad (1)$$

for some $x,v\in R^n$, the equality holds when $Ax=\lambda Av$ for some $\lambda\ge 0$.

Hence if $\langle v, Av\rangle^{1/2}\le 1$, we have

$$\langle x, Av\rangle \le \langle x, Ax\rangle^{1/2}. \qquad (2)$$

In order to relax the convexity assumption in the above problem, we impose the following definitions. Let $f: X\to R$ be a locally Lipschitz function.

Definition 2.6 The function f is said to be (strictly) α-univex at $a\in X$ with respect to $b,\ \phi,\ \alpha$ and η, if there exist $\eta: X\times X\to R^n$, $\alpha: X\times X\to R_+\setminus\{0\}$, nonnegative functions b, also defined on $X\times X$, and function $\phi: R\to R$

such that, for all $x \in X$,
$b(x,a)\phi[f(x)-f(a)] \geq (>) \langle \alpha(x,a)\xi, \eta(x,a)\rangle$,
$\forall \xi \in \partial f(a)$.

Remark 2.1 From Definition 2.5, there are following special cases:

- **(i)** If the function *f* is differentiable at *a*, and $\alpha(x,a)=1$, then we can see that the definition 2.6 implies the definition of univex function given in Bector *et al.* [2].
- **(ii)** Evidently, if we choose $b(x,a)=1$, $\alpha(x,a)=1$ and ϕ as an identity function and *f* is differentiable, then we see that definition 2.6 reduces to definition of invex function given in Hanson [8].
- **(iii)** If the function *f* is differentiable at *a*, then we obtain definition of α-univexity given in Jayswal [10].
- **(iv)** If we define $\phi : R \to R$ with $\phi(V)=V$ and $b(x,a)=1$, then we get the definition of α-type I invex given in Mishra and Rautela [17].

It is noted that, not every α-univex function is α-type I invex function [17]. We have the following counter-example, which shows that the function *f* is α-univex but not α-type I invex.

Example 2.2 Let $x \in R$, $a = 0$ and

$$f(x)=\begin{cases} x, & x \geq 0, \\ 2x, & x < 0. \end{cases}$$

Clearly, $\partial f(a)=[1,\ 2]$. Let $b(x,a)=6/x^2$ and let $\phi : R \to R$ given by $\phi(V)=V^2$. Let $\alpha(x,a)=1/(1+|\sin x|)$ and $\eta(x,a)=|\sin x|$. Then *f* is α-univex at a with respect to b, ϕ, α and η for all $x \in R$.

On the other hand, if we take $x < 0$, we have
$f(x)-f(a) < \langle \alpha(x,a)\xi, \eta(x,a)\rangle,\ \forall \xi \in \partial f(a)$,
which shows that f is not α-type I invex at a with respect to same α and η.

Definition 2.7 The function *f* is said to be pseudo α-univex at $a \in X$ with respect to $b,\ \phi,\ \alpha$ and η, if there exist $\eta : X \times X \to R^n$, $\alpha : X \times X \to R_+ \setminus \{0\}$, nonnegative functions *b*, also defined on $X \times X$, and function $\phi : R \to R$ such that, for all $x \in X$,

$$\langle \alpha(x,a)\xi, \eta(x,a)\rangle \geq 0 \Rightarrow b(x,a)\phi[f(x)-f(a)] \geq 0,\ \ \forall \xi \in \partial f(a)$$

equivalently,

$$b(x,a)\phi[f(x)-f(a)] < 0 \Rightarrow \langle \alpha(x,a)\xi, \eta(x,a)\rangle < 0,\ \forall \xi \in \partial f(a).$$

The following example shows that there exists function which is pseudo α-univex but neither α-type I invex nor pseudo α-type I invex.

Example 2.3 Let $X = R \setminus \{0\}$, $f : X \to R$ be defined by $f(x)=|x|$.

Obviously, $\partial f(x)=\begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$

Let $b(x,a)=|x-a|$ and let $\phi : R \to R$ given by $\phi(V)=V^2$. Let $\alpha(x,a)=|\sin x|$ and $\eta(x,a)=\begin{cases} 1, & x > 0, \\ -1, & x < 0, \end{cases}$. Then f is pseudo α-univex on X with respect to b, ϕ, α and η. But f is neither α-type I invex nor pseudo α-type I invex with respect to same α and η as can be seen by taking $x < a$.

Definition 2.8 The function *f* is said to be strict pseudo α-univex at $a \in X$ with respect to $b,\ \phi,\ \alpha$ and η, if there exist $\eta : X \times X \to R^n$, $\alpha : X \times X \to R_+ \setminus \{0\}$, non-negative functions *b*, also defined on $X \times X$, and function $\phi : R \to R$ such that, for all $x \in X$,

$$\langle \alpha(x,a)\xi, \eta(x,a)\rangle \geq 0 \Rightarrow b(x,a)\phi[f(x)-f(a)] > 0,\ \ \forall \xi \in \partial f(a)$$

equivalently,

$$b(x,a)\phi[f(x)-f(a)] \leq 0 \Rightarrow \langle \alpha(x,a)\xi, \eta(x,a)\rangle < 0,\ \ \forall \xi \in \partial f(a).$$

Example 2.4 Let X, f, b, ϕ, α and η be same as in Example 2.3. By Example 2.3, we know that f is pseudo α-univex on X with respect to b, ϕ, α and η. However, if we assume $x \neq a,\ \forall x, a \in X$, in the above Example 2.3, then f is strictly pseudo α-univex with respect to b,ϕ,α and η.

Definition 2.9 The function *f* is said to be quasi α-univex at $a \in X$ with respect to $b,\ \phi,\ \alpha$ and η, if there exist $\eta : X \times X \to R^n$, $\alpha : X \times X \to R_+ \setminus \{0\}$, nonnegative functions *b*, also defined on $X \times X$, and function $\phi : R \to R$ such that, for all $x \in X$,

$$b(x,a)\phi[f(x) - f(a)] \le 0$$
$$\Rightarrow \langle \alpha(x,a)\xi, \eta(x,a)\rangle \le 0, \quad \forall \xi \in \partial f(a)$$

equivalently,

$$\langle \alpha(x,a)\xi, \eta(x,a)\rangle > 0$$
$$\Rightarrow b(x,a)\phi[f(x) - f(a)] > 0, \ \forall \xi \in \partial f(a).$$

The following example shows that quasi α-univex function exists.

Example 2.5 Let X, f, b and α be same as in Example 2.3. However, if we define $\phi(V) = -V^2$ and $\eta(x,a) = \begin{cases} -1, & x > 0, \\ 1, & x < 0, \end{cases}$.

Then f is quasi α-univex with respect to b,ϕ,α and η.

The following example shows that there exists function which is quasi α-univex but not pseudo α-type I univex not pseudo α-type I invex and not α-type I invex.

