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RESEARCH ARTICLE

On the solutions of boundary value problems

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ABSTRACT

We investigate the nonlinear boundary value problems by reproducing kernel Hilbert space technique in this paper. We construct some reproducing kernel Hilbert spaces. We define a bounded linear operator to obtain the solutions of the problems. We demonstrate our numerical results by some tables. We compare our numerical results with some results exist in the literature to present the efficiency of the proposed method.

1. Introduction

We investigate the following boundary value problems by reproducing kernel method.

$$\big(p(x)y'\big)' = f(x,y) \tag{1}$$

subject to the boundary values

$$y(0) = A, \quad y(1) = B. \tag{2}$$

Reproducing kernel space is a special Hilbert space. Many problems have been investigated by reproducing kernel Hilbert space method in the literature.

Safari et al. [1] have investigated the rainfall-runoff modeling through regression in the reproducing kernel Hilbert space algorithm. Najafi et al. [2] have worked on the combining fractional differential transform method and reproducing kernel Hilbert space method to solve fuzzy impulsive fractional differential equations. Sahihi et al. [3] have searched the system of second-order boundary value problems using a new algorithm based on the reproducing kernel Hilbert space. Agud et al. [4] have investigated the weighted p-regular kernels for reproducing kernel Hilbert spaces. Mundayadan et al. [5] have studied on the linear dynamics in the reproducing kernel Hilbert spaces. Arqub et al. [6] have constructed the modulation of reproducing kernel technique successfully. Emamjome et al. [7] have presented the reproducing kernel pseudospectral technique in details. Foroutan et al. [8] have investigated this technique for the nonlinear three-point boundary value problems. Akgül et al. [9] have worked on the representation for the reproducing kernel Hilbert space method for a nonlinear system. Allahviranloo et al. [14] have investigated the reproducing kernel method to solve parabolic partial differential equations with nonlocal conditions. For more details see [15–23].

We organize our manuscript as: We construct the reproducing kernel Hilbert spaces in Section 2. We apply the reproducing kernel method in this section. We demonstrate the numerical results in Section 3. We give the conclusion in the last section.

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2. Reproducing kernel Hilbert spaces

We define the reproducing kernel Hilbert spaces and find some reproducing kernel functions in these spaces in this section.

Definition 1. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space of real functions defined on a nonempty set E. A function $K : E \times E \to \mathbb{R}$ is called a* reproducing kernel *for H if and only if*

(a) $K(\cdot, z) \in H$ *for each fixed* $z \in E$,
(b) $\langle \varphi, K(\cdot, z)\rangle = \varphi(z)$ *for all* $z \in E$ *and all* $\varphi \in H$.

We will refer to such a Hilbert space H for which there exists a reproducing kernel function K as a reproducing kernel Hilbert space.

Condition (b) is called "the reproducing property" of the kernel K because the value of an arbitrary function $\varphi \in H$ at an arbitrary point $z \in E$ is reproduced by the inner product of φ with $K(\cdot, z)$.

Definition 2. $\mathcal{W}_2^1[0,1]$ *is given as:*

$$\mathcal{W}_2^1[0,1] = \{y : y \in AC[0,1] \text{ and } y' \in L^2[0,1]\},$$

with

$$\langle y, g\rangle_{\mathcal{W}_2^1} = \int_0^1 \left[y(x)g(x) + y'(x)g'(x)\right]\mathrm{d}x, \quad y, g \in \mathcal{W}_2^1[0,1],$$

and

$$\|y\|_{\mathcal{W}_2^1} = \sqrt{\langle y, y\rangle_{\mathcal{W}_2^1}}, \quad y \in \mathcal{W}_2^1[0,1], \tag{3}$$

as the inner product and the norm in $\mathcal{W}_2^1[0,1]$ respectively. Reproducing kernel function $T_x(y)$ of $\mathcal{W}_2^1[0,1]$ is presented as:

$$T_x(y) = \frac{1}{2\sinh(1)}\left[\cosh(x+y-1) + \cosh(|x-y|-1)\right]. \tag{4}$$