Example 2.6 The function $f : R \to R$ is defined by $f(x) = x$. Let

$$b(x,a) = |x-a|, \eta(x,a) = \begin{cases} 1, & x > a, \\ -1, & x = a, \\ 0, & x < a, \end{cases}$$

and $\alpha(x,a) = \begin{cases} 1, & x \ge a, \\ -1, & x < a. \end{cases}$

Further assume that $\phi : R \to R$ be given by $\phi(V) = V$. Then f is quasi α-univex with respect to b,ϕ,α,η and $\partial f(x) = \{f'(x)\} = \{1\}$ for all $x \in R$. But f is neither pseudo α-type I univex with respect to $b,\ \phi,\ \alpha$ and η nor pseudo α-type I invex with respect to α and η on R. Also it can be easily seen that for $x < a$, f is not α-type I invex with respect to α and η on R.

The following result from [12] is needed in the sequel.

Lemma 2.1 *Let x_0 be an optimal solution for* (P) *satisfying $\langle x_0,\ Ax_0\rangle > 0$, $\langle x_0,\ Bx_0\rangle > 0$ and $\partial h_j(x_0)$, $j \in J(x_0)$ are linearly independent. Then there exist $(s, t^*, \bar{y}) \in K(x_0)$, $u,\ v \in R^n$ and $\mu^* \in R_+^p$ such that*

$$0 \in \sum_{i=1}^{s} t_i^*\left(\partial f(x_0, \bar{y}_i) + Au - k_0\left(\partial g(x_0,\ \bar{y}_i) - Bv\right)\right)$$
$$+ \partial\langle \mu^*, h(x_0)\rangle, \tag{3}$$

$$f(x_0, \bar{y}_i) + \langle x_0,\ Ax_0\rangle^{1/2} - k_0\Big(g(x_0,\ \bar{y}_i),$$
$$-\langle x_0,\ Bx_0\rangle^{1/2}\Big) = 0, \quad i = 1,2,\ldots,s \tag{4}$$

$$\langle \mu^*, h(x_0)\rangle = 0, \tag{5}$$

$$t_i^* \in R_+^s \quad \text{with} \quad \sum_{\mathrm{i}=1}^{\mathrm{s}} t_i^* = 1, \tag{6}$$

$$\langle u,\ Au\rangle \le 1, \quad \langle v,\ Bv\rangle \le 1,$$

$$\begin{cases} \langle u,\ Au\rangle \le 1, \quad \langle v,\ Bv\rangle \le 1, \\ \langle x_0,\ Au\rangle = \langle x_0,\ Ax_0\rangle^{1/2}, \\ \langle x_0,\ Bv\rangle = \langle x_0,\ Bx_0\rangle^{1/2}. \end{cases} \tag{7}$$

It should be noted that both the matrices A and B are positive definite at the solution x_0 in the above Lemma. If one of $\langle Ax_0,\ x_0\rangle$ and $\langle Bx_0, x_0\rangle$ is zero, or both A and B are singular at x_0, then for $(s, t^*, \bar{y}) \in K(x_0)$, we can take

$$Z_{\bar{y}}(x_0) = \{z \in R^n : \langle \zeta_j,\ z\rangle \le 0, \forall \zeta_j \in \partial h_j(x_0),$$
$$j \in J(x_0)\},$$

with any one of the following (i) - (iii) holds $\forall v \in \partial f(x_0, \bar{y}_i)$, $\vartheta \in \partial g(x_0, \bar{y}_i)$:

(i) $\langle Ax_0, x_0\rangle > 0$, $\langle Bx_0, x_0\rangle = 0$

$$\Rightarrow \left\langle \sum_{i=1}^{s} t_i^* v + \frac{Ax_0}{\langle Ax_0, x_0\rangle^{1/2}} - k_0 \vartheta,\ z \right\rangle + \langle (k_0^2 B) z, z\rangle^{1/2} < 0,$$

(ii) $\langle Ax_0,\ x_0\rangle = 0,\ \langle Bx_0,\ x_0\rangle > 0$

$$\Rightarrow \left\langle \sum_{i=1}^{s} t_i^* \left(v - k_0 \left(\vartheta - \frac{Bx_0}{\langle Bx_0,\ x_0\rangle^{1/2}} \right)\right),\ z \right\rangle + \langle Bz, z\rangle^{1/2} < 0,$$

(iii) $\langle Ax_0, x_0\rangle = 0,\ \langle Bx_0, x_0\rangle = 0$

$$\Rightarrow \left\langle \sum_{i=1}^{s} t_i^* (v - k_0 \vartheta),\ z \right\rangle + \langle (k_0 B) z, z\rangle^{1/2} + \langle Bz, z\rangle^{1/2} < 0.$$

If we take the condition $Z_{\bar{y}}(x_0) = \phi$ in Lemma 2.1, then the result of Lemma 2.1 still holds.
Throughout the paper, we assume that b_0 and b_1 are nonnegative functions defined on $X \times X$ and $\phi_0, \phi_1 : R \to R$.

3. Sufficient Optimality Condition

We now establish sufficient optimality conditions for (P) under the assumptions of generalized α-univexity discussed in previous section.

Theorem 3.1 *Suppose that* $x_0 \in \Im_P$ *be a feasible solution for (P). Suppose that there exist* $k_0 \in R_+$, $(s, t^*, \bar{y}) \in K(x_0)$, $u,\ v \in R^n$ *and* $\mu^* \in R_+^p$ *satisfying* (3) – (7). *Assume that one of the following conditions holds:*

(a) $\varphi(\cdot) = \sum_{i=1}^{s} t_i^* \big((f(\cdot, \bar{y}_i) + \langle \cdot, Au\rangle) - k_0 (g(\cdot, \bar{y}_i) - \langle \cdot, Bv\rangle) \big)$ *and* $\langle \mu^*, h(\cdot)\rangle$ *are* α*- univex with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha_0$ *and* η *with* $\phi_0(V) \ge 0 \Rightarrow V \ge 0$ *and* $\phi_1(V) \ge V$;

(b) $\varphi(\cdot) = \sum_{i=1}^{s} t_i^* \big((f(\cdot, \bar{y}_i) + \langle \cdot, Au\rangle) - k_0 (g(\cdot, \bar{y}_i) - \langle \cdot, Bv\rangle) \big)$ *is pseudo* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\langle \mu^*, h(\cdot)\rangle$ *is quasi* α*- univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \le 0 \Rightarrow \phi_1(V) \le 0$;

c) $\varphi(\cdot) = \sum_{i=1}^{s} t_i^* \big((f(\cdot, \bar{y}_i) + \langle \cdot, Au\rangle) - k_0 (g(\cdot, \bar{y}_i) - \langle \cdot, Bv\rangle) \big)$ *is quasi* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *and* $\langle \mu^*, h(\cdot)\rangle$ *is strictly pseudo* α*- univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \le 0 \Rightarrow \phi_0(V) \le 0$ *and* $\phi_0(V) \ge 0 \Rightarrow V \ge 0$.

Then x_0 *is an optimal solution of* (P).

Proof. Suppose the contrary that x_0 is not an optimal solution of (P). Then there exists $x_1 \in \Im_P$ such that

$$\sup_{y\in Y} \frac{f(x_1, y) + \langle x_1,\ Ax_1\rangle^{1/2}}{g(x_1, y) - \langle x_1,\ Bx_1\rangle^{1/2}} < \sup_{y\in Y} \frac{f(x_0, y) + \langle x_0,\ Ax_0\rangle^{1/2}}{g(x_0, y) - \langle x_0,\ Bx_0\rangle^{1/2}}.$$

We know that

$$\sup_{y\in Y} \frac{f(x_0, y) + \langle x_0,\ Ax_0\rangle^{1/2}}{g(x_0, y) - \langle x_0,\ Bx_0\rangle^{1/2}} = \frac{f(x_0, \bar{y}_i) + \langle x_0,\ Ax_0\rangle^{1/2}}{g(x_0, \bar{y}_i) - \langle x_0,\ Bx_0\rangle^{1/2}} = k_0,$$

for $\bar{y}_i \in \bar{Y}(x_0), i = 1, 2, ..., s$, and

$$\frac{f(x_1, \bar{y}_i) + \langle x_1,\ Ax_1\rangle^{1/2}}{g(x_1, \bar{y}_i) - \langle x_1,\ Bx_1\rangle^{1/2}} \le \sup_{y\in Y} \frac{f(x_1, y) + \langle x_1,\ Ax_1\rangle^{1/2}}{g(x_1, y) - \langle x_1,\ Bx_1\rangle^{1/2}}.$$

Thus, we have

$$\frac{f(x_1, \bar{y}_i) + \langle x_1,\ Ax_1\rangle^{1/2}}{g(x_1, \bar{y}_i) - \langle x_1,\ Bx_1\rangle^{1/2}} < k_0 \text{ for } i = 1, 2, ..., s.$$

It follows that

$$f(x_1, \bar{y}_i) + \langle x_1,\ Ax_1\rangle^{1/2} - k_0 \left(g(x_1, \bar{y}_i) - \langle x_1,\ Bx_1\rangle^{1/2} \right) < 0, \tag{8}$$

for $i = 1,2, \ldots, s$.

From (2), (4), (6) (7) and (8), we get

$$\varphi(x_1) = \sum_{i=1}^{s} t_i^* \Big(\big(f(x_1, \bar{y}_i) + \langle x_1,\ Au \rangle \big) - k_0 \big(g(x_1, \bar{y}_i) - \langle x_1,\ Bv \rangle \big) \Big)$$

$$\le \sum_{i=1}^{s} t_i^* \Big(\Big(f(x_1, \bar{y}_i) + \langle x_1,\ Ax_1 \rangle^{1/2} \Big) - k_0 \Big(g(x_1, \bar{y}_i) - \langle x_1,\ Bx_1 \rangle^{1/2} \Big) \Big) < 0$$

$$= \sum_{i=1}^{s} t_i^* \Big(\Big(f(x_0, \bar{y}_i) + \langle x_0,\ Ax_0 \rangle^{1/2} \Big) - k_0 \Big(g(x_0, \bar{y}_i) - \langle x_0,\ Bx_0 \rangle^{1/2} \Big) \Big)$$

$$= \sum_{i=1}^{s} t_i^* \Big(\big(f(x_0, \bar{y}_i) + \langle x_0,\ Au \rangle \big) - k_0 \big(g(x_0, \bar{y}_i) - \langle x_0,\ Bv \rangle \big) \Big)$$

$$= \varphi(x_0).$$

That is

$$\varphi(x_1) < \varphi(x_0). \quad (9)$$

If hypothesis (a) holds, then

$$b_0(x_1, x_0)\phi_0 \Bigg[\sum_{i=1}^{s} t_i^* \Big(\big(f(x_1, \bar{y}_i) + \langle x_1,\ Au \rangle \big) - k_0 \big(g(x_1, \bar{y}_i) - \langle x_1,\ Bv \rangle \big) \Big) - \sum_{i=1}^{s} t_i^* \Big(\big(f(x_0, \bar{y}_i) + \langle x_0,\ Au \rangle \big) - k_0 \big(g(x_0, \bar{y}_i) - \langle x_0,\ Bv \rangle \big) \Big) \Bigg]$$

$$\ge \langle \alpha_0(x_1, x_0)\xi, \eta(x_1, x_0) \rangle, \quad \forall \xi \in \partial\varphi(x_0)$$

$$= \Big\langle \alpha_0(x_1, x_0)\big\{ -\langle \mu^*, \zeta \rangle \big\}, \eta(x_1, x_0) \Big\rangle, \quad \forall \zeta \in \partial h(x_0) \quad \text{(by(3))}$$

$\ge -b_1(x_1, x_0)\phi_1 \Big[\langle \mu^*, h(x_1) \rangle - \langle \mu^*, h(x_0) \rangle \Big]$ (by the α-univexity of $\langle \mu^*, h(\cdot) \rangle$)

$\ge \Big[\langle \mu^*, h(x_0) \rangle - \langle \mu^*, h(x_1) \rangle \Big]$ (by the positivity of b_1 and $\phi_1(V) \ge V$)

≥ 0 (by the feasibility of x_1 for (P) and (5)).

Since $\phi_0(V) \ge 0 \Rightarrow V \ge 0$ and $b_0 \ge 0$, we get

$$\varphi(x_1) \ge \varphi(x_0),$$

which contradicts (9).

If hypothesis (b) holds, by the positivity of b_0, $V < 0 \Rightarrow \phi_0(V) < 0$ and from the inequality (9), we get

$$b_0(x_1, x_0)[\varphi(x_1) - \varphi(x_0)] < 0.$$

By the pseudo α-univexity of φ, the above inequality give

$$\langle \alpha_0(x_1, x_0)\xi, \eta(x_1, x_0) \rangle < 0, \forall \xi \in \partial\varphi(x_0). \quad (10)$$

From (10) and (3), we get

$$\Big\langle \alpha_0(x_1, x_0)\big\{ -\langle \mu^*, \zeta \rangle \big\}, \eta(x_1, x_0) \Big\rangle < 0, \quad \forall \zeta \in \partial h(x_0),$$

by the positivity of α_0, we get

$$\Big\langle \langle \mu^*, \zeta \rangle, \eta(x_1, x_0) \Big\rangle > 0, \quad \forall \zeta \in \partial h(x_0). \quad (11)$$

Since $x_1 \in \Im_P, \mu^* \in R_+^P$, from (5), we get

$$\Big[\langle \mu^*, h(x_1) \rangle - \langle \mu^*, h(x_0) \rangle \Big] \le 0. \quad (12)$$

By the condition $V \le 0 \Rightarrow \phi_1(V) \le 0$ and the positivity of b_1, (12) gives

$$b_1(x_1, x_0)\phi_1 \Big[\langle \mu^*, h(x_1) \rangle - \langle \mu^*, h(x_0) \rangle \Big] \le 0.$$

By the quasi α-univexity of $\langle \mu^*, h(\cdot) \rangle$ and the above inequality, we get

$$\Big\langle \alpha_1(x_1, x_0)\langle \mu^*, \zeta \rangle, \eta(x_1, x_0) \Big\rangle \le 0, \ \forall \zeta \in \partial h(x_0).$$

By the positivity of α_1, we get

$$\Big\langle \langle \mu^*, \zeta \rangle, \eta(x_1, x_0) \Big\rangle \le 0, \quad \forall \zeta \in \partial h(x_0),$$

which contradicts (11).

For hypothesis (c) the proof is similar to the proof of case (b). This completes the proof. □

Remark 3.1

(i) If the functions f, g and h are continuous differentiable, then Theorem 3.1 above reduces to Theorem 3.1 given in [10].