Definition 3. *The space ${}^o\mathcal{W}_2^3[0,1]$ is given by*

$${}^o\mathcal{W}_2^3[0,1] = \{y \in AC[0,1] : \; y', y'' \in AC[0,1], \; y^{(3)} \in L^2[0,1], \; y(0) = 0 = y(1)\}.$$

$$\langle y, v\rangle_{{}^o\mathcal{W}_2^3[0,1]} = y(0)v(0) + y'(0)v'(0) + y(1)v(1) + \int_0^1 y^{(3)}(x)v^{(3)}(x)\mathrm{d}x, \quad y, v \in {}^o\mathcal{W}_2^3[0,1],$$

and

$$\|y\|_{{}^o\mathcal{W}_2^3[0,1]} = \sqrt{\langle y, y\rangle_{{}^o\mathcal{W}_2^3[0,1]}}, \quad y \in {}^o\mathcal{W}_2^3[0,1],$$

are the inner product and the norm in ${}^o\mathcal{W}_2^3[0,1]$ respectively.

Theorem 1. *Reproducing kernel function $y(x)$ of ${}^o\mathcal{W}_2^3[0,1]$ is given as*

$$\mathcal{R}_\xi(x) = \begin{cases} -x^2\xi - x\xi^2 + x\xi - \frac{1}{120}x^2\xi^2 + \frac{21}{20}x^2\xi^2 \\ +\frac{1}{24}x^2\xi^4 - \frac{1}{12}x^2\xi^3 \\ +\frac{1}{24}x^4\xi^2 - \frac{1}{24}x^4\xi - \frac{1}{120}x^5\xi^2 + \frac{x^5}{120}, \\ 0 \le x \le \xi \le 1, \\ \\ -\xi^2 x - \xi x^2 + \xi x - \frac{1}{120}\xi^2 x^2 + \frac{21}{20}\xi^2 x^2 \\ +\frac{1}{24}\xi^2 x^4 - \frac{1}{12}\xi^2 x^3 \\ +\frac{1}{24}\xi^4 x^2 - \frac{1}{24}\xi^4 x - \frac{1}{120}\xi^5 x^2 + \frac{\xi^5}{120}, \\ 0 \le \xi < x \le 1. \end{cases} \tag{5}$$

Proof. First, let us suppose

$$\mathcal{R}_\xi(x) = \begin{cases} \sum_{i=1}^6 c_i(\xi)x^{i-1}, & 0 \le x \le \xi \le 1, \\ \sum_{i=1}^6 d_i(\xi)x^{i-1}, & 0 \le \xi < x \le 1. \end{cases} \tag{6}$$

Then from $y \in {}^o\mathcal{W}_2^3[0,1]$, we get

$$\begin{aligned}
\langle y(x), \mathcal{R}_\xi(x)\rangle_{{}^o\mathcal{W}_2^3[0,1]} &= y(0)\mathcal{R}_\xi(0) + y'(0)\mathcal{R}'_\xi(0) + y(1)\mathcal{R}_\xi(1) + \int_0^1 y^{(3)}(x)\frac{\partial^3\mathcal{R}_\xi(x)}{\partial x^3}\mathrm{d}x \\
&= y(0)\mathcal{R}_\xi(0) + y'(0)\frac{\partial\mathcal{R}_\xi(0)}{\partial x} \\
&+ y(1)\mathcal{R}_\xi(1) + y''(1)\frac{\partial^3\mathcal{R}_\xi(1)}{\partial x^3} \\
&- y''(0)\frac{\partial^3\mathcal{R}_\xi(0)}{\partial x^3} - y'(1)\frac{\partial^4\mathcal{R}_\xi(1)}{\partial x^4} \\
&+ y'(0)\frac{\partial^4\mathcal{R}_\xi(0)}{\partial x^4} + y(1)\frac{\partial^5\mathcal{R}_\xi(1)}{\partial x^5} - y(0)\frac{\partial^5\mathcal{R}_\xi(0)}{\partial x^5} \\
&- \int_0^1 y(x)\frac{\partial^6\mathcal{R}_\xi(x)}{\partial x^6}\mathrm{d}x.
\end{aligned}$$