(ii) Evidently, if we choose ϕ_0, ϕ_1 as the identity maps, $b_0 = 1 = b_1$ and if the functions f, g and h are continuous differentiable, then we obtain the Theorem 3.1 given in [16].

(iii) If we take ϕ_0, ϕ_1 as the identity maps, and $b_0 = 1 = b_1$ in the above Theorem 3.1, we get Theorem 3.1 given in [17].

4. First Duality Model

In this section, we consider the following dual to (P):

$$\text{(DI)} \quad \max_{(s,t,\bar{y})\in K} \quad \sup_{(z,t,\bar{y})\in H_1(s,t,\bar{y})} k,$$

subject to

$$0 \in \sum_{i=1}^{s} t_i \Big\{ \partial f(z, \bar{y}_i) + \langle u, Au\rangle^{1/2} - k \Big(\partial g(z, \bar{y}_i) + \langle v, Bv\rangle^{1/2} \Big) \Big\} + \partial\langle \mu, h(z)\rangle, \quad (13)$$

$$\sum_{i=1}^{s} t_i \Big\{ f(z, \bar{y}_i) + \langle z, Au\rangle - k\big(g(z, \bar{y}_i) + \langle z, Bv\rangle\big) \Big\} \geq 0, \quad (14)$$

$$\langle \mu, h(z)\rangle \geq 0, \quad (15)$$

$$\langle z, Az\rangle \leq 1, \langle z, Bz\rangle \leq 1, \quad (16)$$

where $H_1(s,t,\bar{y})$ denotes the set of all triplets $(z, \mu, v) \in R^n \times R_+^P \times R_+$ satisfying (13) - (16) and $(s,t,\bar{y}) \in K(z)$. For a triplet $(s,t,\bar{y}) \in K$, if the set $H_1(s,t,\bar{y})$ is empty, then we define the supremum over it to be -∞. In this section we denote

$$\psi(\cdot) = \sum_{i=1}^{s} t_i \Big(\big(f(\cdot, \bar{y}_i) + \langle \cdot, Au\rangle \big) - k\big(g(., \bar{y}_i) - \langle \cdot, Bv\rangle \big) \Big).$$

Theorem 4.1 (Weak duality). *Let* $x \in \Im_P$ *be a feasible solution for* (P) *and let* $(z, \mu, u, v, s, t, \bar{y})$ *be a feasible solution for* (DI). *Assume that one of the following conditions holds:*

(a) $\psi(\cdot)$ *and* $\langle \mu, h(\cdot)\rangle$ *are* α*-univex with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha_0$ *and* η *with* $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ *and* $\phi_1(V) \geq V$;

(b) $\psi(\cdot)$ *is pseudo* α*- univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\langle \mu, h(\cdot)\rangle$ *is quasi* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \leq 0 \Rightarrow \phi_1(V) \leq 0$;

(c) $\psi(\cdot)$ *is quasi* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\langle \mu, h(\cdot)\rangle$ *is strictly pseudo* α*- univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \leq 0 \Rightarrow \phi_1(V) \leq 0$.

Then $\sup_{y\in Y} \dfrac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} \geq k$.

Proof. Suppose contrary to the result, that is

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} < k.$$

Therefore we get the following relation

$$f(x, \bar{y}_i) + \langle x, Ax\rangle^{1/2} - k\Big(g(x, \bar{y}_i) - \langle x, Bx\rangle^{1/2} \Big) < 0, \quad \forall\, \bar{y}_i \in Y.$$

It follows from $t_i \geq 0, i = 1,2,...,s,$ with $\sum_{i=1}^{s} t_i = 1$, that

$$t_i \Big[f(x, \bar{y}_i) + \langle x, Ax\rangle^{1/2} - k\Big(g(x, \bar{y}_i) - \langle x, Bx\rangle^{1/2} \Big) \Big] \leq 0, \quad i = 1,...,s,$$

with at least one strict inequality because $t = (t_1, t_2, ..., t_s) \neq 0$.

From (2), (14), (16) and the above inequality, we get

$$\psi(x) = \sum_{i=1}^{s} t_i \Big(\big(f(x, \bar{y}_i) + \langle x, Au\rangle \big) - k\big(g(x, \bar{y}_i) - \langle x, Bv\rangle \big) \Big)$$

$$\leq \sum_{i=1}^{s} t_i \left(\left(f(x,\bar{y}_i) + \langle x, \ Ax\rangle^{1/2}\right) - k\left(g(x,\bar{y}_i) - \langle x, \ Bx\rangle^{1/2}\right)\right) < 0$$

$$\leq \sum_{i=1}^{s} t_i \left(\left(f(z,\bar{y}_i) + \langle z, \ Au\rangle\right) - k\left(g(z,\bar{y}_i) - \langle z, \ Bv\rangle\right)\right) = \psi(z).$$

That is,

$$\psi(x) < \psi(z). \qquad (17)$$

If hypothesis (a) holds, then

$$b_0(x,z)\phi_0\left[\sum_{i=1}^{s} t_i \left(\left(f(x,\bar{y}_i) + \langle x, Au\rangle\right) - k\left(g(x,\bar{y}_i) - \langle x, \ Bv\rangle\right)\right) - \sum_{i=1}^{s} t_i \left(\left(f(z,\bar{y}_i) + \langle z, Au\rangle\right) - k\left(g(z,\bar{y}_i) - \langle z, Bv\rangle\right)\right)\right]$$

$$\geq \langle \alpha_0(x,z)v, \eta(x,z)\rangle, \ \forall v \in \partial\psi(z)$$

$$= \langle \alpha_0(x,z)\{-\langle \mu, \zeta\rangle\}, \eta(x,z)\rangle, \quad \forall \zeta \in \partial h(z), \quad \text{(by (13))}$$

$$\geq -b_1(x,z)\phi_1\left[\langle \mu, h(x)\rangle - \langle \mu, h(z)\rangle\right]$$ (by the α-univexity of $\langle \mu, h(\cdot)\rangle$)

$$\geq \left[\langle \mu, h(z)\rangle - \langle \mu, h(x)\rangle\right]$$ (by the positivity of b_1 and $\phi_1(V) \geq V$)

≥ 0 (by the feasibility of x for (P) and (15)).

Since $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ and $b_0 \geq 0$, we get

$$\psi(x) \geq \psi(z),$$

which contradicts (17).

If hypothesis (b) holds, by the positivity of b_0, $V < 0 \Rightarrow \phi_0(V) < 0$ and from the inequality (17), we get

$$b_0(x,z)\phi_0[\psi(x) - \psi(z)] < 0.$$

By the pseudo α-univexity of ψ, the above inequality gives

$$\langle \alpha_0(x,z)v, \eta(x,z)\rangle < 0, \ \forall v \in \partial\psi(z). \quad (18)$$

From (18) and (13), we get

$$\langle \alpha_0(x,z)\{-\langle \mu, \zeta\rangle\}, \eta(x,z)\rangle < 0, \quad \forall \zeta \in \partial h(z),$$

by the positivity of α_0, we get

$$\langle \langle \mu, \zeta\rangle, \eta(x,z)\rangle > 0, \quad \forall \zeta \in \partial h(z). \quad (19)$$

Since $x \in \Im_P$, $\mu \in R_+^P$, from (15), we get

$$\left[\langle \mu, h(x)\rangle - \langle \mu, h(z)\rangle\right] \leq 0.$$

By the condition $V \leq 0 \Rightarrow \phi_1(V) \leq 0$ and the positivity of b_1, the above inequality yield

$$b_1(x,z)\phi_1\left[\langle \mu, h(x)\rangle - \langle \mu, h(z)\rangle\right] \leq 0.$$

By the quasi α-univexity of $\langle \mu, h(\cdot)\rangle$ and from the above inequality, we get

$$\langle \alpha_1(x,z)\{\langle \mu, \zeta\rangle\}, \eta(x,z)\rangle \leq 0, \quad \forall \zeta \in \partial h(z).$$

By the positivity of α_1, we get

$$\langle \langle \mu, \zeta\rangle, \eta(x,z)\rangle \leq 0, \quad \forall \zeta \in \partial h(z),$$

which contradicts (19).

For hypothesis (c) the proof is similar to that of the proof given above for case (b). □

Theorem 4.2 (Strong duality). *Assume that* x^* *is an optimal solution for* (P) *and* x^* *satisfies a constraints qualification for* (P). *Then there exist* $(s^*, t^*, \bar{y}^*) \in K(x^*)$ *and* $(x^*, \mu^*, k^*, u^*, v^*) \in H_1(s^*, t^*, \bar{y}^*)$ *such that* $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ *is feasible for* (DI). *If any of the conditions of Theorem 4.1 holds, then* $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ *is an optimal solution for* (DI), *and problem* (P) *and* (DI) *have the same optimal value.*

Proof. By Lemma 2.1, there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, u^*, v^*) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is a feasible for

(DI), and

$$k^* = \frac{f(x^*, y_i^*) + \langle x^*, Ax^* \rangle^{1/2}}{g(x^*, y_i^*) - \langle x^*, Bx^* \rangle^{1/2}}.$$

The optimality of this feasible solution for (DI) follows from Theorem 4.1. □

Theorem 4.3 (Strict Converse Duality). *Let* x^* *and* $(\bar{z}, \bar{\mu}, \bar{k}, \bar{u}, \bar{v}, \bar{s}, \bar{t}, \bar{y})$ *be optimal for* (P) *and* (DI)*, respectively. Assume that the hypothesis of Theorem 4.2 is fulfilled. Further if any one of the following conditions holds:*

(a) $\sum_{i=1}^{\bar{s}} \bar{t}_i \left(\left(f(\cdot, \bar{y}_i) + \langle \cdot, A\bar{u} \rangle \right) - \bar{k}\left(g(\cdot, \bar{y}_i) - \langle \cdot, B\bar{v} \rangle \right) \right)$ *is strictly* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V \le 0 \Rightarrow \phi_0(V) \le 0$ *and* $\langle \bar{\mu}, h(\cdot) \rangle$ *is* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \le 0 \Rightarrow \phi_1(V) \le 0$;

(b) $\sum_{i=1}^{\bar{s}} \bar{t}_i \left(\left(f(\cdot, \bar{y}_i) + \langle \cdot, A\bar{u} \rangle \right) - \bar{k}\left(g(\cdot, \bar{y}_i) - \langle \cdot, B\bar{v} \rangle \right) \right)$ *is strictly pseudo* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\langle \bar{\mu}, h(\cdot) \rangle$ *is quasi* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \le 0 \Rightarrow \phi_1(V) \le 0$.

Then $x^* = \bar{z}$*; that is,* $\bar{z}$ *is an optimal solution for* (P) *and*

$$\sup_{y \in Y} \frac{f(\bar{z}, \bar{y}^*) + \langle \bar{z}, A\bar{z} \rangle^{1/2}}{g(\bar{z}, \bar{y}^*) - \langle \bar{z}, B\bar{z} \rangle^{1/2}} = \bar{k}.$$

Proof. Suppose on the contrary that $x^* \ne \bar{z}$. From Theorem 4.2, we know that there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, u^*, v^*) \in H_1(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is optimal for (DI) with the optimal value

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + \langle x^*, Ax^* \rangle^{1/2}}{g(x^*, \bar{y}^*) - \langle x^*, Bx^* \rangle^{1/2}} = k^*.$$

Following as in [12], we get

$$\psi(x^*) \le \psi(\bar{z}).$$

Since $V \le 0 \Rightarrow \phi_0(V) \le 0$ and the positivity of b_0, the above inequality yield

$$b_0(x^*, \bar{z})\phi_0[\psi(x^*) - \psi(\bar{z})] \le 0.$$

If condition (a) holds, then by the strict α-univexity $\psi_1(\cdot)$, we get

$$\langle \alpha_0(x^*, \bar{z})\nu, \eta(x^*, \bar{z}) \rangle < 0, \ \forall \nu \in \partial\psi(\bar{z}).$$

Now from (13) and the above inequality, we get

$$\langle \alpha_0(x^*, \bar{z})\{-\langle \bar{\mu}, \zeta \rangle\}, \eta(x^*, \bar{z}) \rangle < 0, \ \forall \zeta \in \partial h(\bar{z}).$$

By the positivity of α_0, we get

$$\langle \langle \bar{\mu}, \zeta \rangle, \eta(x^*, \bar{z}) \rangle > 0, \quad \forall \zeta \in \partial h(\bar{z}). \qquad (20)$$

Since $x^* \in \Im_P, \bar{\mu} \in R_+^P$, from (15), we get

$$[\langle \bar{\mu}, h(x^*) \rangle - \langle \bar{\mu}, h(\bar{z}) \rangle] \le 0.$$

By the condition $V \le 0 \Rightarrow \phi_1(V) \le 0$ and the positivity of b_1, the above inequality yield

$$b_1(x^*, \bar{z})\phi_1[\langle \bar{\mu}, h(x^*) \rangle - \langle \bar{\mu}, h(\bar{z}) \rangle] \le 0.$$

By the α-univexity of $\langle \bar{\mu}, h(\cdot) \rangle$ and from the above inequality, we get

$$\langle \alpha_1(x^*, \bar{z})\{\langle \bar{\mu}, \zeta \rangle\}, \eta(x^*, \bar{z}) \rangle \le 0, \quad \forall \zeta \in \partial h(\bar{z}).$$

By the positivity of α_1, we get

$$\langle \langle \bar{\mu}, \zeta \rangle, \eta(x^*, \bar{z}) \rangle \le 0, \quad \forall \zeta \in \partial h(\bar{z}),$$

which contradicts to (20).
Hence, we get

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + \langle x^*, Ax^* \rangle^{1/2}}{g(x^*, \bar{y}^*) - \langle x^*, Bx^* \rangle^{1/2}} > \bar{k}.$$

The above inequality contradicts the fact that

$$\sup_{y \in Y} \frac{f(x^*, \bar{y}^*) + \langle x^*, Ax^* \rangle^{1/2}}{g(x^*, \bar{y}^*) - \langle x^*, Bx^* \rangle^{1/2}} = k^* = \bar{k}.$$

Therefore, we conclude that $x^* = \bar{z}$.
For hypothesis (b) the proof is similar to that of the proof given above for case (a). □

Remark 4.1

(i) If the functions f, g and h are continuous differentiable, then the above Theorem 4.1 and 4.2 reduces to Theorem 4.1 and 4.2 given in [10].