Solving the coefficients, we get the reproducing kernel function as:

$$\mathcal{R}_\xi(x)=\begin{cases}-x^2\xi-x\xi^2+x\xi-\frac{1}{120}x^2\xi^2+\frac{21}{20}x^2\xi^2\\+\frac{1}{24}x^2\xi^4-\frac{1}{12}x^2\xi^3\\+\frac{1}{24}x^4\xi^2-\frac{1}{24}x^4\xi-\frac{1}{120}x^5\xi^2+\frac{x^5}{120},\\0\le x\le\xi\le 1,\\ \\-\xi^2x-\xi x^2+\xi x-\frac{1}{120}\xi^2x^2\\+\frac{21}{20}\xi^2x^2+\frac{1}{24}\xi^2x^4-\frac{1}{12}\xi^2x^3\\ \\+\frac{1}{24}\xi^4x^2-\frac{1}{24}\xi^4x-\frac{1}{120}\xi^5x^2+\frac{\xi^5}{120},\\0\le\xi<x\le 1.\end{cases} \quad (7)$$

2.1. Solutions in ${}^o\mathcal{W}_2^3[0,1]$

We consider the solution of Eq.(1) in the reproducing kernel space ${}^o\mathcal{W}_2^3[0,1]$ in this section. On defining the operator

$$\mathcal{L}:\ {}^o\mathcal{W}_2^3[0,1]\to\mathcal{W}_2^1[0,1],$$

problem (1) converts as:

$$\begin{cases}\mathcal{L}y=\mathfrak{f}(x,u), & x\in[0,1],\\ y(0)=A, y(1)=B,\end{cases} \quad (8)$$

We should homogenize the conditions. Put

$$u(x)=y(x)+(A-B)x-A,$$

then we can obtain homogeneous boundary-value conditions of problem (1)

$$\mathcal{L}u(x)=p^{'}(x)u^{'}+p(x)u^{''}. \quad (9)$$

With the boundary conditions:

$$\begin{cases}\mathcal{L}u=\mathfrak{g}(x,u), & x\in[0,1],\\ u(0)=u(1)=0,\end{cases} \quad (10)$$

where

$$\mathfrak{g}(x,u)=f(x,u)+p^{'}(x)(A-B). \quad (11)$$

Theorem 2. *The operator $\mathcal{L}$ is a bounded linear operator.*

Proof. Firstly, we present $\|\mathcal{L}u\|^2_{\mathcal{W}_2^1}\le\mathcal{M}\,\|u\|^2_{{}^o\mathcal{W}_2^3}$, with $\mathcal{M}>0$. By
(3) and (3), we get

$$\|\mathcal{L}u\|^2_{\mathcal{W}_2^1}=\langle\mathcal{L}u,\mathcal{L}u\rangle_{\mathcal{W}_2^1}$$
$$=\int_0^1\Big[((\mathcal{L}u)(x))^2+((\mathcal{L}u)'(x))^2\Big]\mathrm{d}x.$$

Moreover, by reproducing property we have:

$$y(x)=\langle y(\cdot),\mathcal{R}_x(\cdot)\rangle_{{}^o\mathcal{W}_2^3}.$$

Then, we get

$$\begin{aligned}\mathcal{L}u(x)&=\langle u(\cdot),\mathcal{L}\mathcal{R}_x(\cdot)\rangle_{{}^o\mathcal{W}_2^3}\\&=\langle u(\cdot),(\mathcal{L}_1+\mathcal{L}_2)\mathcal{R}_x(\cdot)\rangle_{{}^o\mathcal{W}_2^3}\\&=\langle u(\cdot),\mathcal{L}_1\mathcal{R}_x(\cdot)\rangle_{{}^o\mathcal{W}_2^3}+\langle u(\cdot),\mathcal{L}_2\mathcal{R}_x(\cdot)\rangle_{{}^o\mathcal{W}_2^3}.\end{aligned}$$