(ii) Evidently, if we choose ϕ_0, ϕ_1 as the identity maps, $b_0 = 1 = b_1$ and if the functions f, g and h are continuous differentiable, then we obtain the Theorem 4.1 and 4.2 given in [16].

(iii) If we take ϕ_0, ϕ_1 as the identity maps, and $b_0 = 1 = b_1$ in the above Theorem 4.1, and 4.2 we get Theorem 4.1 and 4.2 given in [17].

5. Second Duality Model

In this section, we formulate the Wolfe-type dual model to problem (P) as follows:

$$\textbf{(DII)} \quad \max_{(s,t,\bar{y})\in K(z)} \quad \sup_{(z,\mu,u,v)\in H_2(s,t,\bar{y})} \quad F(z)$$

subject to

$$0 \in \sum_{i=1}^{s} t_i \Big\{ \left(g(z,\bar{y}_i) - \langle z, Bz\rangle^{1/2} \right) \left(\partial f(z,\bar{y}_i) + Au \right) - \left(f(z,\bar{y}_i) + \langle z, Az\rangle^{1/2} \right) \left(\partial g(z,\bar{y}_i) - Bv \right) \Big\} + \partial \langle \mu, h(z) \rangle, \quad (21)$$

$$\langle \mu, h(z)\rangle \geq 0, \quad (22)$$

$$\langle z, Az\rangle \leq 1, \ \langle z, Bz\rangle \leq 1,$$

$$\langle z, Az\rangle^{1/2} = \langle z, Au\rangle, \langle z, Bz\rangle^{1/2} = \langle z, Bv\rangle, \quad (23)$$

where $F(z) = \sup_{y\in Y} \dfrac{f(z,y) + \langle z, Az\rangle^{1/2}}{g(z,y) - \langle z, Bz\rangle^{1/2}}$,

$y_i \in Y(z)$ and $H_2(s,t,\bar{y})$ denotes the set of $(z,\mu,u,v) \in R^n \times R^P_+ \times R^n \times R^n$ satisfying (35) - (37). If the set $H_2(s,t,\bar{y})$ is empty, then we define the supremum over it to be -∞. In this section, we denote

$$\psi_1(\cdot) = \sum_{i=1}^{s} t_i \Big\{ \left(g(z,\bar{y}_i) - \langle z, Bv\rangle \right) \left(f(\cdot,\bar{y}_i) + \langle \cdot, Au\rangle \right) - \left(f(z,\bar{y}_i) + \langle z, Au\rangle \right) \left(g(\cdot,\bar{y}_i) - \langle \cdot, Bv\rangle \right) \Big\}.$$

Theorem 5.1 (Weak duality) *Let* $x \in \Im_P$ *be a feasible solution for* (P) *and let* $(z,\mu,u,v,s,t,\bar{y})$ *be a feasible solution for* (DII). *Assume that one of the following conditions holds:*

(a) $\psi_1(\cdot)$ *and* $\langle \mu, h(\cdot)\rangle$ *are* α*-univex with respect to* $b_0, b_1, \phi_0, \phi_1, \alpha_0$ *and* η *with* $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ *and* $\phi_1(V) \geq V$;

(b) $\psi_1(\cdot)$ *is pseudo* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\langle \mu, h(\cdot)\rangle$ *is quasi* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \leq 0 \Rightarrow \phi_1(V) \leq 0$;

(c) $\psi_1(\cdot)$ *is quasi* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V < 0 \Rightarrow \phi_0(V) < 0$ *and* $\langle \mu, h(\cdot)\rangle$ *is strictly pseudo* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \leq 0 \Rightarrow \phi_1(V) \leq 0$. *Then*

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} \geq F(z).$$

Proof. Suppose contrary to the result that for each $x \in \Im_P$,

$$\sup_{y\in Y} \frac{f(x,y) + \langle x, Ax\rangle^{1/2}}{g(x,y) - \langle x, Bx\rangle^{1/2}} < F(z). \quad (24)$$

Since $\bar{y}_i \in \bar{Y}(z)$, $i = 1,2,...,s$, we have

$$F(z) = \frac{f(z,\bar{y}_i) + \langle z, Az\rangle^{1/2}}{g(z,\bar{y}_i) - \langle z, Bz\rangle^{1/2}}, \ i = 1,2,...,s. \quad (25)$$

Following as in [12], we get

$$\psi_1(x) < \psi_1(z). \quad (26)$$

Now if condition (a) holds, then

$$b_0(x,z)\phi_0[\psi_1(x) - \psi_1(z)]$$
$$\geq \langle \alpha_0(x,z)\nu, \eta(x,z)\rangle, \quad \forall \nu \in \partial\psi_1(z)$$
$$= \langle \alpha_0(x,z)\{-\langle \mu, \zeta\rangle\}, \eta(x,z)\rangle, \quad \forall \zeta \in \partial h(z)$$

(by (21))

$\geq -b_1(x,z)\phi_1[\langle \mu, h(x)\rangle - \langle \mu, h(z)\rangle]$ (by the α-univexity of $\langle \mu, h(\cdot)\rangle$)

$\geq [\langle \mu, h(z)\rangle - \langle \mu, h(x)\rangle]$ (by the positivity of b_1 and $\phi_1(V) \geq V$)

≥ 0 (by the feasibility of x for (P) and (22)).

Since $\phi_0(V) \geq 0 \Rightarrow V \geq 0$ and $b_0 \geq 0$, we get

$$\psi_1(x) \geq \psi_1(z),$$

which contradicts (26).

If hypothesis (b) holds, by the positivity of b_0, $V < 0 \Rightarrow \phi_0(V) < 0$ and from the inequality (26), we get

$$b_0(x,z)\phi_0[\psi_1(x) - \psi_1(z)] < 0.$$

By the pseudo α-univexity of ψ_1, the above inequality gives

$$\langle \alpha_0(x,z)\nu, \eta(x,z)\rangle < 0, \quad \forall \nu \in \partial\psi_1(z). \quad (27)$$

From (21) and (27), we get

$$\langle \alpha_0(x,z)\{-\langle \mu, \zeta\rangle\}, \eta(x,z)\rangle < 0, \quad \forall \zeta \in \partial h(z),$$

by the positivity of α_0, we get

$$\langle -\langle \mu, \zeta\rangle, \eta(x,z)\rangle < 0, \quad \forall \zeta \in \partial h(z).$$

i.e. $\langle \langle \mu, \zeta\rangle, \eta(x,z)\rangle > 0, \quad \forall \zeta \in \partial h(z). \quad (28)$

Since $x \in \Im_P$, $\mu \in R_+^P$, from (22), we get

$$[\langle \mu, h(x)\rangle - \langle \mu, h(z)\rangle] \leq 0. \quad (29)$$

By the condition $V \leq 0 \Rightarrow \phi_1(V) \leq 0$ and the positivity of b_1, (29) gives

$$b_1(x,z)\phi_1[\langle \mu, h(x)\rangle - \langle \mu, h(z)\rangle] \leq 0.$$

By the quasi α-univexity of $\langle \mu, h(\cdot)\rangle$ and from the above inequality, we get

$$\langle \alpha_1(x,z)\{\langle \mu, \zeta\rangle\}, \eta(x,z)\rangle \leq 0, \quad \forall \zeta \in \partial h(z).$$

By the positivity of α_1, we get

$$\langle \langle \mu, \zeta\rangle, \eta(x,z)\rangle \leq 0, \quad \forall \zeta \in \partial h(z),$$

which contradicts (28).

The proof is similar when hypothesis (c) holds. This completes the proof. □

Theorem 5.2 (Strong duality). *Assume that* x^* *is an optimal solution for* (P) *and* x^* *satisfies a constraints qualification for* (P). *Then there exist* $(s^*, t^*, \bar{y}^*) \in K(x^*)$ *and* $(x^*, \mu^*, k^*, u^*, v^*) \in H_2(s^*, t^*, \bar{y}^*)$ *such that* $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ *is feasible for* (DII). *If any of the conditions of Theorem 5.1 holds, then* $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ *is an optimal solution for* (DII), *and problem* (P) *and* (DII) *have the same optimal value.*

Proof. By Lemma 2.1, there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, k^*, u^*, v^*) \in H_2(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, k^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is feasible for (DII), and

$$k_0 = \frac{f(x^*, \bar{y}^*) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, \bar{y}^*) - \langle x^*, Bx^*\rangle^{1/2}}.$$

The optimality of this feasible solution for (DII) follows from Theorem 5.1. □

Theorem 5.3 (Strict Converse Duality). *Let* x^* *and* $(z, \mu, u, v, s, t, \bar{y})$ *be optimal for* (P) *and* (DII), *respectively. Assume that the hypothesis of Theorem 5.2 is fulfilled. Further if any one of the following conditions holds:*

(a) $\psi_1(\cdot)$ is *strictly* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V \leq 0 \Rightarrow \phi_0(V) \leq 0$ *and* $\langle \mu, h(\cdot)\rangle$ *is* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \leq 0 \Rightarrow \phi_1(V) \leq 0$;

(b) $\psi_1(\cdot)$ *is strictly pseudo* α*-univex with respect to* b_0, ϕ_0, α_0 *and* η *with* $V \leq 0 \Rightarrow \phi_0(V) \leq 0$ *and* $\langle \mu, h(\cdot)\rangle$ *is quasi* α*-univex with respect to* b_1, ϕ_1, α_1 *and* η *with* $V \leq 0 \Rightarrow \phi_1(V) \leq 0$.

Then $x^* = z$*; that is,* z *is an optimal solution for* (P).

Proof. Suppose on the contrary that $x^* \neq z$. Similar to the proof of Theorem 5.1, we get

$$\sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}} \leq F(z). \quad (30)$$

Following as in [12], we get

$$\psi_1(x^*) \leq \psi_1(z). \quad (31)$$

By the positivity of b_0, $V \leq 0 \Rightarrow \phi_0(V) \leq 0$ and from the inequality (31), we get

$$b_0(x^*, z)\phi_0[\psi_1(x^*) - \psi_1(z)] \leq 0.$$

If hypothesis (a) holds, then by the strict α-univexity of $\psi_1(\cdot)$ and from the above inequality, we get

$$\langle \alpha_0(x^*, z)\nu, \eta(x^*, z)\rangle < 0, \ \forall \nu \in \partial\psi_1(z). \quad (32)$$

Now from (32) and (21), we get

$$\langle \alpha_0(x^*, z)\{-\langle \mu, \zeta\rangle\}, \eta(x^*, z)\rangle < 0, \ \forall \zeta \in \partial h(z).$$

By the positivity of α_0, we get

$$\langle \{-\langle \mu, \zeta\rangle\}, \eta(x^*, z)\rangle < 0, \ \ \forall \zeta \in \partial h(z).$$

$$\text{i.e.} \langle \langle \mu, \zeta\rangle, \eta(x^*, z)\rangle > 0, \quad \forall \zeta \in \partial h(z). \quad (33)$$

Since $x^* \in \Im_P$, $\mu \in R_+^P$, from (22), we get

$$[\langle \mu, h(x^*)\rangle - \langle \mu, h(z)\rangle] \leq 0. \quad (34)$$

By the condition $V \leq 0 \Rightarrow \phi_1(V) \leq 0$ and the positivity of b_1, (34) gives

$$b_1(x^*, z)\phi_1[\langle \mu, h(x^*)\rangle - \langle \mu, h(z)\rangle] \leq 0.$$

By the α-univexity of $\langle \mu, h(\cdot)\rangle$, from the above inequality, we get

$$\langle \alpha_1(x^*, z)\{\langle \mu, \zeta\rangle\}, \eta(x^*, z)\rangle \leq 0, \ \ \forall \zeta \in \partial h(z).$$

By the positivity of α_1, we get

$$\langle \langle \mu, \zeta\rangle, \eta(x^*, z)\rangle \leq 0, \quad \forall \zeta \in \partial h(z),$$

which contradicts (33). Hence (30) is false, and we have

$$\sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}} > F(z). \quad (35)$$

Since x^* is an optimal solution for (P), from Theorem 5.2 there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, u^*, v^*) \in H_2(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is an optimal solution for (DII) with the optimal value

$$\sup_{y\in Y} \frac{f(x^*, y) + \langle x^*, Ax^*\rangle^{1/2}}{g(x^*, y) - \langle x^*, Bx^*\rangle^{1/2}} = F(x^*) = F(z),$$

which contradicts (35). Hence $x^* = z$; that is, z is an optimal solution for (P).

Since $V \leq 0 \Rightarrow \phi_0(V) \leq 0$ and the positivity of b_0, from (31), we get

$$b_0(x^*, z)\phi_0[\psi_1(x^*) - \psi_1(z)] \leq 0.$$

If hypothesis (b) holds, then by the strict pseudo α-univexity of ψ_1 and from the above inequality, we get

$$\langle \alpha_0(x^*, z)\nu, \eta(x^*, z)\rangle < 0, \ \ \forall \nu \in \partial\psi_1(z).$$

The remaining part of the proof is similar to the case of case (a). This completes the proof. □

Remark 5.1

(i) If the functions f, g and h are continuous differentiable, then the above Theorem 5.1, 5.2 and 5.3 reduces to Theorem 5.1, 5.2 and 5.3 given in [10].

(ii) Evidently, if we choose ϕ_0, ϕ_1 as the identity maps, $b_0 = 1 = b_1$ and if the functions f, g and h are continuous differentiable, then we obtain the Theorem 5.1, 5.2 and 5.3 given in [16].

(iii) If we take ϕ_0, ϕ_1 as the identity maps, and $b_0 = 1 = b_1$ in the above Theorem 5.1, 5.2 and 5.3 we get Theorem 5.1, 5.2 and 5.3 given in [17].