With condition to $p(x)\in C^2[0,1]$, $\mathcal{M}_p=\max\{|p(x)|,\left|p^{'}(x)\right|,\left|p^{''}(x)\right|\quad 0\le x\le 1\}$, $\mathcal{M}_1=\max\{\frac{\partial}{\partial x}\mathcal{R}_x(\xi)\quad 0\le\xi\le 1\}$, and $\mathcal{M}_2=\max\{\frac{\partial}{\partial x^2}\mathcal{R}_x(\xi)\quad 0\le\xi\le 1\}$ then

$$\begin{aligned}|\mathcal{L}u(x)|&\le\|u\|_{{}^o\mathcal{W}_2^3}\|\mathcal{L}_1\mathcal{R}_x\|_{{}^o\mathcal{W}_2^3}+\|u\|_{{}^o\mathcal{W}_2^3}\|\mathcal{L}_2\mathcal{R}_x\|_{{}^o\mathcal{W}_2^3}\\&=\mathcal{M}_1\mathcal{M}_p\|u\|_{{}^o\mathcal{W}_2^3}+\mathcal{M}_2\mathcal{M}_p\|u\|_{{}^o\mathcal{W}_2^3}\\&=(\mathcal{M}_1+\mathcal{M}_2)\mathcal{M}_p\|u\|_{{}^o\mathcal{W}_2^3}\end{aligned}$$

where $\mathcal{M}_1>0,\mathcal{M}_2>0,\mathcal{M}_p>0$. Therefore

$$\int_0^1[(\mathcal{L}y)(x)]^2\,\mathrm{d}x\le(\mathcal{M}_1+\mathcal{M}_2)^2\mathcal{M}_p^2\|u\|^2_{{}^o\mathcal{W}_2^3}.$$

Also, from

$$\begin{aligned}(\mathcal{L}u)'(x)&=\left\langle u(\cdot),(\mathcal{L}\mathcal{R}_x)'(\cdot)\right\rangle_{{}^o\mathcal{W}_2^3}\\&=\left\langle u(\cdot),((\mathcal{L}_1+\mathcal{L}_2)\mathcal{R}_x)'(\cdot)\right\rangle_{{}^o\mathcal{W}_2^3}\\&=\left\langle u(\cdot),(\mathcal{L}_1\mathcal{R}_x)'(\cdot)\right\rangle_{{}^o\mathcal{W}_2^3}+\left\langle u(\cdot),(\mathcal{L}_2\mathcal{R}_x)'(\cdot)\right\rangle_{{}^o\mathcal{W}_2^3}\end{aligned}$$

we have condition to $p(x)\in C^2[0,1]$, $\mathcal{M}_p=\max\{|p(x)|,\left|p^{'}(x)\right|,\left|p^{''}(x)\right|0\le x\le 1\}$ $\mathcal{M}_1=\max\{\frac{\partial}{\partial x}\mathcal{R}_x(\xi)\quad 0\le\xi\le 1\},\mathcal{M}_2=\max\{\frac{\partial}{\partial x^2}\mathcal{R}_x(\xi)\quad 0\le\xi\le 1\}$ and $\mathcal{M}_3=\max\{\frac{\partial}{\partial x^2}\mathcal{R}_x(\xi)\quad 0\le\xi\le 1\}$ then

$$\begin{aligned}\left|(\mathcal{L}u)'(x)\right|&\le\|u\|_{{}^o\mathcal{W}_2^3}\left\|(\mathcal{L}_1\mathcal{R}_x)'\right\|_{{}^o\mathcal{W}_2^3}\\&+\|u\|_{{}^o\mathcal{W}_2^3}\left\|(\mathcal{L}_2\mathcal{R}_x)'\right\|_{{}^o\mathcal{W}_2^3}\\&=(\mathcal{M}_p\mathcal{M}_1+\mathcal{M}_p\mathcal{M}_2)\|u\|_{{}^o\mathcal{W}_2^3}\\&+(\mathcal{M}_p\mathcal{M}_2+\mathcal{M}_p\mathcal{M}_3)\|u\|_{{}^o\mathcal{W}_2^3}\\&=\mathcal{M}_p(\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3)\|u\|_{{}^o\mathcal{W}_2^3}\end{aligned}$$

where $\mathcal{M}_1>0,\mathcal{M}_2>0,\mathcal{M}_3>0,\mathcal{M}_p>0$. Thus, we get

$$\left[(\mathcal{L}y)'(x)\right]^2\le\mathcal{M}_p^2(\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3)^2\|u\|^2_{{}^o\mathcal{W}_2^3}$$

and

$$\int_0^1\left[(\mathcal{L}u)'(x)\right]^2\mathrm{d}x\le\mathcal{M}_p^2(\mathcal{M}_1+\mathcal{M}_2+\mathcal{M}_3)^2\|u\|^2_{{}^o\mathcal{W}_2^3},$$

that is

$$\|\mathcal{L}y\|^2_{\mathcal{W}^1_2} = \int_0^1 \left[[(\mathcal{L}u)(x)]^2 + \left[(\mathcal{L}u)'(x)\right]^2 \right] \mathrm{d}x$$
$$\leq (\mathcal{M}_1 + \mathcal{M}_2)^2 \mathcal{M}_p^2 \|u\|^2_{{}^o\mathcal{W}^3_2}$$
$$+ \mathcal{M}_p^2(\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)^2 \|u\|^2_{{}^o\mathcal{W}^3_2}$$
$$= \mathcal{M}_p^2((\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)^2 + (\mathcal{M}_1 + \mathcal{M}_2)^2) \|u\|^2_{{}^o\mathcal{W}^3_2}$$
$$= \mathcal{M} \|u\|^2_{{}^o\mathcal{W}^3_2}$$

where $\mathcal{M} = \mathcal{M}_p^2((\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3)^2 + (\mathcal{M}_1 + \mathcal{M}_2)^2) > 0$. □

2.2. Solutions of the problems

Obviously, defined operator in (9) as $\mathcal{L} : {}^o\mathcal{W}^3_2[0,1] \to \mathcal{W}^1_2[0,1]$ is a bounded linear operator.

Let us define $\varphi_i(x) = T_{x_i}(x)$ and $\psi_i(x) = \mathcal{L}^*\varphi_i(x)$, where $\mathcal{L}^*$ is conjugate operator of $\mathcal{L}$. The orthonormal system $\{\hat{\psi}_i(x)\}_1^\infty \subseteq {}^o\mathcal{W}^3_2[0,1]$ can be attained by the Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_1^\infty$:

$$\hat{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \ (\beta_{ii} > 0, i = 1, 2, ...). \quad (12)$$

Theorem 3. *If $y(x)$ is the exact solution of (10), then*

$$y(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\mathfrak{f}(x_k, y_k)\hat{\psi}_i(x), \quad (13)$$

where $\{x_i\}_1^\infty$ is dense in $[0,1]$.

Proof. By the (12) and uniqueness of solution of (10) we obtain:

$$y(x) = \sum_{i=1}^{\infty} \left\langle y(x), \hat{\psi}_i(x) \right\rangle_{{}^o\mathcal{W}^3_2} \hat{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle y(x), \psi_k(x) \right\rangle_{{}^o\mathcal{W}^3_2} \hat{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle y(x), \mathcal{L}^*\varphi_k(x) \right\rangle_{{}^o\mathcal{W}^3_2} \hat{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle \mathcal{L}y(x), \varphi_k(x) \right\rangle_{\mathcal{W}^1_2} \hat{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} \left\langle \mathfrak{f}(x,y), T_{x_k} \right\rangle_{\mathcal{W}^1_2} \hat{\psi}_i(x)$$
$$= \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\mathfrak{f}(x_k, y_k)\hat{\psi}_i(x).$$

□

Finite terms of (13) concludes the approximate solution:

$$y_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik}\mathfrak{f}(x_k, y_k)\hat{\psi}_i(x). \quad (14)$$