6. Third Duality Model

In this section we take the following form of Lemma 2.1:

Lemma 6.1 *Let x^* be an optimal solution for* (P). *Assume that $\partial h_j(x^*),\ j \in J(x^*)$ are linearly independent. Then there exist $(s, t^*, \bar{y}) \in K$ and $\mu^* \in R_+^p$ such that*

$$0 \in \partial\left(\frac{\sum_{i=1}^{s^*} t_i^*\left(f(x^*, \bar{y}_i) + \langle x^*, Au\rangle\right) + \langle \mu^*, h(x^*)\rangle}{\sum_{i=1}^{s^*} t_i^*\left(g(x^*, \bar{y}_i) - \langle x^*, Bv\rangle\right)}\right), \tag{36}$$

$$\langle \mu^*, h(x^*)\rangle = 0, \tag{37}$$

$$\begin{cases} \langle u,\ Au\rangle \le 1, \quad \langle v,\ Bv\rangle \le 1, \\ \langle x^*,\ Ax^*\rangle^{1/2} = \langle x^*,\ Au\rangle, \\ \langle x^*,\ Bx^*\rangle^{1/2} = \langle x^*,\ Bv\rangle, \end{cases} \tag{38}$$

$$t_i^* \ge 0, \sum_{i=1}^{s} t_i^* = 1,\ y_i \in Y(x^*), i = 1,2,\dots,s^*. \tag{39}$$

Now we consider the following parameter free dual problem for (P):

(DIII)

$$\max_{(s,t,\bar{y})\in K(z)} \ \sup_{(z,\mu,u,v)\in H_3(s,t,\bar{y})} \left(\frac{\sum_{i=1}^{s^*} t_i^*\left(f(z, \bar{y}_i) + \langle z, Au\rangle\right) + \langle \mu, h(z)\rangle}{\sum_{i=1}^{s^*} t_i^*\left(g(z, \bar{y}_i) - \langle z, Bv\rangle\right)}\right) = 0$$

subject to

$$0 \in \partial\left(\frac{\sum_{i=1}^{s^*} t_i^*\left(f(z, \bar{y}_i) + \langle z, Au\rangle\right) + \langle \mu, h(z)\rangle}{\sum_{i=1}^{s^*} t_i^*\left(g(z, \bar{y}_i) - \langle z, Bv\rangle\right)}\right), \tag{40}$$

$$\langle u, \mathrm{A}u\rangle \le 1, \quad \langle v, B\mathrm{v}\rangle \le 1,$$
$$\langle z, Az\rangle^{1/2} = \langle z, Au\rangle, \langle z, Bz\rangle^{1/2} = \langle z, B\mathrm{v}\rangle, \tag{41}$$

where $H_3(s,t,\bar{y})$ denotes the set of $(z,\mu,u,v) \in R^n \times R_+^P \times R^n \times R^n$ satisfying (40). If the set $H_3(s,t,\bar{y})$ is empty, then we define the supremum over it to be -∞. Throughout this section for the sake of simplicity, we denote by $\psi_2(\cdot)$

$$\left[t_i^*\left(g(z, \bar{y}_i) - \langle z,\ Bv\rangle\right)\right]\left[\sum_{i=1}^{s} t_i\ f(\cdot, y_i) + \sum_{j=1}^{P} \mu_j g_j(\cdot)\right] - \left[\sum_{i=1}^{s} t_i^*\left(f(z, \bar{y}_i) + \langle z, Au\rangle\right) + \langle \mu, h(z)\rangle\right]\left[t_i^*\left(g(\cdot, \bar{y}_i) - \langle \cdot,\ Bv\rangle\right)\right].$$

Now we shall state weak, strong and strict converse duality theorems without proof as they can be proved in thelight of Theorem 5.1 to Theorem 5.3, proved in the previous section.

Theorem 6.1 (Weak duality) *Let $x \in \Im_P$ be a feasible solution for* (P) *and let $(z,\mu,u,v,s,t,\bar{y})$ be a feasible solution for* (DIII). *If $\psi_2(\cdot)$ is pseudo α-univex with respect to b_0, ϕ_0, α_0 and η with $V \le 0 \Rightarrow \phi_0(V) \le 0$, then*

$$\sup_{y\in Y} \frac{f(x,y) + \langle x,\ Ax\rangle^{1/2}}{g(x,y) - \langle x,\ Bx\rangle^{1/2}} \ge \left(\frac{\sum_{i=1}^{s^*} t_i\left(f(z, \bar{y}_i) + \langle z, Au\rangle\right) + \langle \mu, h(z)\rangle}{\sum_{i=1}^{s^*} t_i\left(g(z, \bar{y}_i) - \langle z, Bv\rangle\right)}\right).$$

Theorem 6.2 (Strong duality). *Assume that x^* is an optimal solution for* (P) *satisfying the hypothesis of Theorem 6.1. Then there exist $(s^*, t^*, \bar{y}^*) \in K(x^*)$ and $(x^*, \mu^*, u^*, v^*) \in H_3(s^*, t^*, \bar{y}^*)$ such that $(x^*, \mu^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is feasible for* (DIII). *If any of the conditions of Theorem 6.1 holds, then $(x^*, \mu^*, u^*, v^*, s^*, t^*, \bar{y}^*)$ is an optimal solution for* (DIII) *and problem* (P) *and* (DIII) *have the same optimal value.*

Theorem 6.3 (Strict Converse Duality). *Let x^* be an optimal solution for* (P) *and $(z,\mu,u,v,s,t,\bar{y})$ be an optimal solution for* (DIII). *Assume that the hypothesis of Theorem 6.2 is fulfilled and $\psi_2(\cdot)$* is *strictly pseudo α-univex with respect to b_0, ϕ_0, α_0 and η with $V \le 0 \Rightarrow \phi_0(V) \le 0$. Then $z = x^*$ is an optimal solution of* (P).

Remark 6.1

(i) If the functions f, g and h are continuous differentiable, then the above Theorem 6.1, 6.2 and 6.3 reduces to Theorem 6.1, 6.2 and 6.3 given in [10].

(ii) Evidently, if we choose ϕ_0, ϕ_1 as the identity maps, $b_0 = 1 = b_1$ and if the functions f, g and h are continuous differentiable, then we obtain the Theorem 6.1, 6.2 and 6.3 given in [16].

(iii) If we take ϕ_0, ϕ_1 as the identity maps, and $b_0 = 1 = b_1$ in the above Theorem 6.1, 6.2 and 6.3 we get Theorem 6.1, 6.2 and 6.3 given in [17].

7. Conclusion and Further Developments

In this paper, we have introduced the classes of α -univex and generalized α -univex functions where the involved functions are locally Lipschitz, and have used these different classes of functions to derive sufficient optimality conditions and three types of duality results for nondifferentiable minimax fractional programming problems. The results developed in this paper improve and generalize a number of existing results in the literature. In fact, some researchers have paid much attention on extending some known results for univex functions. Hence, for this purpose, we may conclude that this paper enriched optimization theory in the view of mathematics.

Furthermore, the results developed in this paper can be generalized to the following nondifferentiable multiobjective programming problem:

$$\text{(MOP)} \quad \text{Minimize } \left(f_1(x)+\left(x^t B_1 x\right)^{1/2}, f_2(x)+\left(x^t B_2 x\right)^{1/2}, \ldots, f_k(x)+\left(x^t B_k x\right)^{1/2}\right)$$

subject to $x \in S = \{x \in X : g(x) \leqq 0\},$

where X is an open subset of R^n, $f_i : X \to R,\ i = 1,2,\ldots,k,\ g : X \to R^m$ and $B_i,\ i = 1,2,\ldots,k$ is an $n \times n$ positive semidefinite symmetric matrix. This will orient the future research of the authors.

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