Lemma 1. *If $\|y_n - y\|_{{}^o\mathcal{W}^3_2} \to 0$, $x_n \to x$,($n \to \infty$) and $\mathfrak{f}(x,y)$ is continuous w.r.t. $x \in [0,1]$, then*

$$\mathfrak{f}(x_n, y_{n-1}(x_n)) \to \mathfrak{f}(x, u(x)), \quad \textit{as } n \to \infty.$$

Theorem 4. *Let for any fixed $y_0(x) \in {}^o\mathcal{W}^3_2[0,1]$ we have*

(i)
$$y_n(x) = \sum_{i=1}^{n} A_i\hat{\psi}_i(x), \quad (15)$$
where
$$A_i = \sum_{k=1}^{i} \beta_{ik}\mathfrak{f}(x_k, y_{k-1}(x_k)), \quad (16)$$

(ii) *$\|y_n\|_{{}^o\mathcal{W}^3_2}$ is bounded;*

(iii) *$\{x_i\}_1^\infty$ is dense in $[0,1]$;*

(iv) *$\mathfrak{f}(x,y) \in \mathcal{W}^1_2[0,1]$ for any $y(x) \in {}^o\mathcal{W}^3_2[0,1]$. Then the approximate solution $y_n(x)$ converges to the exact solution of (13) in ${}^o\mathcal{W}^3_2$ and we have*
$$y(x) = \sum_{i=1}^{\infty} A_i\hat{\psi}_i(x).$$

Proof. First, we prove the convergence of $y_n(x)$. From (15), we have

$$y_{n+1}(x) = y_n(x) + A_{n+1}\hat{\psi}_{n+1}(x). \quad (17)$$

Also, orthonormality of $\{\hat{\psi}_i\}_{i=1}^\infty$, yields

$$\|y_{n+1}\|^2_{{}^o\mathcal{W}^3_2} = \|y_n\|^2_{{}^o\mathcal{W}^3_2} + A^2_{n+1}$$
$$= \|y_{n-1}\|^2_{{}^o\mathcal{W}^3_2} + A^2_n + A^2_{n+1} = ... = \sum_{i=1}^{n+1} A_i^2, \quad (18)$$

and from boundedness of $\|y_n\|_{{}^o\mathcal{W}^3_2}$, we obtain

$$\sum_{i=1}^{\infty} A_i^2 < \infty,$$

i.e.,

$$\{A_i\} \in l^2 \quad (i = 1, 2, ...).$$

Let $m > n$, in view of $(y_m - y_{m-1}) \perp (y_{m-1} - y_{m-2}) \perp ... \perp (y_{n+1} - y_n)$, we get

$$\|y_m - y_n\|^2_{^o\mathcal{W}_2^3} = \|y_m - y_{m-1} + y_{m-1} - y_{m-2} + ... + y_{n+1} - y_n\|^2_{^o\mathcal{W}_2^3}$$
$$= \|y_m - y_{m-1}\|^2_{^o\mathcal{W}_2^3} + ... + \|y_{n+1} - y_n\|^2_{^o\mathcal{W}_2^3}$$
$$= \sum_{i=n+1}^{m} A_i^2 \to 0, \quad m, n \to \infty.$$

Considering the completeness of $^o\mathcal{W}_2^3[0,1]$, there exists $y(x) \in {^o\mathcal{W}_2^3}[0,1]$, such that

$$y_n(x) \to y(x) \quad as\ n \to \infty.$$

(ii) Now, we show $y(x)$ is the exact solution of (10). Tacking limits in (15) we get

$$y(x) = \sum_{i=1}^{\infty} A_i \hat{\psi}_i(x).$$

Thus, we reach

$$(\mathcal{L}y)(x_2) = \mathfrak{f}(x_2, y_1(x_2)).$$

Moreover, by induction we conclude

$$(\mathcal{L}y)(x_j) = \mathfrak{f}(x_j, y_{j-1}(x_j)). \tag{19}$$

From $\overline{\{x_i\}_{i=1}^{\infty}} = [0,1]$, it can be presented that for any $\xi \in [0,1]$, there exists $\{x_{n_j}\}_1^{\infty} \subseteq \{x_i\}_1^{\infty}$, such that $\lim_{j\to\infty} x_{n_j} = \xi$. Therefore, convergence of $y_n(x)$ and Lemma 4.3 yields

$$(\mathcal{L}y)(\xi) = \mathfrak{f}(\xi, y(\xi)).$$

So, $y(x)$ is the exact solution of (10) given by

$$y(x) = \sum_{i=1}^{\infty} A_i \hat{\psi}_i(x),$$

where A_i are given by (16). □

Theorem 5. *If* $y \in {^o\mathcal{W}_2^3}[0,1]$ *then*

$$\|y_n - y\|_{^o\mathcal{W}_2^3} \to 0, \quad n \to \infty.$$

Moreover a sequence $\|y_n - y\|_{^o\mathcal{W}_2^3}$ *is monotonically decreasing in* n.

Proof.
By (13) and (14), we acquire

$$\|y_n - y\|_{^o\mathcal{W}_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \mathfrak{f}(x_k, y_k) \hat{\psi}_i \right\|_{^o\mathcal{W}_2^3}.$$

Therefore, we obtain

$$\|y_n - y\|_{^o\mathcal{W}_2^3} \to 0, \quad n \to \infty.$$

In addition

$$\|y_n - y\|^2_{\mathcal{W}_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \beta_{ik} \mathfrak{f}(x_k, y_k) \hat{\psi}_i \right\|^2_{^o\mathcal{W}_2^3}$$
$$= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} \mathfrak{f}(x_k, y_k) \hat{\psi}_i \right)^2.$$

□

3. Numerical examples

We consider the following problems by reproducing kernel Hilbert space method in this section. We computed our results by MAPLE. We showed our results by tables.

Example 1. We investigate:

$$\left(\frac{1}{1+t} u(t)'\right)' = 2\exp(3u(t)) \quad 0 \le t \le 1 \tag{20}$$

$$u(0) = 0 \quad u(1) = -\log_e(2)$$

We have the exact solution of the above problem as:

$$u(t) = \log_e(1/(1+t)).$$

We searched the boundary value problem (20) by the proposed method and gave corresponding error-norms by Table 1.

Example 2. We solved the following problem for the second example in the reproducing kernel Hilbert space.

$$((1+t^2)u(t)')' - (1+t-t^2)u(t) = h(t) \tag{21}$$

$$u(0) = 0 \quad u(1) = 0$$

We get the exact solution of the above problem as:

$$u(t) = 1 + (t-1)\exp(-t) - t\exp(-(1-t)).$$

In Table 2, we computed absolute errors for (21).

Table 1. Maximum absolute errors (MAE) of the first example.

	$N = 64,\ \sigma = 1.02$	$N = 64,\ \sigma = 1.02$
RKHSM	$5.46E-12$	$5.46E-12$
[10]	$1.70E-08$	$2.85E-06$
[11]	$8.49E-04$	$1.21E-02$
[12]	$2.43E-03$	$1.88E-02$
[13]	$5.63E-03$	$2.70E-02$

Table 2. Maximum absolute errors (MAE) of the second example.

	$N = 64,\ \sigma = 1.02$	$N = 64,\ \sigma = 1.02$
RKHSM	$9.20E-11$	$9.20E-11$
[10]	$8.07E-06$	$6.09E-05$
[11]	$1.86E-03$	$1.08E-02$
[12]	$2.16E-03$	$9.87E-03$
[13]	$5.19E-04$	$2.64E-03$

4. Conclusions

In this work, we gave a new application of the reproducing kernel Hilbert space method. We obtained very useful reproducing kernel functions in the reproducing kernel Hilbert spaces. We proved the accuracy of the method. We compared the reproducing kernel Hilbert space method with the techniques existed in the literature. We concluded that the proposed technique is very effective for solving nonlinear two-point boundary value problems.

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